Smallest defining sets for 2-(9,4,3) and 3-(10,5,3) designs

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ABSTRACT

A set of blocks which can be completed to exactly one t-(v,k,λ) design is called a *defining set* of that design. A known algorithm is used to determine all smallest defining sets of the 11 non-isomorphic 2-(9,4,3) designs. Nine of the designs have smallest defining sets of eight blocks each; the other two have smallest defining sets of six blocks each.

Various methods are then used to find all smallest defining sets of the seven non-isomorphic 3-(10,5,3) designs, all of which are extensions of 2-(9,4,3) designs. Four of the 3-(10,5,3) designs have smallest defining sets of eight blocks each; the other three have smallest defining sets of six blocks each.

Whereas in previous computations of sizes of smallest defining sets of classes of non-isomorphic designs with the same parameters, the size of smallest defining sets was found to be non-decreasing as automorphism group order increases, both of these classes of designs provide cases which show that this is not a universal rule.

1. Introduction

A t-(v,k,λ) **design** is a collection of k-subsets (called **blocks**) of a v-set, V, such that every t-subset of V occurs in exactly λ blocks. Sometimes such designs will be referred to as t-**designs**. A t-(v,k,λ) design is **simple** if it contains no repeated blocks.

If a set, S, of blocks is a subset of the set of blocks of a t-(v,k,λ) design, D, then it is said that S **completes to** D or that D **is a completion of** S to a t-(v,k,λ) design. If S completes to D but to no other design with the same parameters (t, v, k and λ), then S is a **defining set** of D (notation $d(D)$). A defining set of D such that no other defining set has smaller cardinality is called a **smallest defining set** of D (notation $d_s(D)$). The **size** of a defining set (notation $|d(D)|$) is its number of blocks.

For any subset, X, of the blocks of a design, D, on a v-set, V, a permutation of the elements of V which preserves the blocks of X is an **automorphism** of X. The set of all automorphisms of X is the **automorphism group** of X, denoted by G_X; hence G_D denotes the automorphism group of the whole design, D. If G_D contains no permutations which are single transpositions of elements, then D is said to be **single-transposition-free** or STF.

In the following, given a set, S, of blocks from a design and a permutation, ρ, on the elements of the underlying set of the design, ρS will denote the image of S under the action of ρ. If T_1 is a sub-collection of the blocks of a t-(v, k, λ) design, D, and if there exists a disjoint collection, T_2, of k-sets such that any t-set which occurs in T_1 occurs with the same frequency in T_2, then T_1 and T_2 are said to constitute a **trade**. For ease of reference, the single collection T_1 will henceforth also be called a trade. Clearly, if T_1 is a trade in D and $\rho \in G_D$, then ρT_1 is also a trade in D. The **volume** of a trade is the number of blocks in the trade. A **minimal trade** is a trade, no proper sub-collection of which is also a trade. It is noted that Hwang [13] uses the term **minimal trade** to mean a trade containing the smallest possible number of elements and the smallest possible number of blocks.

If D is a t-(v, k, λ) design and the set of blocks of D containing a particular element x is chosen, the deletion of the element x from each block leaves a set of blocks, D^x, which is called the **restriction** of D on x. It is well known that each such D^x is a $(t$-$1)$-$(v$-$1,k$-$1,\lambda)$ design.

In the following, given a set, S, of blocks, $S(x)$ will denote the set of blocks formed by adding a new element x to each block of S. Given a t-(v, k, λ) design, D, with underlying v-set, V, it may be possible to create another set, M, of blocks such that $D(x) \cup M = E$ is a $(t+1)$-$(v+1,k+1,\lambda)$ design. Then E is called an **extension** of D. If the set, M, can be chosen in more than one way, then D has more than one extension to a $(t+1)$-$(v+1,k+1,\lambda)$ design. If $\widetilde{D(x)}$ is the set of complements in $V \cup \{x\}$ of the blocks of $D(x)$, and if $M = \widetilde{D(x)}$, then E is called an **extension by complementation** of D. Clearly no design can have more than one extension by complementation; also any extension by complementation is a self-complementary design and any self-complementary design is an extension by complementation of its restrictions.

2. Useful results

The theory of defining sets was first studied by Gray [7], [8]. The following four lemmata were stated and proven, *inter alia*, by Gray.

Lemma 1 [7] *A defining set of a design, D, intersects every trade in D.* □

Lemma 2 [8] *If S is a defining set of a simple STF design, D, and if D contains precisely n configurations isomorphic to S, then $n = |G_D|/|G_S|$.* □

Lemma 3 [8] *If S is a defining set of a design, D, and if ρ is a permutation on the elements of the underlying set of D such that $\rho S \subseteq D$, then $\rho \in G_D$ and ρS is a defining set of D.* □

Lemma 4 [8] *If S is a defining set of a design, D, based on the set V, then $\tilde{S}$ is a defining set of $\tilde{D}$, the design comprising the complements of the blocks of D with respect to V.* □

Gray [8] showed that a converse of Lemma 2 applies in certain circumstances, as explained in the next Lemma.

Lemma 5 [8] *Let D be a simple STF t-(v,k,λ) design and let S be a subset of the set of blocks of D, containing at least $v-1$ distinct elements, such that there are n configurations in D isomorphic to S and $n = |G_D|/|G_S|$. If there is no set of blocks isomorphic to S contained in any design with the same parameters (t, v, k and λ) as D, but not isomorphic to D, then S is a $d(D)$.* □

Gray also produced several results which can help to put a lower bound on the size of smallest defining sets of simple STF designs. One such result follows.

Lemma 6 [8] *If S is a defining set of a STF t-(v,k,λ) design, D, if $s = |S|$ and if $k^* = min(k, v-k)$, then*

$$s \geq \frac{2(v-1)}{k^*+1}.$$ □

It is clear that a necessary condition for a t-(v,k,λ) design to have an extension by complementation is that $v = 2k+1$. The following lemma, guaranteeing the existence of certain extensions, was proven by Alltop [1].

Lemma 7 *Any t-$(2n$-$1,n$-$1,\lambda)$ design, where t is even, has an extension by complementation to a $(t+1)$-$(2n, n, \lambda)$ design.* □

It is easy to extend Alltop's proof to show the following lemma.

Lemma 8 *A t-$(2n$-$1,n$-$1,\lambda)$ design, where t is odd, has an extension by complementation to a $(t+1)$-$(2n, n, \lambda)$ design if and only if either $t = n-1$ (i.e. the design is a multiple of the full design) or the t-design is also a $(t+1)$-design.* □

The following lemmata, relating the defining sets of a design to the defining sets of its extension(s) or restriction(s), generalize and extend results of Gray [7]; Gray proved the special cases of Lemmata 9, 10 and 12 for which $t = 2$ and all 3-designs with the given parameters are self-complementary.

Lemma 9 *If a t-(v,k,λ) design, D, has exactly one extension to a $(t+1)$-$(v+1,k+1,\lambda)$ design, $E = D(x) \cup M$, then for each defining set S of D, there is a defining set $S(x)$ of E.*

Proof: Let $S(x) \subseteq E_1$, a $(t+1)$-$(v+1,k+1,\lambda)$ design.

Let $D_1 = E_1^x$; then D_1 is a t-(v,k,λ) design and $S \subseteq D_1$.

But S is a $d(D)$, so $D_1 = D$.

But D has only one extension, so $E_1 = E$ and $S(x)$ is a $d(E)$. □

The following corollary is immediately clear.

Corollary 9.1 *If a t-(v,k,λ) design D, has exactly one extension, E, then*

$$|d_s(D)| \geq |d_s(E)|.$$ □

At the end of this paper, a case is noted in which $|d_s(D)| > |d_s(E)|$.

