On the Integrity of Distance Domination in Graphs

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Abstract

Let n and k be positive integers and let G be a graph. A set $\mathcal{D}$ of vertices of G is defined to be an (n,k)-dominating set of G if every vertex of $V(G)-\mathcal{D}$ is within distance n from at least k vertices of $\mathcal{D}$. The minimum cardinality among all (n,k)-dominating sets of G is called the (n,k)-domination number of G and is denoted by $\gamma_{n,k}(G)$. A set I of vertices of G is defined to be an (n,k)-independent set in G if every vertex of I is within distance n from at most $k-1$ other vertices of I in G. We denote by $\beta_{n,k}(G)$ the maximum cardinality of an (n,k)-independent set of G. We show that the problem of computing $\gamma_{n,k}$ is in the NP-complete class, even when restricted to bipartite graphs and chordal graphs. We prove that in every graph there exist some subsets of vertices that are both (n,k)-independent and (n,k)-dominating, so $\gamma_{n,k} \leq \beta_{n,k}$. We also investigate lower and upper bounds on $\gamma_{n,k}$.

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1 Introduction

Let $n \geq 1$ be an integer. The *open n-neighbourhood* $N_n(v)$ of a vertex v in a graph G is the set of all vertices, different from v, which are within distance n from v, that is to say, $N_n(v) = \{u \mid 0 < d(u,v) \leq n\}$. The *n-degree* $\deg_n v$ of v in G is given by $|N_n(v)|$, while $\Delta_n(G)(\delta_n(G))$ denotes the maximum (respectively, minimum) n-degree among all the vertices of G. For A a subset of vertices of G, let us denote by $m_n(A)$ the number of pairs (u,v) of vertices with $u, v \in A$ and $d_G(u,v) \leq n$. Further, we let $\deg_n(x,A) = |\{a \in A \mid 0 < d_G(x,a) \leq n\}|$ and $\Delta_n(A) = \max_{x \in A} \deg_n(x,A)$. For other graph theory terminology, we follow [11]. Specifically, $p(G)$ denotes the number of vertices (order) and $q(G)$ denotes the number of edges (size) of a graph G with vertex set $V(G)$ and edge set $E(G)$.

Let n and k be positive integers and let G be a graph. We define a set $\mathcal{D}$ of vertices of G to be an *(n,k)-dominating set* of G if every vertex of $V(G) - \mathcal{D}$ is within distance n from at least k vertices of $\mathcal{D}$. The minimum cardinality among all (n,k)-dominating sets of G is called the *(n,k)-domination number* of G and is denoted by $\gamma_{n,k}(G)$. We note that (1,1)-dominating sets are the classical dominating sets, that is, $\gamma_{1,1}(G) = \gamma(G)$. When $n = 1$, our definition of (n,k)-domination coincides with the notion of k-domination, introduced by Fink and Jacobson [15, 16] and further studied by Cockayne, Gamble and Shepherd [12], Favaron [13, 14], Hopkins and Staton [24] and Jacobson and Peters [25]. When $k = 1$, our definition of (n,k)-domination coincides with the notion of n-domination, results on which have been presented by, among others, Bacsó and Tuza [1,2], Beineke and Henning [3], Bondy and Fan [4], Chang [7], Chang and Nemhauser [8 - 10], Fraisse [17], Fricke, Hedetniemi and Henning [18], Henning, Oellermann and Swart [20 - 23], Meir and Moon [26], Mo and Williams [27] and Topp and Volkmann [28, 29].

The vertices of G may represent centres, some pairs of which are in direct communication with each other (represented by adjacent vertices), and $\mathcal{D}$ a set of centres from which signals may be sent, where a signal may be reliably transmitted along a route between centres corresponding to a path in G of length at most n. A breakdown in reliable communication may occur for a number of reasons. For example, an erroneous massage may be sent from one or more of the transmitting centres, or a transmitter may fail. To retain the integrity of the communication network in the event of such failures, further conditions must be imposed on the set of transmitting centres represented by $\mathcal{D}$. One may require the each non-transmitting centre be able to receive messages from at least k transmitters, where k is a positive integer sufficiently large to allow for adequate security of transmission in all likely events of a breakdown in reliable communications as mentioned above. The set of transmitting centres then corresponds to an (n,k)-dominating set of G.

