A construction of vertex-transitive non-Cayley graphs

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Abstract We present a new construction of infinite families of (finite as well as infinite) vertex-transitive graphs that are not Cayley graphs; many of these turn out even to be arc-transitive. The construction is based on representing vertex-transitive graphs as coset graphs of groups, and on a simple but powerful necessary arithmetic condition for Cayley graphs.

1 Introduction

Vertex-transitive graphs are interesting from both combinatorial as well as group-theoretical point of view, and have been studied extensively for more than a century. Although the well-known Cayley graphs have played a prominent role here, there has been an increasing interest in the other side of the fence - that is, in vertex-transitive graphs that are not Cayley graphs (we borrow the acronym VTNCG for such objects from [8]). By [6] , the problem of constructing VTNCG's is equivalent to the widely studied problem of the existence of certain permutation groups that do not have regular subgroups. And yet, only a few infinite families of VTNCG's have been described in the literature before 1980.

The situation has changed dramatically since then; a great deal of the activity was prompted by Marušič's question [3] of characterizing the values of n for which there exists a VTNCG on n vertices. This resulted in quite a variety of constructions of infinite families of VTNCG's; the most recent ones appear in [4] together with a large bibliography of other constructions. The general question of characterizing all VTNCG's is probably beyond our reach in the foreseeable future, but much progress has been done for orders that have only a few prime factors (see again [4] for references). New constructions of infinite families of VTNCG's are therefore of growing interest.

Basically, there seem to be two main approaches to the problem. The first assumes that we have enough information on the automorphism group of a given graph to show that it is transitive and cannot contain a regular subgroup. Examples with this property are mostly found among graphs that are related to some of the well-known families of finite groups, and most of the constructions listed or cited in [4] would fall in this category. The second approach consists in trying to reveal (without invoking the automorphism group) some structural conditions that a Cayley graph has to satisfy, and then show that these are not met by a particular class of vertex-transitive graphs. For such necessary conditons and corresponding constructions the reader is referred to [1, 2, 8].

In this paper we present a new construction of infinite families of (finite as well as infinite) VTNCG's. Moreover, imposing additional conditions we even obtain arc-transitive non-Cayley graphs (ATNCG's, for short). In a way, our method is a combination of the above ones. First, we represent a vertex-transitive graph by means of a suitable coset graph as in [5, 7, 9] (this is, in fact, equivalent to having a certain information about the structure of some transitive subgroup of the automorphism group of the graph). Then, we show that under some restrictions, our coset graphs do not satisfy a simple but efficient necessary condition for Cayley graphs, given in [1].

2 Preliminaries

Graphs considered in our paper may be finite or infinite, but are always locally finite (i.e., every vertex has finite valency), loopless and without multiple edges.

Let G be a (finite or infinite) group and X be a unit-free symmetric subset of G, that is, $1 \notin X$ and $x^{-1} \in X$ whenever $x \in X$. The *Cayley graph* $\Gamma = C(G, X)$ has G as its vertex set, and two vertices $a, b \in G$ are adjacent if and only if $a^{-1}b \in X$. Note that we do not require the set X to be a generating set for G and therefore we allow also disconnected Cayley graphs. The graph Γ is locally finite if and only if the set X is finite. In any case, the group G acts regularly (as a subgroup of automorphisms) on the vertex set of $\Gamma = C(G, X)$ by left multiplication, which shows that every Cayley graph is vertex-transitive. This necessary condition is not sufficient, and it is therefore reasonable to ask for more conditions imposed by the Cayley graph structure. The following simple fact, proved in [1], seems to be quite powerful in applications. It focuses on oriented closed walks of prime length p based at a fixed vertex, that is, on ordered sequences $(a_0, a_1, \ldots, a_p = a_0)$ of (not necessarily distinct) vertices such that a_{i-1} and a_i are adjacent for each i, $1 \le i \le p$.