Lemma 10 *If D is a t-(v,k,λ) design, where t is even, such that $E = D(x) \cup \widetilde{D(x)}$ is an extension of D and if S is a defining set of E, then $(S \cup \tilde{S})^x$ is a defining set of D.*

Proof: Let $(S \cup \tilde{S})^x \subseteq D_1$, a t-(v,k,λ) design.

Since D has an extension by complementation, so does D_1. Let $E_1 = D_1(x) \cup \widetilde{D_1(x)}$. Then $S \cup \tilde{S} \subseteq E_1$ and so $S \subseteq E_1$.

But S is a $d(E)$, so $E_1 = E$.

But $E^x = D$, so $E_1^x = D_1 = D$.

Hence $S \cup \tilde{S}$ is a $d(D)$. □

The following corollary is now clear.

Corollary 10.1 *If D is a t-design, where t is even, such that $E = D(x) \cup \widetilde{D(x)}$ is an extension of D, then*

$$|d_s(D)| \leq |d_s(E)|.$$ □

It should be noted that Lemma 10 and its Corollary do not necessarily apply in the case that t is odd, since it is conceivable that, whereas there may be several t-designs with the same parameters, not all of them are also $(t+1)$-designs, and so not all are extendable by complementation.

Lemma 11 *If D_1 and D_2 are t-designs, where t is even, such that there exists a common extension, $E = D_1(x) \cup \widetilde{D_1(x)} = D_2(y) \cup \widetilde{D_2(y)}$, then $|d_s(D_1)| = |d_s(D_2)|$ and D_1 and D_2 have the same number of smallest defining sets.*

Proof: Let S be a $d_s(D_2)$ and let D_2 have an extension $E_1 = D_2(y) \cup M$, such that $S(y) \cup \widetilde{S(y)} \subseteq E_1$. Then M is a t-design with the same parameters as $\widetilde{D_2(y)} (= \widetilde{D_2})$.

Now S is a $d(D_2)$, so by Lemma 4, $\widetilde{S(y)} (= \tilde{S})$ is a $d(\widetilde{D_2})$. So, if $S(y) \cup \widetilde{S(y)} \subseteq E_1$, then $\widetilde{S(y)} \subseteq M$ and $M = \widetilde{D_2}$. Hence $E_1 = E$ and $S(y) \cup \widetilde{S(y)}$ is a $d(E)$.

So, by Lemma 10, $(S(y) \cup \widetilde{S(y)})^x$ is a $d(D_1)$. But $|(S(y) \cup \widetilde{S(y)})^x| = |S|$, so there is a $d(D_1)$ of the same size as the $d_s(D_2)$.

Similarly, there is a $d(D_2)$ of the same size as a $d_s(D_1)$.

Hence $|d_s(D_1)| = |d_s(D_2)|$. □

Now any $d_s(D_1)$, say S_1, yields a unique set $(S_1(x) \cup \widetilde{S_1(x)})^y$, which, by the above argument, is a $d_s(D_2)$. Similarly any $d_s(D_2)$ yields a unique $d_s(D_1)$.

Hence D_1 and D_2 have the same number of smallest defining sets. □

Lemma 12 *If D is a t-design, where t is even, such that $E = D(x) \cup \widetilde{D(x)}$ is the only extension of D, then $|d_s(D)| = |d_s(E)|$.*

Proof: The result follows immediately from Corollary 9.1 and Corollary 10.1. □

Lemma 13 *If D is a t-design, where t is even, such that $E = D(x) \cup \widetilde{D(x)}$ is the only extension of D, and if there are precisely n $d_s(D)$ (each comprising q blocks) and precisely m $d_s(E)$, then*

$$n \leq m \leq 2^q n,$$

with the upper bound being attained if all designs with the same parameters as E are self-complementary.

Proof: Since $E = D(x) \cup \widetilde{D(x)}$ is the only extension of D, if S_1 and S_2 are $d_s(D)$, then by Lemma 9, $S_1(x)$ and $S_2(x)$ are $d(E)$ and, by Lemma 12, $d_s(E)$. Further, $S_1(x) = S_2(x)$ only if $S_1 = S_2$. Hence, if there are precisely n $d_s(D)$ and precisely m $d_s(E)$, then $n \leq m$.

If S is a $d_s(D)$ and $|S| = q$, then there are 2^q sets S^* such that $S^* \cup \widetilde{S^*} = S(x) \cup \widetilde{S(x)}$; each such S^* contains exactly one block from each set $\{\mathbf{b}, \tilde{\mathbf{b}}\}$, where $\mathbf{b}$ is any block of $S(x)$. There are $2^q \times n$ such sets arising from the n $d_s(D)$; no other set can be a $d_s(E)$, since if S' is a $d_s(E)$, there is, by Lemmata 10 and 12 a $d_s(D)$, namely $S = (S' \cup \widetilde{S'})^x$, such that $S' \cup \widetilde{S'} = S(x) \cup \widetilde{S(x)}$. Thus $m \leq 2^q \times n$.

If all designs with the parameters of E are self-complementary, then for any $d_s(D)$, S, if $S^* \cup \widetilde{S^*} = S(x) \cup \widetilde{S(x)}$ then S^* forces $\widetilde{S^*}$, $S \cup \widetilde{S^*}$ contains $S(x)$ and $S(x)$ is a $d(E)$. Hence each of the $2^q \times n$ such sets S^* is a $d_s(E)$. □

3. The algorithm

Greenhill [10] [11] used Lemmata 1 to 4 and 6 above to construct an algorithm to determine all smallest defining sets of simple STF designs; the algorithm is implementable for small designs, but as the number of blocks and the block size increase, the computer time necessary to implement it in full becomes vast. For a given simple STF t-(v, k, λ) design, D, the steps in the algorithm are as follows.

STEP 1: Using Lemma 6, or otherwise, estimate a lower bound, l, on the number of blocks in any $d(D)$.

STEP 2: List any known minimal trades in D and use G_D to generate, if possible, more minimal trades.

STEP 3: Generate all subsets of l blocks of D and sort them into m isomorphism classes (under the symmetric group of all permutations on v elements). Take one representative of each isomorphism class; by Lemma 3, if the representative is a defining set, then all isomorphs of the representative under the action of G_D are also defining sets of D. If S_i is the representative of the ith isomorphism class, record n_i (the number of sets of l blocks in the isomorphism class) and $|G_{S_i}|$.

STEP 4: From the list (S_1, S_2, ..., S_m), eliminate any S_i for which either

(a) $n_i \times |G_{S_i}| \neq |G_D|$ (using Lemma 2) or
(b) $S_i \cap T = \emptyset$ for any trade, T, in D (using Lemma 1).

If all S_i are eliminated, then there are no defining sets of size l. Hence the value of l should be increased by one and the algorithm restarted at STEP 3.

If any S_i remain, they are called **feasible sets**.

STEP 5: Determine all completions to t-(v, k, λ) designs of each S_i. If any S_i completes uniquely, it is a $d(D)$; if the original value of l was determined using Lemma 6, then S_i is a $d_s(D)$. If the original value of l was estimated by other means, the value of l should be decreased by one and the algorithm restarted at STEP 3.

If no S_i completes uniquely, then the different completions may be used to determine more trades. The value of l is increased by one and the algorithm restarted at STEP 2.

Whether the value of l has to be iteratively increased or decreased, both the size of a $d_s(D)$ and a complete list of $d_s(D)$ can be theoretically determined by the above algorithm.

Greenhill [10] gave computer programs which facilitate STEPS 2 to 5 above; in the program for STEP 3 she made use of the program *nauty* by McKay [14]. Delaney [4], [5] subsequently modified and improved Greenhill's programs and it is Delaney's versions (which still use *nauty*) which have been used here. The automation of the program for STEP 5, allowing for the input of sometimes several thousand sets of blocks for completion, is due to Sharry [15].