Next we define the set I of G to be an (n,k)*-independent set* in G if every vertex of I is within distance n from at most $k-1$ other vertices of I in G, that is to say, $\Delta_n(I) < k$. Let $\beta_{n,k}(G)$ denote the maximum cardinality of an (n,k)-independent set of G. We note that (1,1)-independent sets are the classical independent sets, that is, $\beta_{1,1}(G) = \beta(G)$. For $n = 1$, our definition of (n, k)-independence coincides with the notion of k-independence (also called $(k-1)$-small in [24]) introduced by Fink and Jacobson [15, 16] and further studied by Favaron [13, 14], Hopkins and Staton [24] and Jacobson and Peters [25].

In this paper we show the problem of determining $\gamma_{n,k}$ and $\beta_{n,k}$ is in the NP-complete class, even when restricted to bipartite graphs and chordal graphs. We prove that for any graph G, and for all positive integers n and k, $\gamma_{n,k}(G) \leq \beta_{n,k}(G)$. Finally we present bounds on $\gamma_{n,k}$ that do not involve $\beta_{n,k}$.

2 Complexity Issues

Jacobsen and Peters [25] showed that the problem of determining γ_k for an arbitrary graph is in the NP-complete class. In this section, we show that even when restricted to bipartite graphs and chordal graphs the problem of determining γ_k is in the NP-complete class. We also show that the problem of determining $\gamma_{n,k}$ for bipartite graphs and chordal graphs is NP-complete. The following decision problem for the domination number of a bipartite graph is known to be NP-complete (see [19]).

Problem: Bipartite Domination (BDM)
INSTANCE: A bipartite graph G and a positive integer m.
QUESTION: Is $\gamma(G) \leq m$?

We will demonstrate a polynomial time reduction of this problem to the bipartite k-domination problem. For notational convenience we will write γ_k instead of $\gamma_{1,k}$.

Problem: Bipartite k-Domination (BkDM)
INSTANCE: A bipartite graph G^* and positive integers $k \geq 2$ and m^*.
QUESTION: Is $\gamma_k(G^*) \leq m^*$?

Theorem 1 *Problem BkDM is NP-complete.*

Proof. It is obvious that BkDM is a member of NP since we can, in polynomial time, guess a subset of vertices $\mathcal{D}$ and then verify, in polynomial time, whether or not $\mathcal{D}$ is a k-dominating set of G^* and that $|\mathcal{D}| \leq m^*$.

We next show how a polynomial time algorithm for BkDM could be used to solve BDM in polynomial time. Given a graph G and positive integer m, construct the graph G^* by adding to each $v \in V(G)$ a set of $k-1$ paths of length 1. Let $p = |V(G)|$ and $q = |E(G)|$. We have $|V(G^*)| = pk$ and $|E(G^*)| = q + (k-1)p$, and so G^* can be constructed from G in polynomial time. Note that if G is bipartite, then G^* is also bipartite.

We will show that G has a dominating set $\mathcal{D}$ with $|\mathcal{D}| \leq m$ if and only if G^* has a k-dominating set $\mathcal{D}^*$ with $|\mathcal{D}^*| \leq m^* = m + p(k-1)$. Let $\mathcal{D}^*$ be a k-dominating set of

G^* with $|\mathcal{D}^*| \le m^* = m+p(k+1)$. Note that every vertex of G^* of degree less than k must be in $\mathcal{D}^*$. In particular, each end-vertex of G^* must be in $\mathcal{D}^*$. Consider the set $\mathcal{D} = \mathcal{D}^* \cap V(G)$. We claim that $\mathcal{D}$ is a dominating set of G. Suppose $v \in V(G) - \mathcal{D}$. Since v is adjacent to only $k-1$ vertices of $V(G^*) - V(G)$, it follows that v is adjacent to at least one vertex of $\mathcal{D}$. Thus $\mathcal{D}$ is a dominating set of G of cardinality $|\mathcal{D}^*| - p(k-1) \le m$, so $\gamma(G) \le |\mathcal{D}| \le m$. Next suppose that G has a dominating set $\mathcal{D}$ with $|\mathcal{D}| \le m$. Then it is evident that $\mathcal{D}$, together with the set $V(G^*) - V(G)$, forms a k-dominating set of G^* of cardinality $|\mathcal{D}| + p(k-1) \le m + p(k-1) = m^*$, so $\gamma_k(G^*) \le m^*$. □

The following decision problem for the domination number of a chordal graph is known to be NP-complete (see [5,6]).

Problem: Chordal Domination (CDM)
INSTANCE: A chordal graph G and a positive integer m.
QUESTION: Is $\gamma(G) \le m$?

Using the same construction as that in the proof of Theorem 1, we may demonstrate a polynomial time reduction of this problem to the chordal k-domination problem.