Lemma 1 *Let $\Gamma = C(G,X)$ be a locally finite Cayley graph and p be an odd prime. Then the number of closed oriented walks of length p, based at any fixed vertex of Γ, is congruent $(\mathrm{mod}\ p)$ to the number of elements in X of order p.* $\square$

We note that Lemma 1 was proved in [1] for finite graphs only, but the same proof is valid for the locally finite case as well.

The central concept of this paper is the one of a coset graph. Such graphs have apparently been known as "folklore" for decades (see [9] , or [5] for a more recent treatment; they happen to be a special case of the two-sided coset graphs introduced in [7]). Let G be a group, H a subgroup of G and X a symmetric subset of elements of G such that $H \cap X = \emptyset$. The vertex set of the *coset graph* $Cos(G,H,X)$ is the set of all left cosets of H in G; two vertices (cosets) aH and bH are adjacent in $Cos(G,H,X)$ if and only if $a^{-1}b \in HXH = \{hxh';\ x \in X \text{ and } h,h' \in H\}$. It is easy to check that this definition is correct, i.e., it does not depend on the choice of coset representatives and it produces graphs without loops and parallel edges.

An alternate way to define the incidence relation on $Cos(G,H,X)$ is by referring to the *associated Cayley graph* $C(G,X)$: Two cosets aH, bH are adjacent in $Cos(G,H,X)$ provided that there exist $h,h' \in H$ such that ah and bh' are adjacent vertices in the associated Cayley graph $C(G,X)$. The coset graph $Cos(G,H,X)$ can therefore be viewed as a graph obtained by "factoring" the associated Cayley graph $C(G,X)$ by the subgroup H. It is an easy exercise to show that the coset graph $Cos(G,H,X)$ is connected if and only if the set HXH is a generating set for the group G. Observe that in the special case when $H = \{1\}$, the coset graph reduces to a Cayley graph. For more information on coset graphs we refer the reader to [5].

As in the case of Cayley graphs, the group G acts transitively as a group of automorphisms of the coset graph $Cos(G,H,X)$ by left multiplication, and therefore every coset graph is vertex-transitive. (However, the action is no longer regular in general.) The converse has been proved in [5, 7, 9] for finite graphs, but the same proof applies also to infinite graphs (possibly of infinite valency): Given a vertex-transitive graph Γ, take a transitive subgroup of its automorphisms for G, the G-stabilizer of a fixed vertex for H, and define X to be the subset of automorphisms of G that are sending the fixed vertex to its neighbours. Then Γ is isomorphic to $Cos(G,H,X)$.

Lemma 2 *A graph Γ is vertex-transitive if and only if it is isomorphic to some coset graph $Cos(G,H,X)$.* $\square$

In some cases we can guarantee even a higher degree of symmetry of the coset graphs, namely, their arc-transitivity. The following simple observation shows how the existence of suitable *group automorphisms* can be used in this context. For notational convenience, if $Cos(G,H,X)$ is a coset graph, let $Aut_{H;X}(G)$ be the group of all the automorphisms of G which fix both X and the subgroup H setwise.

Lemma 3 *Let $\Gamma = Cos(G,H,X)$ be a coset graph. Assume that the group $Aut_{H;X}(G)$ contains a subgroup that acts transitively on X. Then Γ is an arc-transitive graph.*

Proof. Since $\Gamma = Cos(G, H, X)$ is vertex-transitive, it is sufficient to show that for any two neighbors aH and bH of the vertex H there is an automorphism of the graph Γ that fixes H and sends aH onto bH. According to the definition of a coset graph, the fact that aH and bH are neighbors of H means that there exist $x, y \in X$ such that, say, $a \in HxH$ and $b \in HyH$. (Note: It may happen that $x = y$ and yet $aH \neq bH$.) Since we are interested in *cosets* aH and bH, we may without loss of generality assume that $a \in Hx$ and $b \in Hy$. Thus, $a = h_1x$ and $b = h_2y$ for suitable $h_1, h_2 \in H$.