Completion of large numbers of feasible sets can use much computer time, especially for t-designs with $t \geq 3$. Lemma 5 provides an alternative to STEP 5 which can be much quicker; this is discussed in Section 6.

4. The 2-(9,4,3) and 3-(10,5,3) designs

There are exactly eleven non-isomorphic 2-(9,4,3) designs; these were enumerated and constructed by Stanton, Mullin and Bate [16] and independently by Gibbons [6] (by means of a computer algorithm) and by Breach [2]; the result was also claimed by van Lint, van Tilborg and Wiekama [17] though without specific details of the constructions. Each design, by Lemma 7 above, has at least one extension (by complementation) to a 3-(10,5,3) design.

There are exactly seven non-isomorphic 3-(10,5,3) designs; these were also listed, with their relations to the 2-(9,4,3) designs, by Gibbons [6]. Breach [2], [3] produced an independent theoretical verification of these results, along with other useful properties of the designs. Breach's tabulation, which neatly shows the relations between the 2-designs and the 3-designs, will be used here.

Each block in a 2-(9,4,3) design either is disjoint from exactly one other block in the design or intersects exactly one other block in exactly three elements, but not both; Breach consequently called each pair of blocks **disjoint** or **friendly**. The designs below are tabulated in these pairs, with a space separating the fiendly pairs (above) from the disjoint pairs (below). The 2-(9,4,3) designs are grouped according to their common extensions to 3-(10,5,3) designs. In each case, the symbol used for the extension is that one missing from the set $\{0,1,2,3,4,5,6,7,8,9\}$ in the 2-(9,4,3) design. In the listing of the eleven 2-(9,4,3) designs in Tables 1, 2, 3, 4 and 5, the block numbers, in bold type, precede the blocks in each case.

				1:	1247	**10:**	2478
1:	0247	**10:**	3569	**2:**	1259	**11:**	2569
2:	0259	**11:**	3478	**3:**	1268	**12:**	2368
3:	0268	**12:**	4579	**4:**	1346	**13:**	3469
4:	0346	**13:**	2578	**5:**	1358	**14:**	3578
5:	0358	**14:**	2469	**6:**	1379	**15:**	2379
6:	0379	**15:**	4568	**7:**	1489	**16:**	4589
7:	0489	**16:**	2367	**8:**	1567	**17:**	4567
8:	0567	**17:**	2389				
9:	6789	**18:**	2345	**9:**	2345	**18:**	6789
M_1				M_2			

Table 1: The 2-(9,4,3) designs M_1 and M_2

Designs M_1 and M_2, listed in Table 1, have a common extension by complementation, as do designs M_3 and M_4, listed in Table 2. M_1 is the only 2-(9,4,3) design comprising nine pairs of disjoint blocks.

Designs M_5, M_6 and M_7, listed in Table 3, all have a common extension by complementation.

Designs M_8 and M_9, listed in Table 4, have a common extension by complementation, as do designs M_{10} and M_{11}, listed in Table 5. M_{11} is the only 2-(9,4,3) design comprising nine pairs of friendly blocks; it can be developed from blocks **1** and **10** by cycling (mod 9). Each of M_8 and M_9 contains four pairs of friendly blocks with a single element common to all eight blocks (the element 1 in M_8 and the element 8 in M_9).

All eleven designs are simple and all except M_8 are STF: M_8 has the permutation (89) as an automorphism.

1:	1246	**10**:	1268	**1**:	0246	**10**:	0268
2:	1347	**11**:	1379	**2**:	0347	**11**:	0379
3:	1489	**12**:	4589	**3**:	2578	**12**:	4578
4:	1358	**13**:	3568	**4**:	3569	**13**:	4569
5:	1567	**14**:	4567				
6:	1259	**15**:	2579	**5**:	0259	**14**:	3468
7:	2369	**16**:	3469	**6**:	0358	**15**:	2479
8:	2378	**17**:	2478	**7**:	0489	**16**:	2367
				8:	0567	**17**:	2389
9:	2345	**18**:	6789	**9**:	2345	**18**:	6789
M_3				M_4			

Table 2: The 2-(9,4,3) designs M_3 and M_4

1:	1236	**10**:	1268	**1**:	0236	**10**:	0268	**1**:	0158	**10**:	0189
2:	1259	**11**:	2579	**2**:	0347	**11**:	0379	**2**:	0235	**11**:	1235
3:	1347	**12**:	1379	**3**:	0458	**12**:	0489	**3**:	0269	**12**:	0369
4:	1458	**13**:	1489	**4**:	2467	**13**:	2479	**4**:	1267	**13**:	1279
5:	1567	**14**:	4567	**5**:	2578	**14**:	3578	**5**:	1368	**14**:	3678
6:	2378	**15**:	2478	**6**:	3569	**15**:	4569	**6**:	3579	**15**:	5789
7:	2469	**16**:	3469								
8:	3568	**17**:	3589	**7**:	0259	**16**:	3468	**7**:	0137	**16**:	2568
				8:	0567	**17**:	2389	**8**:	0278	**17**:	1569
9:	2345	**18**:	6789	**9**:	2345	**18**:	6789	**9**:	0567	**18**:	2389
M_5				M_6				M_7			

Table 3: The 2-(9,4,3) designs M_5, M_6, M_7

1:	1238	**10**:	1239	**1**:	0148	**10**:	0158
2:	1267	**11**:	1367	**2**:	0678	**11**:	1678
3:	1456	**12**:	1457	**3**:	2368	**12**:	2378
4:	1489	**13**:	1589	**4**:	2458	**13**:	3458
5:	2468	**14**:	2469				
6:	2578	**15**:	2579	**5**:	0123	**14**:	4567
7:	3478	**16**:	3479	**6**:	0246	**15**:	1357
8:	3568	**17**:	3569	**7**:	0257	**16**:	1346
				8:	0347	**17**:	1256
9:	2345	**18**:	6789	**9**:	0356	**18**:	1247
M_8				M_9			

Table 4: The 2-(9,4,3) designs M_8 and M_9

1:	1237	10:	2367	1:	0124	10:	0146
2:	1249	11:	1469	2:	1235	11:	1257
3:	1389	12:	3589	3:	2346	12:	2368
4:	1457	13:	1578	4:	3457	13:	3470
5:	2345	14:	2458	5:	4568	14:	4581
6:	2569	15:	5679	6:	5670	15:	5602
7:	3468	16:	4678	7:	6781	16:	6713
				8:	7802	17:	7824
8:	1268	17:	3479	9:	8013	18:	8035
9:	1356	18:	2789				
M_{10}				M_{11}			

Table 5: The 2-(9,4,3) designs M_{10} and M_{11}

The orders of the automorphism groups of the eleven non-isomorphic 2-(9,4,3) designs together with their extensions to 3-(10,5,3) designs are given in Table 6; the blocks of the extensions are given in later tables.

Design(D)	$\|G_D\|$	Extensions
M_1	144	N_1, N_1^*, N_2
M_2	16	N_2
M_3	2	N_3
M_4	8	N_3
M_5	1	N_4
M_6	2	N_4
M_7	6	N_4
M_8	8	N_6
M_9	32	N_5, N_6
M_{10}	1	N_7
M_{11}	9	N_7

Table 6: Automorphism group orders and extensions of the 2-(9,4,3) designs

Breach [2] showed that only two types of blocks, classified according to their intersections with the other 35 blocks in the design, are possible in a 3-(10,5,3) design; these will be called Type I and Type II here. Breach referred to Type I blocks as blocks of type (0,5,10,20,0) and to Type II blocks as blocks of type (1,1,16,16,1); these ordered quintuples give the numbers of other blocks whose intersections with the block in question are 0,1,2,3,4 elements respectively. The self-complementary 3-(10,5,3) designs clearly contain only blocks of Type II.