Problem: Chordal k-Domination (CkDM)
INSTANCE: A chordal graph G^* and positive integers $k \ge 2$ and m^*.
QUESTION: Is $\gamma_k(G^*) \le m^*$?

Hence we have the following result.

Theorem 2 *Problem CkDM is NP-complete.*

Next we demonstrate a polynomial time reduction of the problem BDM to our bipartite (n,k)-domination problem.

Problem: Bipartite (n,k)-Domination (BnkDM)
INSTANCE: A bipartite graph G^* and integers $n, k \ge 2$ and m^*.
QUESTION: Is $\gamma_{n,k}(G^*) \le m^*$?

Theorem 3 *Problem BnkDM is NP-complete.*

Proof. Clearly there exists a nondeterministic-polynomial algorithm for deciding whether or not a graph G^* has a subset $\mathcal{D}$ of $V(G^*)$ that is an (n,k)-dominating set with $|\mathcal{D}| \le m^*$. So *BnkDM* is in the class *NP*.

We next show how a polynomial time algorithm for BnkDM could be used to solve *BDM* in polynomial time. Given a graph G with vertex set $\{v_1, v_2, ..., v_p\}$ and size q and a positive integer m, we construct a graph G^* as follows. Consider a complete bipartite graph $K_{k-1,k-1}$ with partite sets U and W. Add three new vertices u, v and w to this graph, join u to every vertex in U, join w to every vertex in W, and join v and w. Now subdivide each of the $k-1$ edges incident with u $n-1$ times, and subdivide the edge vw $n-2$ times. Let H denote the resulting graph. Further, let $H_1, H_1, ..., H_p$ be p (disjoint) copies of H. Let W_i be the name of the set in H_i corresponding to W, and let u_i, v_i and w_i be the names of the vertices of H_i that are named u, v and w, respectively, in H. For each $i = 1, 2, ..., p$, identify the vertex v_i of G and the vertex v_i of H_i. Let G^* be the graph so constructed from G. We have $|V(G^*)| = |V(H)| \cdot p = k(n+1)p$ and $|E(G^*)| = q + |E(H)| \cdot p = q + [k^2 + (n-1)k - 1]p$, so G^* can be constructed from G in polynomial time. An example is presented in Figure 1 with $n = 3$ and $k = 4$, and where G is the 4-cycle v_1, v_2, v_3, v_4, v_1. Note that if G is bipartite, then G^* is also bipartite.

G^*:

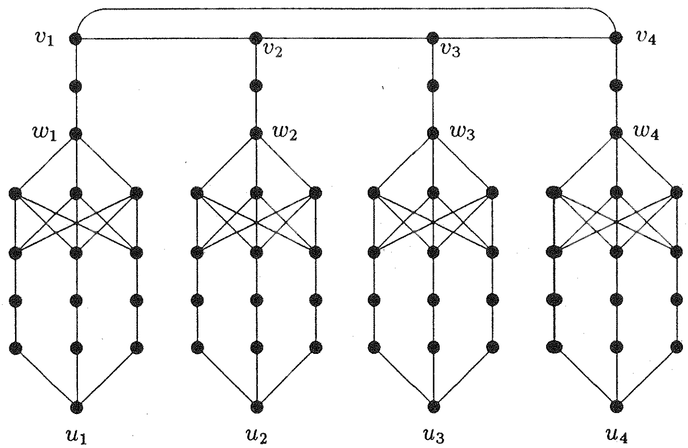

Figure 1: G is a (4,4) graph; G^* is a (64,96) graph.

We will show that G has a dominating set $\mathcal{D}$ with $|\mathcal{D}| \leq m$ if and only if G^* has an (n, k)-dominating set $\mathcal{D}^*$ with $|\mathcal{D}^*| \leq m^* = m + kp$. Suppose first the G has a dominating set $\mathcal{D}$ with $|\mathcal{D}| \leq m$. Then it is evident that $\mathcal{D} \cup (W_1 \cup W_2 \cup \ldots \cup W_p) \cup \{u_1, u_2, \ldots, u_p\}$ is an (n, k)-dominating set of G^* of cardinality $|\mathcal{D}| + p(k-1) + p \leq m + kp = m^*$.