By our assumption, there exists a subgroup $K < Aut_{H;X}(G)$ and an automorphism $\phi \in K$ such that $\phi(x) = y$ (in the case when $x = y$ we simply take the identity automorphism). The properties of ϕ (an automorphism of G, leaving H and X invariant as sets) imply that the mapping $\phi^* : cH \mapsto \phi(c)H$ is a well defined automorphism of the graph $\Gamma = Cos(G, H, X)$. Clearly, ϕ^* maps the arc (H, aH) onto the arc $(H, \phi(a)H)$. Further, from $a = h_1x$ we have $\phi(a) = \phi(h_1)\phi(x) = \phi(h_1)y$, where $\phi(h_1) \in H$. Combining this with $b = h_2y$, we see that $b = h\phi(a)$ for $h = h_2\phi(h_1^{-1}) \in H$. Consider now the mapping $\psi_h : cH \mapsto hcH$. It is easy to see that ψ_h is again an automorphism of the coset graph $\Gamma = Cos(G, H, X)$. Note that ψ_h fixes H and sends $\phi(a)H$ onto bH. Hence, the composition $\psi_h\phi^*$ is an automorphism of Γ that fixes H and maps aH onto bH, as desired. □

As a consequence, we see that the graph $Cos(G, H, X)$ is arc-transitive if $|X| = 1$; this was also shown in [5, Theorem 3].

3 Main results

The coset graph construction is general enough to yield all vertex-transitive graphs. In order to obtain VTNCG's, we need to impose certain restrictions on the triple (G, H, X). Applying Lemma 1, we will then be able to prove that the resulting graphs are not Cayley.

Theorem 1 *Let G be a group, let H be a finite subgroup of G, and let X be a finite symmetric unit-free subset of G such that $XHX \cap H = \{1\}$. Further, suppose that there are at least $|X| + 1$ distinct ordered pairs $(x, h) \in X \times H$ such that $(xh)^p = 1$ for some fixed prime $p > |X||H|^2$. Then the coset graph $\Gamma = Cos(G, H, X)$ is a vertex-transitive non-Cayley graph.*

Proof. Observe that the condition $XHX \cap H = \{1\}$ implies $H \cap X = \emptyset$, and so our coset graph is well defined. Let us begin by showing that the valency of Γ is $|H||X|$. This could be done using [5, Theorem 2] but we prefer here a different way in order to obtain more information about our coset graph.

Let aH be an arbitrary vertex of Γ and let bH be a neighbor of aH, that is, $a^{-1}b \in HXH$ (or, $b \in aHXH$). We claim that there exists a *unique* $x \in X$ such that $b \in aHxH$, and that x is independent of the coset representatives (we shall later refer to this x as the *color* of the edge emanating from aH and terminating at bH). Indeed, let $a'H = aH$, $b'H = bH$, and assume that there are two elements $x, x' \in X$

such that $b \in aHxH$ and $b' \in a'Hx'H$. Since $b^{-1}b' \in H$ and $a^{-1}a' \in H$, we have $(Hx^{-1}Hx'H) \cap H \neq \emptyset$, which implies that $x^{-1}Hx' \cap H \neq \emptyset$. From our assumption $XHX \cap H = \{1\}$ we now have $x^{-1}Hx' \cap H = \{1\}$, that is, $x^{-1}hx' = 1$ for some $h \in H$. But then, $h = xx'^{-1} \in XHX$. Invoking our assumption again, we see that $h = 1 = xx'^{-1}$, and so $x' = x$, as claimed.

Now we show that for each $x \in X$ there are exactly $|H|$ edges of color x emanating from aH. For any endvertex bH of such an edge we have $b \in aHxH$. Therefore the number of cosets bH such that $b \in aHxH$ is equal to the number of elements in the set aHx, which is $|H|$ (here we also use the fact that $XHX \cap H = \{1\}$). Hence the valency of Γ is $|H||X|$, as stated above.