Breach also showed that there are three completions of $M_1(1)$ and two completions of $M_9(9)$ to 3-(10,5,3) designs; the following information about these designs is adapted from his listings. Each of the three sets of eighteen blocks, A, B and C, given in Table 7, combines with $M_1(1)$ to form a 3-(10,5,3) design. A, B and C are mutually disjoint. $M_1(1) \cup A$ and $M_1(1) \cup B$ are isomorphic; the design N_1 is chosen here to be $M_1(1) \cup A$, while $M_1(1) \cup B$ is referred to in Table 6 as N_1^*. N_1 consists entirely of blocks of Type I. $M_1(1) \cup C$ is the extension by complementation of M_1, which is referred to here as N_2.

23479	24567	35689
24678	24789	34678
23480	23490	34579
24560	24580	25789
23570	23780	24679
27890	26790	24568
45780	34570	23567
46790	46780	23489
23690	23560	67890
34590	45690	24780
56890	57890	25690
24589	25689	23680
25679	23579	34690
36780	36890	35780
23568	23468	23790
34567	34679	45890
34689	34589	45670
35789	35678	23450
A	B	C

Table 7: Sets of blocks which combine with $M_1(1)$ to form 3-(10,5,3) designs

To facilitate consideration of the extensions of M_9, $M_9(9)$ is partitioned into two sets of blocks:
$G = \{\mathbf{1, 2, 3, 4, 5, 10, 11, 12, 13, 14}\}$ and $H = \{\mathbf{6, 7, 8, 9, 15, 16, 17, 18}\}$.

Now, let $F = \{12578, 12468, 13568, 13478, 03468, 03578, 02478, 02568\}$.
It is noted that the application of either the permutation (23) or the permutation (67) to the elements of the blocks of $\tilde{H}$ gives F. Then $G \cup \tilde{G} \cup H \cup \tilde{H}$ is N_6, the extension by complementation of M_9, while $G \cup \tilde{G} \cup H \cup F$ is the other extension of M_9, here called N_5. N_5 comprises 20 blocks of Type II and 16 blocks of Type I.

The design $G \cup \tilde{G} \cup F \cup \tilde{F}$, although not an extension of M_9 as written, is isomorphic to N_6, while $G \cup \tilde{G} \cup \tilde{F} \cup \tilde{H}$ is isomorphic to N_5.

In the listings of the seven non-isomorphic 3-(10,5,3) designs in Tables 8 and 9, the block numbers, in bold type, precede the blocks in each case. The blocks are arranged so that the restriction on the common element in blocks **1-18** in each case gives one of the 2-(9,4,3) designs.

1:	10247	**19**:	23479	**1**:	10247	**19**:	35689
2:	10259	**20**:	24678	**2**:	10259	**20**:	34678
3:	10268	**21**:	23480	**3**:	10268	**21**:	34579
4:	10346	**22**:	24560	**4**:	10346	**22**:	25789
5:	10358	**23**:	23570	**5**:	10358	**23**:	24679
6:	10379	**24**:	27890	**6**:	10379	**24**:	24568
7:	10489	**25**:	45780	**7**:	10489	**25**:	23567
8:	10567	**26**:	46790	**8**:	10567	**26**:	23489
9:	16789	**27**:	23690	**9**:	16789	**27**:	02345
10:	13569	**28**:	34590	**10**:	13569	**28**:	02478
11:	13478	**29**:	56890	**11**:	13478	**29**:	02569
12:	14579	**30**:	24589	**12**:	14579	**30**:	02368
13:	12578	**31**:	25679	**13**:	12578	**31**:	03469
14:	12469	**32**:	36780	**14**:	12469	**32**:	03578
15:	14568	**33**:	23568	**15**:	14568	**33**:	02379
16:	12367	**34**:	34567	**16**:	12367	**34**:	04589
17:	12389	**35**:	34689	**17**:	12389	**35**:	04567
18:	12345	**36**:	35789	**18**:	12345	**36**:	06789
N_1				N_2			

1:	01489	**19**:	23567	**1**:	01489	**19**:	23567
2:	06789	**20**:	12345	**2**:	06789	**20**:	12345
3:	23689	**21**:	01457	**3**:	23689	**21**:	01457
4:	24589	**22**:	01367	**4**:	24589	**22**:	01367
5:	01239	**23**:	45678	**5**:	01239	**23**:	45678
6:	02469	**24**:	13478	**6**:	02469	**24**:	13578
7:	02579	**25**:	13568	**7**:	02579	**25**:	13468
8:	03479	**26**:	12578	**8**:	03479	**26**:	12568
9:	03569	**27**:	12468	**9**:	03569	**27**:	12478
10:	01589	**28**:	23467	**10**:	01589	**28**:	23467
11:	16789	**29**:	02345	**11**:	16789	**29**:	02345
12:	23789	**30**:	01456	**12**:	23789	**30**:	01456
13:	34589	**31**:	01267	**13**:	34589	**31**:	01267
14:	45679	**32**:	01238	**14**:	45679	**32**:	01238
15:	13579	**33**:	02478	**15**:	13579	**33**:	02468
16:	13469	**34**:	02568	**16**:	13469	**34**:	02578
17:	12569	**35**:	03468	**17**:	12569	**35**:	03478
18:	12479	**36**:	03578	**18**:	12479	**36**:	03568
N_5				N_6			

Table 8: The 3-(10,5,3) designs which are extensions of M_1 and M_9

1:	01246	**19**:	35789	**1**:	01458	**19**:	23679	**1**:	01249	**19**:	356789
2:	01347	**20**:	25689	**2**:	02345	**20**:	16789	**2**:	12359	**20**:	04678
3:	12578	**21**:	03469	**3**:	02469	**21**:	13578	**3**:	23469	**21**:	01578
4:	13569	**22**:	02478	**4**:	12467	**22**:	03589	**4**:	34579	**22**:	01268
5:	01259	**23**:	34678	**5**:	13468	**23**:	02579	**5**:	45689	**23**:	01237
6:	01358	**24**:	24679	**6**:	34579	**24**:	01268	**6**:	05679	**24**:	12348
7:	01489	**25**:	23567	**7**:	01347	**25**:	25689	**7**:	16789	**25**:	02345
8:	01567	**26**:	23489	**8**:	02478	**26**:	13569	**8**:	02789	**26**:	13456
9:	12345	**27**:	06789	**9**:	04567	**27**:	12389	**9**:	01389	**27**:	24567
10:	01268	**28**:	34579	**10**:	01489	**28**:	23567	**10**:	01469	**28**:	23578
11:	01379	**29**:	24568	**11**:	12345	**29**:	06789	**11**:	12579	**29**:	03468
12:	14578	**30**:	02369	**12**:	03469	**30**:	12578	**12**:	23689	**30**:	01457
13:	14569	**31**:	02378	**13**:	12479	**31**:	03568	**13**:	03479	**31**:	12568
14:	13468	**32**:	02579	**14**:	34678	**32**:	01259	**14**:	14589	**32**:	02367
15:	12479	**33**:	03568	**15**:	45789	**33**:	01236	**15**:	02569	**33**:	13478
16:	12367	**34**:	04589	**16**:	24568	**34**:	01379	**16**:	13679	**34**:	02458
17:	12389	**35**:	04567	**17**:	14569	**35**:	02378	**17**:	24789	**35**:	01356
18:	16789	**36**:	02345	**18**:	23489	**36**:	01567	**18**:	03589	**36**:	12467
N_3				N_4				N_7			

Table 9: The 3-(10,5,3) designs N_3, N_4, N_7

Design (D)	$\|G_D\|$	Restrictions
N_1	720	$10 \times M_1$
N_2	144	$1 \times M_1$; $9 \times M_2$
N_3	16	$8 \times M_3$; $2 \times M_4$
N_4	6	$6 \times M_5$; $3 \times M_6$; $1 \times M_7$
N_5	320	$10 \times M_9$
N_6	64	$8 \times M_8$; $2 \times M_9$
N_7	9	$9 \times M_{10}$; $1 \times M_{11}$

Table 10: Automorphism group orders and restrictions of the 3-(10,5,3) designs

The orders of the automorphism groups of the 3-(10,5,3) designs, together with their restrictions to 2-(9,4,3) designs, are given in Table 10; this information is given by Gibbons [6].

5. Smallest defining sets of the 2-(9,4,3) designs

In order to determine all smallest defining sets of each of the 2-(9,4,3) designs, it is strictly only necessary to determine those of one of each of the following sets of designs: $\{M_1, M_2\}$, $\{M_3, M_4\}$, $\{M_5, M_6, M_7\}$, $\{M_8, M_9\}$, $\{M_{10}, M_{11}\}$. All smallest defining sets of the other designs in each set can then be determined by the extension and restriction process described in the proof of Lemma 11. For checking purposes, however, the algorithm was used on all eleven designs.

Since there are no trades of volumes one, two, three or five in any t-design, for $t \geq 2$ (see Hwang [13]), all trades of volumes four or six were determined for each design. There are two possible structures of trades of volume four and ten structures of minimal trades of volume six in these designs. These structures are shown in Tables 11 and 12.

abcd	*abce*	*abcd*	*abce*
abef	*abdf*	*abef*	*abdf*
agce	*agcd*	*ghce*	*ghcd*
agdf	*agef*	*ghdf*	*ghef*

Table 11: The two types of trade of volume 4 in 2-(9,4,3) designs

abde	*abdf*	*abcd*	*abce*	*acde*	*acdh*	*acde*	*acdf*	*acde*	*acdg*
abfg	*abeh*	*abef*	*abdg*	*acfg*	*acfi*	*acfg*	*aceh*	*acfg*	*achi*
abhi	*abgi*	*abgh*	*abfh*	*adhi*	*adeg*	*adeh*	*adeg*	*adhi*	*adef*
acdf	*acde*	*cdeg*	*cdef*	*bcdh*	*bcde*	*bcdf*	*bcde*	*bcdg*	*bcde*
aceh	*acfg*	*cefh*	*cdgh*	*bcfi*	*bcfg*	*bceh*	*bcfg*	*bchi*	*bcfg*
acgi	*achi*	*dfgh*	*efgh*	*bdeg*	*bdhi*	*bdeg*	*bdeh*	*bdef*	*bdhi*
acde	*acdg*	*acde*	*acdh*	*acde*	*aceh*	*acde*	*acdh*	*acde*	*acdh*
acdf	*acef*	*acfg*	*acef*	*acdf*	*acdg*	*acfg*	*acef*	*acfg*	*acef*
aegh	*adeh*	*adfh*	*adfg*	*aghi*	*adfi*	*adfh*	*adfg*	*adhi*	*adgi*
bcfg	*bcdf*	*bcdh*	*bcde*	*bcdg*	*bcde*	*bceh*	*bceg*	*bcdh*	*bcde*
bdgh	*begh*	*bcef*	*bcfg*	*bceh*	*bcdf*	*bdgh*	*bdeh*	*bcef*	*bcfg*
befh	*bfgh*	*bdfg*	*bdfh*	*bdfi*	*bghi*	*befg*	*bfgh*	*bdgi*	*bdhi*

Table 12: The 10 types of minimal trade of volume six in 2-(9,4,3) designs

All isomorphs of these structures were selected from the lists of all 4-sets and 6-sets of blocks for each design. Each of designs M_8 and M_9 contains eight trades of volume four and no trades of volume six. Each of the other nine designs contains 18 trades of volume four and 36 minimal trades of volume six. The intuitive expectation, then, is that since smallest defining sets of M_8 and M_9 need to intersect fewer small trades, they will be smaller than the smallest defining sets of the other nine designs.

For each design except M_8 (which is not STF), Lemma 6 gives a lower bound of four on the size of the smallest defining sets. Hence the starting value of l in the application of the algorithm to design M_1 was four; there were, however, no feasible sets (given the 54 trades mentioned above) of size seven or less. The starting value of l in the application of the algorithm to designs M_2, M_3, M_4, M_5, M_6, M_7, M_{10} and M_{11} was consequently taken to be seven; while there were feasible sets of seven blocks for most of these designs, none of them was a defining set. In Table 13, n_7 denotes the number of isomorphism classes of 7-sets of blocks and f_7 the number of feasible 7-sets of blocks, given the 54 trades found for each design; n_8 denotes the number of isomorphism classes of 8-sets of blocks, f_8 the number of feasible 8-sets of blocks, d_8 the number of isomorphism classes of defining sets of eight blocks and Δ_8 the total number of defining sets of eight blocks.

Design	n_7	f_7	n_8	f_8	d_8	Δ_8
M_1	264	0	360	30	25	3276
M_2	2036	0	2862	248	209	3276
M_3	10519	48	19463	1992	1644	3276
M_4	3416	12	5367	506	417	3276
M_5	14030	130	32741	4100	3222	3222
M_6	9846	72	18789	2071	1617	3222
M_7	4730	26	7173	693	539	3222
M_{10}	14217	174	32903	4182	3204	3204
M_{11}	3188	18	4764	446	356	3204

Table 13: Summary of algorithm output for nine of the 2-(9,4,3) designs

For design M_9, the starting value of l in the application of the algorithm was taken to be four. There were no feasible sets of four blocks and just seven feasible sets of five blocks, given the eight trades of volume four; none of the feasible sets of five blocks has a unique completion to a 2-(9,4,3) design. There were 113 feasible sets of six blocks given the eight trades of volume four but completion from six blocks proved very time-consuming, so 122 further trades of volume eight were derived from some early completions. These trades reduced the number of feasible sets to 36, just six of which have unique completions to M_9. Although M_8 is not STF, the algorithm was applied to it with the knowledge that $|d_s(M_8)| = 6$, by Lemma 11. Given the eight trades of volume four, there were 32 feasible sets of five blocks, none of which has a unique completion, and 461 feasible sets of six blocks. When 272 trades of volume eight were taken into account, there were 90 feasible sets of 6 blocks, 23 of which completed uniquely to M_8. It is of interest that M_8 behaved just the same as the STF designs in the application of the algorithm. In Table 14, n_5 denotes the number of isomorphism classes of 5-sets of blocks and f_5 the number of feasible 5-sets of blocks, given the eight trades of volume four found for each design. Similarly, n_6 denotes the number of isomorphism classes of 6-sets of blocks, f_6 the number of feasible sets of blocks given just the trades of volume four, f_6^* the number of feasible sets of 6-sets of blocks given also the trades of volume eight, d_6 the number of isomorphism classes of defining sets of six blocks and Δ_6 the total number of defining sets of six blocks.

Design	n_5	f_5	n_6	f_6	f_6^*	d_6	Δ_6
M_8	376	32	1393	461	90	23	80
M_9	222	7	590	113	36	6	80

Table 14: Summary of algorithm output for M_8 and M_9

In previous computations of sizes of smallest defining sets for classes of designs with the same parameters (see [7], [9], [10], [11], [12]), the size of the smallest defining set has been non-decreasing as the size of the automorphism group of the design increases. The existence of smallest defining sets of six blocks for M_8 and M_9 shows, however, that this does not always apply, since there are several 2-(9,4,3) designs with smaller automorphism groups but larger smallest defining sets than M_8 and M_9. There does, however, seem to be a relationship between the number of small trades in a design and the size of the smallest defining sets.