As the next step, we investigate the closed oriented walks of length p in Γ, based at a fixed vertex a_0H. Since the graph does not have multiple edges, every such walk is represented by a sequence $W = (a_0H, a_1H, \ldots, a_pH = a_0H)$, where $a_{i-1}H$ is adjacent to a_iH for $1 \leq i \leq p$. Let $x_i \in X$ be the (uniquely determined) color of the edge emanating from $a_{i-1}H$ and terminating at a_iH, $1 \leq i \leq p$. This means that there exist elements $r_i \in H$ and $s_i \in H$ such that

$$a_i = a_{i-1}r_ix_is_i, \quad 1 \leq i \leq p\ . \tag{1}$$

This recursion yields $a_p = a_0 \prod_{i=1}^{p} r_ix_is_i$. Moreover, $a_pH = a_0H$ implies that $a_p = a_0t$ for some $t \in H$, and so $\prod_{i=1}^{p} r_ix_is_i = t$. Letting $h_i = s_ir_{i+1}$ for $1 \leq i \leq p-1$ and $h_p = s_pt^{-1}r_1$, the last product equation reduces to

$$\prod_{i=1}^{p} x_ih_i = 1\ . \tag{2}$$

Now let $b_0 = a_0r_1$ and $b_i = b_{i-1}x_ih_i$ for $1 \leq i \leq p$. It is easy to show (by induction) that $b_i = a_ir_{i+1}$ for $0 \leq i \leq p-1$. (This is obvious for $i = 0$, and the induction step uses (1): $b_i = b_{i-1}x_ih_i = a_{i-1}r_ix_is_ir_{i+1} = a_ir_{i+1}$.) Consider the oriented closed walk $(b_0H, b_1H, \ldots, b_pH)$. We claim that this walk is *identical* with our walk $W = (a_0H, a_1H, \ldots, a_pH)$. To see it, we again apply induction. Clearly, $b_0H = a_0r_1H = a_0H$. For $1 \leq i \leq p$ the induction step yields (recall that $b_{i-1} = a_{i-1}r_i$ and observe that $h_iH = s_iH$):

$$b_iH = b_{i-1}x_ih_iH = b_{i-1}x_is_iH = a_{i-1}r_ix_is_iH = a_iH\ .$$

We call this walk $(b_0H, b_1H, \ldots, b_pH)$ the *canonical form* of W; we point out that the corresponding parameters b_i, $0 \leq i \leq p$ and h_i, $1 \leq i \leq p$ satisfy the conditions

$$h_i \in H, \quad b_0 \in a_0H, \quad b_i = b_{i-1}x_ih_i, \quad \prod_{i=1}^{p} x_ih_i = 1\ . \tag{3}$$

In what follows we show that every closed walk in Γ has a *unique* canonical form, that is, the elements x_i, h_i and b_i are uniquely determined by W and the conditions (3). To see this, let $(b_0'H, b_1'H, \ldots, b_p'H)$ be another canonical form for the same walk W as above. That is, the associated parameters b_i', $0 \leq i \leq p$ and h_i', $1 \leq i \leq p$

satisfy (3): $h'_i \in H$, $b'_0 \in a_0H$, $b'_i = b'_{i-1}x_ih'_i$ $(1 \leq i \leq p)$ and $\prod_{i=1}^{p} x'_ih'_i = 1$. (Note that we already know that the colors x_i are uniquely determined.) In addition, from the fact that both canonical forms represent the same W we know that $b'_iH = b_iH$ (or, $b_i^{-1}b'_i \in H$) for $0 \leq i \leq p$. Using the recursions for b_i and b'_i we obtain for $1 \leq i \leq p$:

$$h_i^{-1}x_i^{-1}b_{i-1}^{-1}b'_{i-1}x_ih'_i = b_i^{-1}b'_i \in H \ .$$

This is equivalent to

$$x_i^{-1}b_{i-1}^{-1}b'_{i-1}x_i = h_ib_i^{-1}b'_ih_i'^{-1} \in H \ .$$

By our assumption $XHX \cap H = \{1\}$ we have

$$x_i^{-1}b_{i-1}^{-1}b'_{i-1}x_i = 1 = h_ib_i^{-1}b'_ih_i'^{-1} \ . \tag{4}$$

The left-hand side of (4) implies immediately that $b'_{i-1} = b_{i-1}$ for $1 \leq i \leq p$; but then it follows from the right-hand side of (4) that also $h'_i = h_i$ for $1 \leq i \leq p-1$. Now, from the fact that $\prod_{i=1}^{p} x_ih_i = \prod_{i=1}^{p} x_ih'_i$ we see that $h'_p = h_p$ as well; invoking the right-hand side of (4) again we at last have $b'_p = b_p$.

We thus have established a 1-1 correspondence between the set $\mathbf{W}_p$ of all oriented closed walks of length p in Γ, based at a_0H, and the set I of all ordered $(p+1)$-tuples $(b_0; (x_1, h_1), (x_2, h_2), \ldots, (x_p, h_p))$ consisting of b_0 and pairs of elements of the group G such that $b_0 \in a_0H$, $x_i \in X$, $h_i \in H$ for $1 \leq i \leq p$, and $\prod_{i=1}^{p} x_ih_i = 1$. Namely, if $W \in \mathbf{W}_p$ is such a walk, then x_i are the edge colors, and b_0 and the h_i's are determined by the unique canonical form for W. Conversely, any $(p+1)$-tuple $(b_0; (x_1, h_1), (x_2, h_2), \ldots, (x_p, h_p))$ with the above properties determines (already in canonical form) the closed walk $(b_0H, b_1H, \ldots, b_pH) \in \mathbf{W}_p$ where $b_i = b_{i-1}x_ih_i$ for $1 \leq i \leq p$. Therefore the number $|\mathbf{W}_p|$ of all oriented a_0H-based closed walks of length p in Γ is equal to $|I|$, the number of $(p+1)$-tuples in I.

Note that $\prod_{i=1}^{p} x_ih_i = 1$ implies $(\prod_{i=2}^{p} x_ih_i)x_1h_1 = 1$. This innocently looking observation leads to the following basic property of the set I: If $(b_0; (x_1, h_1), (x_2, h_2), \ldots, (x_p, h_p))$ belongs to I then also the $(p+1)$-tuple $(b_0; (x_2, h_2), \ldots, (x_p, h_p), (x_1, h_1))$ is in I. We thus have an action Φ of the cyclic group Z_p generated by the permutation $\pi = (12...p)$ on the set I. If θ is any power of π, then Φ_θ sends a $(p+1)$-tuple $(b_0; (x_1, h_1), (x_2, h_2), \ldots, (x_p, h_p))$ to the $(p+1)$-tuple $(b_0; (x_{\theta(1)}, h_{\theta(1)}), (x_{\theta(2)}, h_{\theta(2)}), \ldots, (x_{\theta(p)}, h_{\theta(p)}))$. Note that Φ leaves b_0 invariant. Since p is a prime, each orbit of Φ on I has length 1 or p. Moreover, if a $(p+1)$-tuple $(b_0; (x_1, h_1), (x_2, h_2), \ldots, (x_p, h_p))$ constitutes a length 1 orbit of Φ, then $x_1 = \ldots = x_p = x$, $h_1 = \ldots = h_p = h$, and $(xh)^p = 1$. Conversely, if $x \in X$ and $h \in H$ are such that $(xh)^p = 1$, then these elements (together with an arbitrary $b_0 \in a_0H$) determine a length 1 orbit.

Let m denote the number of length 1 orbits of Φ. According to the above analysis, $|I| \equiv m \pmod p$, and since $|\mathbf{W}_p| = |I|$, also $|\mathbf{W}_p| \equiv m \pmod p$. Moreover, as we have seen, m is equal to the number of ordered pairs $(b_0, (x, h))$, where $b_0 \in a_0H$ and $x \in X$, $h \in H$ are such that $(xh)^p = 1$. Therefore,

$$(|X| + 1)|H| \leq m \leq |X||H|^2 \ . \tag{5}$$

Indeed, while the upper bound is obvious ($|a_0H| = |H|$), the lower bound follows from our assumption that there are at least $|X| + 1$ pairs $(x, h) \in X \times H$ for which $(xh)^p = 1$.