In Tables 15-18 which follow, the smallest defining sets of the 2-(9,4,3) designs are classified according to the orders of their automorphism groups. For each group order, the number of isomorphism classes of defining sets with that group order (n_c) and the number of defining sets in each isomorphism class (n_i) are also given. Hence the sum of the entries n_c in the table for each design is the number of non-isomorphic smallest defining sets for that design, while the sum of the products, $n_i \times n_c$ for each group order, is the total number of smallest defining sets for the relevant design. For each group order, examples of defining sets are given, sufficient to show the diversity of structures, with respect to the numbers of pairs of friendly blocks (n_{fp}) and pairs of disjoint blocks (n_{dp}) contained in the defining sets.

The results of this section are summarized in the following theorem.

Theorem 1 *The* 2-$(9,4,3)$ *designs* M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_{10} *and* M_{11} *have smallest defining sets of eight blocks. The remaining two* 2-$(9,4,3)$ *designs,* M_8 *and* M_9*, have smallest defining sets of six blocks.* □

Design (D)	$\lvert G_S\rvert$	n_{fp}	n_{dp}	example of $d_s(D)$								n_i	n_c
M_1	4	0	4	**1**	**2**	**4**	**6**	**10**	**11**	**13**	**15**	36	1
	2	0	2	**1**	**2**	**3**	**4**	**7**	**12**	**14**	**16**	72	3
	1	0	3	**1**	**2**	**3**	**4**	**5**	**7**	**12**	**14**	144	21
		0	2	**1**	**2**	**3**	**4**	**6**	**10**	**12**	**13**		
M_2	4	4	0	**1**	**3**	**5**	**6**	**10**	**12**	**14**	**15**	4	1
	2	2	0	**1**	**2**	**3**	**4**	**7**	**12**	**14**	**16**	8	7
	1	2	0	**1**	**2**	**3**	**4**	**5**	**7**	**12**	**14**	16	201
		3	0	**1**	**2**	**3**	**4**	**6**	**10**	**12**	**13**		
		4	0	**1**	**2**	**4**	**6**	**10**	**11**	**13**	**15**		
		1	1	**1**	**2**	**3**	**7**	**9**	**11**	**15**	**18**		
		2	1	**1**	**2**	**3**	**7**	**9**	**10**	**12**	**18**		
		3	1	**1**	**2**	**7**	**9**	**10**	**11**	**16**	**18**		
M_3	2	2	0	**1**	**2**	**5**	**10**	**11**	**12**	**13**	**15**	1	12
	1	4	0	**1**	**2**	**4**	**5**	**10**	**11**	**13**	**14**	2	1632
		3	0	**1**	**2**	**3**	**4**	**5**	**10**	**13**	**14**		
		2	0	**1**	**2**	**3**	**4**	**5**	**8**	**13**	**17**		
		1	0	**1**	**2**	**3**	**4**	**5**	**7**	**13**	**17**		
		3	1	**1**	**2**	**7**	**9**	**10**	**11**	**16**	**18**		
		2	1	**1**	**2**	**3**	**7**	**9**	**10**	**16**	**18**		
		1	1	**1**	**2**	**3**	**7**	**9**	**10**	**15**	**18**		
		0	1	**1**	**3**	**7**	**8**	**9**	**11**	**15**	**18**		
M_4	2	2	0	**1**	**2**	**3**	**4**	**8**	**10**	**11**	**16**	4	15
		0	2	**1**	**2**	**3**	**5**	**7**	**13**	**14**	**16**		
		2	2	**1**	**3**	**5**	**6**	**10**	**12**	**14**	**15**		
	1	1	0	**1**	**2**	**3**	**4**	**5**	**6**	**10**	**17**	8	402
		2	0	**1**	**2**	**3**	**4**	**5**	**7**	**10**	**12**		
		3	0	**1**	**2**	**3**	**4**	**5**	**10**	**11**	**12**		
		0	1	**1**	**2**	**3**	**5**	**6**	**7**	**15**	**17**		
		1	1	**1**	**2**	**3**	**4**	**5**	**7**	**13**	**14**		
		2	1	**1**	**2**	**3**	**4**	**5**	**11**	**12**	**14**		
		3	1	**1**	**2**	**3**	**5**	**10**	**11**	**12**	**14**		
		0	2	**1**	**2**	**3**	**4**	**5**	**8**	**14**	**17**		
		1	2	**1**	**2**	**3**	**5**	**7**	**10**	**14**	**16**		
		2	2	**1**	**2**	**5**	**8**	**10**	**11**	**14**	**17**		
		0	3	**1**	**3**	**5**	**6**	**7**	**14**	**15**	**16**		
		1	3	**1**	**5**	**6**	**7**	**10**	**14**	**15**	**16**		

Table 15: Some smallest defining sets of M_1, M_2, M_3 and M_4

Design (D)	$\|G_S\|$	n_{fp}	n_{dp}	example of $d_s(D)$	n_i	n_c
M_5	1	4	0	1 2 3 5 10 11 12 14	1	3222
		3	0	1 2 3 4 7 11 13 16		
		2	0	1 2 3 4 5 10 14 16		
		1	0	1 2 3 4 7 8 13 15		
		0	0	1 2 4 7 8 12 14 18		
		3	1	1 2 5 9 10 11 14 18		
		2	1	1 2 5 8 9 10 14 18		
		1	1	1 2 5 9 11 12 17 18		
		0	1	1 2 9 12 13 16 17 18		
M_6	2	2	0	1 2 3 4 5 10 14 17	1	12
		0	2	1 7 8 9 11 14 16 18		
	1	4	0	1 2 4 6 10 11 13 15	2	1605
		3	0	1 2 3 4 6 10 11 13		
		2	0	1 2 3 4 5 6 10 11		
		1	0	1 2 3 5 8 12 15 16		
		0	0	1 2 4 5 8 12 15 16		
		3	1	1 2 4 7 10 11 13 16		
		2	1	1 2 3 5 7 12 14 16		
		1	1	1 2 3 5 6 8 11 17		
		0	1	1 2 3 5 6 8 13 17		
		2	2	1 2 7 8 10 11 16 17		
		1	2	1 2 5 7 8 10 16 17		
		0	2	1 2 5 6 7 8 16 17		
		1	3	1 7 8 9 10 16 17 18		
		0	3	1 5 7 8 9 16 17 18		
M_7	2	2	0	1 2 4 5 11 14 16 17	3	4
	1	4	0	1 2 3 5 10 11 12 14	6	535
		3	0	1 2 3 4 10 11 12 16		
		2	0	1 2 3 4 5 10 11 16		
		1	0	1 2 3 4 5 6 12 16		
		0	0	1 2 3 6 8 13 14 16		
		3	1	1 2 3 7 10 11 12 16		
		2	1	1 2 3 5 7 10 11 16		
		1	1	1 2 3 5 6 7 11 16		
		0	1	1 2 3 5 8 9 16 17		
		2	2	1 2 7 8 10 11 16 17		
		1	2	1 2 5 7 8 11 16 17		
		0	2	1 2 3 7 8 9 16 17		

Table 16: Some smallest defining sets of M_5, M_6 and M_7

Design (D)	$\|G_S\|$	n_{fp}	n_{dp}	example of $d_s(D)$						n_i	n_c
M_8	4	2	0	**2**	**5**	**6**	**14**	**15**	**18**	2	6
	2	3	0	**1**	**2**	**5**	**10**	**11**	**14**	4	17
		2	0	**2**	**3**	**5**	**7**	**14**	**16**		
		1	0	**2**	**3**	**4**	**5**	**14**	**18**		
		2	1	**1**	**5**	**9**	**10**	**14**	**18**		
M_9	4	2	1	**1**	**3**	**6**	**10**	**12**	**15**	8	2
		0	2	**1**	**3**	**8**	**9**	**17**	**18**		
	2	2	1	**1**	**2**	**6**	**10**	**11**	**15**	16	4
		0	1	**1**	**2**	**3**	**4**	**8**	**17**		
		1	2	**1**	**5**	**6**	**10**	**14**	**15**		
		0	2	**1**	**2**	**6**	**8**	**15**	**17**		