All the information about the structure of our graph $\Gamma = Cos(G, H, X)$ that we have obtained above is now going to be combined with Lemma 1 to show that Γ is a non-Cayley graph if the prime p is sufficiently large. Indeed, assume the contrary and let the coset graph $\Gamma = Cos(G, H, X)$ be a Cayley graph $C(G', X')$ for some group G' and a symmetric unit-free subset $X' \subset G'$. Let g denote the number of elements of prime order p in X'. Recalling the set $\mathbf{W}_p$ of all oriented a_0H-based closed walks of length p and invoking Lemma 1, we see that $|\mathbf{W}_p| \equiv g \pmod p$. On the other hand, as we have seen in the preceding paragraph, we also have $|\mathbf{W}_p| \equiv m \pmod p$. This yields

$$g \equiv m \ (mod\ p)\ . \tag{6}$$

Further, the number g cannot exceed the valency of Γ, and so

$$g \leq |X||H|\ . \tag{7}$$

However, an easy inspection shows that the relations (5), (6) and (7) are contradictory if $p > |X||H|^2$, and so Γ cannot be a Cayley graph. Hence (cf. Lemma 2), Γ is a VTNCG, as claimed. □

Combining Theorem 1 with Lemma 3 we obtain a means of constructing not only vertex-transitive, but even *arc-transitive* non-Cayley graphs:

Theorem 2 *Let a group G, a subgroup $H < G$, and a subset $X \subset G$ satisfy all assumptions of Theorem 1. Moreover, suppose that the group $Aut_{H;X}(G)$ contains a subgroup that acts transitively on X. Then the coset graph $\Gamma = Cos(G, H, X)$ is an arc-transitive non-Cayley graph.* □

4 Examples

This section is devoted to showing that the above theorems are suitable for constructing a variety of vertex-transitive as well as arc-transitive non-Cayley graphs. It should be said that our methods are generally producing VTNCG's and ATNCG's of large order but comparatively small valency. Also, it is not clear whether one can, in this way, obtain VTNCG's whose orders would have only a small number of prime factors. Nevertheless, we believe that our sample of constructions will be interesting from the point of view of a further study of VTNCG's and ATNCG's.

We start with the simplest case when the set X contains only one element (which is necessarily an involution). Note that the coset graphs built with a one-element set X are automatically arc-transitive (see Lemma 3).

Example 1. Let G be a (finite or infinite) quotient of the *triangle group* $(2, r, p)$, that is, $G = <x, y | x^2 = y^r = (xy)^p = \ldots = 1>$. Assume that the presentation of

G contains no relation of type $xy^ix = y^j$. Further, let $r \geq 3$ and let p be a prime greater than r^2. Then, Theorem 2 implies that the graph $Cos(G, <y>, \{x\})$ is a (connected) arc-transitive non-Cayley graph. Indeed, let $H = <y>$ and $X = \{x\}$. It is easy to see that the set HXH generates G. The absence of relations of the above type guarantees that $XHX \cap H = \{1\}$. Now, since $x^2 = 1$, $(xy)^p = 1$ implies that also $(xy^{-1})^p = 1$. Hence if $r \geq 3$ then there are at least 2 $(= |X| + 1)$ pairs $(x, h) \in X \times H$ such that $(xh)^p = 1$ for the prime $p > r^2$ $(= |X||H|^2)$. The rest follows from Theorem 2. □

Note that any graph constructed in this way is an underlying graph of a (finite or infinite) regular map. We thus obtained a special case of a more general result of [2], where it is shown that the underlying graph of any r-valent p-covalent regular map is an ATNCG provided that $r \geq 3$ and p is a prime greater than $r(r-1)$. (The argument in [2] is finer in that it uses some facts from the theory of regular maps, and applies also to graphs with loops and multiple edges.)