Table 17: Some smallest defining sets of M_8 and M_9

Design (D)	$\|G_S\|$	n_{fp}	n_{dp}	example of $d_s(D)$								n_i	n_c
M_{10}	1	4	0	**1**	**2**	**3**	**5**	**10**	**11**	**12**	**14**	1	3204
		3	0	**1**	**2**	**3**	**4**	**5**	**10**	**12**	**13**		
		2	0	**1**	**2**	**3**	**4**	**5**	**7**	**13**	**16**		
		1	0	**1**	**2**	**3**	**4**	**5**	**9**	**10**	**16**		
		0	0	**1**	**2**	**4**	**5**	**6**	**8**	**9**	**16**		
		3	1	**1**	**2**	**4**	**9**	**10**	**11**	**13**	**18**		
		2	1	**1**	**2**	**3**	**4**	**9**	**12**	**13**	**18**		
		1	1	**1**	**2**	**3**	**4**	**5**	**8**	**13**	**17**		
		0	1	**1**	**2**	**3**	**5**	**6**	**8**	**9**	**17**		
		2	2	**1**	**4**	**8**	**9**	**10**	**13**	**17**	**18**		
		1	2	**1**	**2**	**3**	**8**	**9**	**10**	**17**	**18**		
		0	2	**1**	**2**	**3**	**4**	**8**	**9**	**17**	**18**		
M_{11}	1	4	0	**1**	**2**	**3**	**4**	**10**	**11**	**12**	**13**	9	356
		3	0	**1**	**2**	**3**	**4**	**6**	**10**	**12**	**13**		
		2	0	**1**	**2**	**3**	**4**	**5**	**7**	**11**	**14**		
		1	0	**1**	**2**	**3**	**4**	**5**	**7**	**8**	**11**		
		0	0	**1**	**2**	**4**	**5**	**7**	**12**	**15**	**17**		

Table 18: Some smallest defining sets of M_{10} and M_{11}

6. Smallest defining sets of the 3-(10,5,3) designs

The total numbers of smallest defining sets of each of the 3-(10,5,3) designs are now determined. The five self-complementary 3-(10,5,3) designs, N_2, N_3, N_4, N_6 and N_7, are considered first. Because they are extensions by complementation and they are the only extensions of the 2-(9,4,3) designs M_2, M_4, M_7, M_8 and M_{11}, respectively, these 3-(10,5,3) designs can be said, by Lemma 12, to have smallest defining sets of the same cardinality as those of their restrictions.

By Lemma 9, smallest defining sets of these self-complementary 3-designs can be formed by adding the appropriate extra element to each block of the smallest defining sets of the restrictions which have unique extensions. By the proof of Lemma 13, any other smallest defining set of one of these 3-designs must be formed by replacing one or more blocks of one of these established smallest defining sets by their complements.

Given two isomorphic defining sets, the replacement of corresponding blocks in each by their complements yields two sets of blocks which are also isomorphic. Hence, in searching for smallest defining sets of the 3-designs, it is sufficient to consider all 2^s sets of s blocks arising from the representatives of the isomorphism classes of smallest defining sets of the 2-(9,4,3) designs. Clearly, it is more efficient to use the restriction which has the fewest isomorphism classes of smallest defining sets.

Hence the representatives of the isomorphism classes of smallest defining sets (augmented by the appropriate extra element) of 2-(9,4,3) designs M_1, M_4, M_7, M_9 and M_{11} were used as the original sets in searching for smallest defining sets of the 3-(10,5,3) designs N_2, N_3, N_4, N_6 and N_7 respectively. Now, let S be a smallest defining set of the 2-(9,4,3) design M_i, whose extension by complementation is N_j. Let C be a set of blocks formed from the set $S(x)$ by replacing some blocks by their complements. Then $C \cup \tilde{C} = S(x) \cup \widetilde{S(x)}$. But $S(x) \cup \widetilde{S(x)}$ cannot be contained in any other self-complementary 3-(10,5,3) design because it contains $S(x)$, whose restriction on x is a defining set of M_i, which extends by complementation uniquely. If either N_1 or N_5 contains a subset of blocks isomorphic to C, then C cannot be a defining set of N_j; if neither N_1 nor N_5 contains such a subset, then C must be a smallest defining set of N_j.

The algorithm used for finding all smallest defining sets of a self-complementary 3-(10,5,3) design is, therefore, as follows.

STEP A. For the restriction which has the fewest isomorphism classes of smallest defining sets, take one representative of each isomorphism class.

STEP B. If S is a smallest defining set (of size q) of the restriction, form all 2^q sets of q blocks containing exactly one of each complementary pair of blocks in $S(x) \cup \widetilde{S(x)}$.

STEP C. Obtain the *nauty* signature of each such q-set of blocks; also obtain the *nauty* signatures of each non-isomorphic q-subset of blocks of N_1 and N_5.

STEP D. Compare the lists of signatures. If a set, C, of blocks in the self-complementary design, for which $C \cup \tilde{C} = S(x) \cup \widetilde{S(x)}$, has a signature which does not occur in the relevant signature lists of N_1 or N_5, then C has no isomorph in either of those designs, so is a smallest defining set of the self-complementary design.

STEP E. If C's signature occurs in the signature lists of either N_1 or N_5, an isomorphism check is done. If C has an isomorph in either N_1 or N_5, it is not a defining set; otherwise it is a smallest defining set.

The numbers of smallest defining sets of the self-complementary 3-(10,5,3) designs, as found by the above algorithm, are given in Table 19. The smallest defining sets of N_6 comprise six blocks, since it is the only extension of M_8, whose smallest defining sets have six blocks. The smallest defining sets of the other four self-complementary 3-(10,5,3) designs comprise eight blocks, as do those of their restrictions. A comparison of the numbers of smallest defining sets of the 2-(9,4,3) designs, as given in Tables 13 and 14, with the numbers of smallest defining sets of their self-complementary extensions, as given in Table 19, shows that STEP E eliminated a small proportion (less than 1%) of potential defining sets in each of the cases N_2, N_3, N_4 and N_7, but almost 40% of such sets in the case of N_6.

The cases of N_2 and N_6 merit further comment. It is clear that, if S is a defining set of M_1, then $S(1)$ is not a defining set of N_2, since M_1 has three extensions. Also, since Breach [3] showed that $\widetilde{N_1}$ is isomorphic to N_1, $\widetilde{S(1)}$ cannot be a defining set of N_2. The application of STEP E shows that for each representative, S, of an isomorphism class of smallest defining sets of M_1, all of the remaining 254 sets S', such that $S' \cup \widetilde{S'} = S(1) \cup \widetilde{S(1)}$, are defining sets of N_2, since none of them occurs in any isomorph of N_1 or N_5.

Similarly, if S is a defining set of M_9, then $S(9)$ is not a defining set of N_6, since M_9 has two extensions; neither is $\widetilde{S(9)}$ a defining set of N_6 since $\widetilde{S(9)} \subseteq \widetilde{N_5} = G \cup \tilde{G} \cup \tilde{H} \cup \tilde{F}$, which, as was mentioned earlier, is isomorphic to N_5. Further, each of the smallest defining sets of M_9, augmented by the element 9, can be partitioned into a subset of G and a subset of H. Any set S' for which $S' \cup \widetilde{S'} = S(9) \cup \widetilde{S(9)}$, where S is a defining set of M_9 and for which $S' \subset G \cup \tilde{G} \cup H$ or $S' \subset G \cup \tilde{G} \cup \tilde{H}$ cannot be a defining set of N_6 since both complete to more than one design. The application of STEP E shows that any such S' which intersects both H and $\tilde{H}$ is a defining set of N_6.