Example 2. Let $r \geq 2$ and $s \geq 2$ be such that $r + s$ is odd. Let S_p be the full symmetric group acting on the set $\{1, 2, \dots, p\}$ where $p \geq r + s + 2$ is a prime. Let $H = <y, z>$ be the subgroup of S_p generated by the permutations $y = (1\ 2\ \dots\ r)$ and $z = (r+1\ r+2\ \dots\ r+s)$. Obviously, $H \simeq Z_r \times Z_s$, and so $|H| = rs$. Further, let $x = (1\ 2\ \dots\ p)$ be a cyclic permutation of the entire underlying set and let $X = \{x, x^{-1}\}$. Let us consider the coset graph $\Gamma_p = Cos(S_p, H, X)$.

It is easy to see that if $r + s$ is odd then $<x, y, z> = S_p$. For instance, if r is even, then the permutation $w = (xy^{-1}x^{-1}yxy^{-1})^{p-r}$ is just a transposition of the elements 1 and r (the composition is to be read from the right to the left). Since p is a prime, the p-cycle x together with the transposition w are sufficient to generate S_p. Consequently, the coset graph Γ_p is connected.

A routine checking shows that $XHX \cap H = id$, the identity permutation. Moreover, if r is odd then the permutation xy is a p-cycle; the same is true for xz if s is odd. But $r + s$ is odd, and so there are at least three pairs in $X \times H$ such that the p-th power of their product is id; namely, x, x^{-1}, and one of xy and xz. By Theorem 1, if $p > 2r^2s^2$, then the coset graph $\Gamma_p = Cos(S_p, H, X)$ is a VTNCG (of order $p!/(rs)$ and valency $2rs$). □

An inspection of the above construction shows that we could have enlarged the set X by adjoining any number of new p-cycles x' to X (together with their inverses) such that $x'(i) = x(i)$ for $1 \leq i \leq r + s$ and $i = p$. Also, one can modify the construction to obtain infinite locally finite VTNCG's (details will be clear from Example 4). Still another modification of Example 2 yields ATNCG's as coset graphs of symmetric groups, as shown below.

Example 3. Again, let S_p be the symmetric group on the set $\{1, 2, \dots, p\}$ where $p \geq 2(r + s) + 1$ is a prime and $r + s$ is odd $(r, s \geq 2)$. We consider the same $X = \{x, x^{-1}\}$ where $x = (1, 2, \dots, p)$. However, this time we pick a larger subgroup of S_p: Let $H' = <y, y', z, z'>$ where $y = (1\ 2\ \dots\ r)$, $y' = (p-1\ p-2\ \dots\ p-r)$, $z = (r+1\ r+2\ \dots\ r+s)$, and finally, $z' = (p-r-1\ p-r-2\ \dots\ p-r-s)$. Now,

$H' \simeq Z_r \times Z_r \times Z_s \times Z_s$, and $|H'| = (rs)^2$.

The arguments used in the preceding example imply readily that the coset graph $\Gamma'_p = Cos(S_p, H', X)$ is a (connected) VTNCG if $p > |X||H'|^2 = 2(rs)^4$. But we can show more. Let $p = 2k+1$ and let $\sigma \in S_p$ be the involution $(1\ p-1)(2\ p-2)\dots(k\ k+1)$. Denote by ξ_σ the inner automorphism of S_p defined by σ, that is, $\xi_\sigma(w) = \sigma w \sigma$ for every $w \in S_p$. It is easy to verify that the subgroup $K = \{id, \xi_\sigma\} \subset Aut(S_p)$ fixes both H' and X and (obviously) acts transitively on X. It follows from Theorem 2 that Γ'_p is an ATNCG if $p > 2(rs)^4$; it has order $p!/(rs)^2$ and valency $2(rs)^2$. □

We have seen how to construct coset graphs that are VTNCG's (and also ATNCG's) using a cyclic group or some products of cyclic groups in place of H. Our last example presents a sufficiently general principle that can be adopted to construct finite as well as infinite VTNCG's by means of fairly arbitrary (abstract) groups H, and with arbitrarily large sets X.