Design (D)	$\|d_s(D)\|$	Number of $d_s(D)$
N_2	8	832104
N_3	8	837616
N_4	8	824744
N_6	6	3136
N_7	8	820156

Table 19: Size and number of smallest defining sets of the self-complementary 3-(10,5,3) designs

The smallest defining sets of N_1 and N_5 were found by means of an adaptation of the algorithm described in Section 3. Since the completion of feasible sets to all possible 3-(10,5,3) designs is very time consuming whether done by hand or by computer, the following step, which uses Lemma 5, was substituted.

STEP 5: Given a feasible set of l blocks, the *nauty* signature of that set is determined. A list of signatures of all l-subsets of blocks for each other 3-(10,5,3) design is determined. If the signature of a feasible l-set does not occur in the list of l-subset signatures for any other 3-(10,5,3) design, then there is no subset of blocks isomorphic to this feasible set in any of the other designs. Hence, by Lemma 5, the feasible set is a smallest defining set. If signatures match, an isomorphism check is carried out; verification of the existence of an isomorph in another 3-(10,5,3) design discounts the feasible set as a defining set.

This adaptation of Greenhill's algorithm was used by Gray and Street [9] in determining the smallest defining sets of the 2-(15,7,3) designs.

Taking $N_1 = M_1(1) \cup A$, it is easy to see that A is a trade, since $M_1(1) \cup B$ is also a 3-(10,5,3) design. Breach [3] showed that $M_1(1)$ is also a trade. Using only these two trades, the algorithm applied to finding smallest defining sets of N_1 yields 99 non-isomorphic 5-sets of blocks as feasible sets. But STEP 5 shows that each of these 5-sets is isomorphic to a 5-set of blocks in N_7; hence there are no defining sets of just five blocks for N_1. The algorithm then yields 1643 non-isomorphic 6-sets of blocks of N_1 as feasible sets. All except 344 of these are isomorphic to 6-sets of blocks occuring in at least one other 3-(10,5,3) design. Hence, by Lemma 5, there are 344 non-isomorphic smallest defining sets of six blocks for N_1; the information in Table 20 shows that the total number of smallest defining sets of N_1 is 243600.

$\|G_S\|$	example of $d_s(N_1)$						n_i	n_c
3	1	2	6	16	31	33	240	1
2	1	2	4	10	27	36	360	10
1	1	2	3	4	9	34	720	333

Table 20: Some smallest defining sets of N_1

Table 9 shows that N_5 and N_6 intersect in 28 blocks; the remaining eight blocks of N_5, which are the eight Type II blocks containing the element 0, therefore constitute a trade of volume eight. Since the automorphism group of N_5 was shown by Breach [3] to be transitive, any set of eight blocks of Type II with a single common element must also constitute a trade in N_5.

Given these ten trades, there are 26 feasible sets of five blocks in N_5; each set has, however, an isomorph in N_1, so none is a defining set. There are 851 feasible sets of six blocks; each of these sets contains a subset of three blocks, one of which intersects the other two in exactly one element and is hence a Type II block. Since none of the self-complementary designs contains any Type II blocks, it is sufficient to check for the occurrence of isomorphs of the feasible sets in design N_1. This check reveals that there are, up to isomorphism, 638 smallest defining sets of six blocks and altogether 200640 smallest defining sets for design N_5; these results are summarized in Table 21.

$\lvert G_S \rvert$	example of $d_s(N_5)$						n_i	n_c
2	**1**	**2**	**3**	**14**	**25**	**31**	160	22
1	**1**	**2**	**3**	**13**	**17**	**25**	320	616

Table 21: Some smallest defining sets of N_5

As with the 2-(9,4,3) designs, the sizes of the smallest defining sets of the 3-(10,5,3) designs cannot be seen to bear any direct relation to the orders of the automorphism groups of the designs. Designs N_2, N_3, N_4 and N_7 all have smaller automorphism groups but larger smallest defining sets than either N_1 or N_5; on the other hand, N_2 has a larger automorphism group and larger smallest defining sets than N_6.

The results of this section are summarized in the following Theorem.

Theorem 2 *The* 3-(10, 5, 3) *designs N_2, N_3, N_4 and N_7 have smallest defining sets of eight blocks each, while the remaining* 3-(10, 5, 3) *designs, N_1, N_5 and N_6 have smallest defining sets of six blocks.* □

Finally, a case is noted in which the strict inequality of Corollary 9.1 holds. The 3-(10,5,3) design, N_1, was shown by Breach [3] to have exactly one extension, to the unique 4-(11,6,3) design. So, by Lemma 9, since there are defining sets of six blocks for N_1, there are also defining sets of six blocks for the 4-(11,6,3) design.

But Greenhill [10] showed that the unique 4-(11,5,1) design has a smallest defining set of 5 blocks. Hence, by Lemma 4, the complementary design, which is the 4-(11,6,3) design, also has a smallest defining set of five blocks.

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References

[1] W.O.Alltop, *An infinite class of 5-designs,* Journal of Combinatorial Theory Series A **12** (1972), 390-395.

[2] D.R.Breach, The 2-(9,4,3) and 3-(10,5,3) Designs, Research Report CORR 77-11 (1977), Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ont., Canada.

[3] D.R.Breach, *The* 2-(9, 4, 3) *and* 3-(10, 5, 3) *designs,* Journal of Combinatorial Theory, Series A **27** (1979), 50-73.

[4] Cathy Delaney, User Manual - *Complete*, Department of Mathematics, The University of Queensland, 1993.

[5] Cathy Delaney, Instructions for semi-automatic execution of Catherine Greenhill's algorithm for finding smallest defining sets, Department of Mathematics, The University of Queensland, 1993.

[6] P.Gibbons, Computing techniques for the construction and analysis of block designs, Department of Computer Science, University of Toronto, Technical Report 92/76, 1976.

[7] Ken Gray, *On the minimum number of blocks defining a design,* Bulletin of the Australian Mathematics Society **41** (1990), 97-112.

[8] Ken Gray, *Defining sets of single-transposition-free designs,* Utilitas Mathematica **38** (1990), 97-103.

[9] Ken Gray and Anne Penfold Street, *Smallest defining sets of the five non-isomorphic* 2-(15, 7, 3) *designs,* Bulletin of the Institute of Combinatorics and its Applications **9** (1993), 96-102.

[10] Catherine S. Greenhill, An algorithm for finding smallest defining sets of t-designs, M.Sc. Thesis, The University of Queensland, 1993.

[11] Catherine S. Greenhill, *An algorithm for finding smallest defining sets of t-designs,* Journal of Combinatorial Mathematics and Combinatorial Computing **14** (1993), 39-60.

[12] Catherine S. Greenhill and Anne Penfold Street, *Smallest defining sets of some small t-designs and relations to the Petersen graph,* submitted.

[13] H.L. Hwang, *On the structure of* (v, k, t) *trades,* Journal of Statistical Planning and Inference **13** (1986), 171-191.

[14] B.D. McKay, *nauty* User's Guide (Version 1.5), Australian National University Computer Science Technical Report TR-CS-90-02, 1990.

[15] M.J.Sharry, private communication (1992).

[16] R.G. Stanton, R.C.Mullin and J.A.Bate, *Isomorphism classes of a set of prime BIBD parameters,* Ars Combinatoria **2** (1976), 251-264.

[17] J.H.van Lint, H.C.A. van Tilborg and J.R.Wiekma, *Block designs with* $v = 10$, $k = 5$, $\lambda = 4$, Journal of Combinatorial Theory Series A **23** (1977), 105-115.

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