Example 4. Let $m \geq 1$ and let M_i $(-m \leq i \leq m)$ be a system of pairwise disjoint sets of equal cardinality $|M_i| = q$ where $q \geq 3$ is an odd number. Let $p \geq (2m+1)q$ be a prime number. Take a finite set M' disjoint from all M_i such that $|M'| = p - (2m+1)q$. Let $L = (\bigcup_{-m\leq i\leq m} M_i) \cup M'$; clearly, $|L| = p$. Further, let M'' be an arbitrary (finite or infinite) set disjoint from L and let $\Omega = L \cup M''$. Denote by S_Ω the (full) symmetric group on the set Ω. For $1 \leq i \leq m$ let $x_i \in S_\Omega$ be a permutation of order p (i.e., $x_i^p = id$) such that its restriction to the set L is a cyclic permutation of L with the property that $x_i(M_{-i}) = M_0$ and $x_i(M_0) = M_i$ (that is, the images of the set M_0 under x_i and x_i^{-1} are M_i and M_{-i}, respectively).

Consider next the action of the permutation x_1 on the set M_0. Let $M_0 = \{a_1, a_2, \dots, a_q\}$ and let the restriction of x_1 on L be the cyclic permutation $(a_{j_1}\ \dots\ a_{j_2}\ \dots\ \dots\ a_{j_q}\ \dots)$, where the dots represent the remaining $p-q$ elements of the set L. This way, x_1 defines a unique permutation $x_0 \in S_\Omega$ whose restriction to M_0 is the cyclic permutation $x_0 = (a_{j_1}\ a_{j_2}\ \dots a_{j_q})$ of M_0, and such that x_0 fixes every element in $\Omega \setminus M_0$. The important fact to observe is that $(x_1x_0)^p = id$ (this is the place where we use the fact that q is odd).

Now, let $X = \{x_i, x_i^{-1};\ 1 \leq i \leq m\}$; note that $|X| = 2m$. Let H be an arbitrary subgroup of S_Ω fixing the set $\Omega \setminus M_0$ pointwise and such that $x_0 \in H$. Let $G_{H,X}$ be the subgroup of S_Ω generated by the elements in $H \cup X$. Since $X^{-1} = X$, the coset graph $\Gamma_{H,X} = Cos(G_{H,X}, H, X)$ is well defined and connected. Moreover, if the set M'' (and hence Ω) is infinite, the group $G_{H,X}$ may be infinite as well (observe that we did not restrict the action of the permutations in X on the set $\Omega \setminus L$ in any other way except for the requirement that $x_i^p = id$). But in any case, the group H and the set X are finite, and so our coset graph is always locally finite.

The properties of permutations in X imply that for any $y, z \in X$, $y \neq z$ we have $y(M_0) \cap z(M_0) = \emptyset$ and $y(M_0) \cap M_0 = \emptyset$. Using these facts, it is easy to show that $XHX \cap H = \{id\}$. (Briefly, if $h \in H$ is such that $h = z^{-1}h'y$ for some $h' \in H$ and $y, z \in X$ and if $h(a) = a'$ for $a, a' \in M_0$ then, recalling that $y(a) \notin M_0$ and that h' fixes $\Omega \setminus M_0$ pointwise, we have $z(a') = zh(a) = h'y(a) = y(a)$. The above intersection facts now imply that $z = y$, and so $a = a'$ and $h = id$.) We also know that $y^p = id$ for every $y \in X$ and that $(x_1 x_0)^p = id$, having thus $|X| + 1$ pairs $(y, h) \in X \times H$ such that $(yh)^p = id$. Applying Theorem 1, we conclude that the coset graph $\Gamma_{H,X} = Cos(G_{H,X}, H, X)$ is a (possibly infinite but locally finite) VTNCG whenever $p > 2m|H|^2$. □

The above construction clearly has a large degree of freedom; the restrictions on the subgroup H and on the set X are localized just in a "small" subset of Ω. It can also be adapted to produce ATNCG's using the same trick as in Example 3, that is, by choosing X in such way that both X and H are fixed by a subgroup (transitive on X) of inner automorphisms of S_Ω.

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