Now let $\mathcal{D}^*$ be an (n, k)-dominating set of G^* with $|\mathcal{D}^*| \leq m^* = m + kp$. Before proceeding further, we introduce some notation. Let $\mathcal{D}_i = \mathcal{D}^* \cap V(H_i)$ and let P_i denote the $v_i - w_i$ path. For each $j = 0, 1, 2, ..., 2n+1$, let U_{ij} be the set of all vertices of H_i at distance j from v_i. Note that $U_{io} = \{v_i\}$, $U_{in} = W_i$, and $U_{i,2n+1} = \{u_i\}$. Further, let U_i be the set of all vertices, distinct from u_i, that are within distance n from u_i, so $U_i = \bigcup_{j=n+1}^{2n} U_{ij}$. We now prove five claims.

Claim 1 *$|\mathcal{D}_i| \geq k$ for all i.*

Proof. If $u_i \notin \mathcal{D}_i$, then the vertex u_i is within distance n from at least k vertices of $\mathcal{D}_i$, so $|\mathcal{D}_i| \geq k$. On the other hand, if $u_i \in \mathcal{D}_i$, then consider the set of vertices $U_{i,2n}$. If some vertex of $U_{i,2n}$ does not belong to $\mathcal{D}_i$, then $|\mathcal{D}_i| \geq k$. If every vertex of $U_{i,2n}$ belongs to $\mathcal{D}_i$, then $\mathcal{D}_i$ contains these $k-1$ vertices in addition to the vertex u_i, so $|\mathcal{D}_i| \geq k$. □

Claim 2 *If $u_i \notin \mathcal{D}_i$, then $|\mathcal{D}_i| \geq k+1$.*

Proof. Assume, to the contrary, that $|\mathcal{D}_i| < k+1$. Then, by Claim 1, $|\mathcal{D}_i| = k$. Since $u_i \notin \mathcal{D}_i$, the vertex u_i is within distance n from each of the k vertices of $\mathcal{D}_i$, so $|\mathcal{D}_i \cap U_i| = k$ and $\mathcal{D}_i \subset U_i$. Note that by the way in which H_i is constructed, each vertex $v \in U_i$ belongs to exactly $k-2$ cycles of length $2n+2$ that contain no chords. Hence, for each $v \in U_i$, there is a (unique) set S_v of $k-2$ vertices of U_i at distance $n+1$ from v, and $S_v \subset U_{ij}$ for some j with $n+1 \leq j \leq 2n$. Now consider a vertex $v \in \mathcal{D}_i$. Then each vertex of S_v is within distance n from at most $|\mathcal{D}_i - \{v\}| = k-1$ vertices of $\mathcal{D}_i$. This means that $S_v \subset \mathcal{D}_i$. Since $|\mathcal{D}_i| = k$, and $|S_v \cup \{v\}| = k-1$, there is a vertex $u \in \mathcal{D}_i$ that does not belong to $S_v \cup \{v\}$, so $\mathcal{D}_i = S_v \cup \{u, v\}$. Since $v \neq u$, $S_v \neq S_u$. Let $u^* \in S_u - S_v$. Since $u \notin S_v$, we note that $d(u,v) \leq n$, so since $d(u,u^*) = n+1$ we know that $u^* \neq v$. Hence $u^* \notin \mathcal{D}_i$. But this means that u^* must be within distance n from each of the k vertices of $\mathcal{D}_i$. This contradicts the fact that $d(u,u^*) = n+1$ and $u \in \mathcal{D}_i$. We deduce, therefore, that $|\mathcal{D}_i| \geq k+1$. □

Claim 3 *If $|\mathcal{D}_i| = k$, then $\mathcal{D}_i = U_{in} \cup \{u_i\}$.*

Proof. Necessarily, $u_i \in \mathcal{D}_i$ by Claim 2. Now consider a vertex $v \in U_{in}$. Then $d(u_i, v) = n+1$. Furthermore, $d(v, v_i) = n$, so $d(v, V(G^*) - V(H_i)) \geq n+1$. Hence if v does not belong to $\mathcal{D}_i$, then $|\mathcal{D}_i - \{u_i\}| \geq k$, so that $|\mathcal{D}_i| \geq k+1$, which produces a contradiction. We deduce that every vertex of U_{in} belongs to $\mathcal{D}_i$, so $U_{in} \cup \{u_i\} \subseteq \mathcal{D}_i$. But $|\mathcal{D}_i| = k$, and $|U_{in} \cup \{u_i\}| = k$, whence $U_{in} \cup \{u_i\} = \mathcal{D}_i$.□

As an immediate consequence of Claim 3, we have the following result.

Claim 4 *If $\mathcal{D}_i$ contains some vertex of P_i, then $|\mathcal{D}_i| \geq k+1$.*

For each $i = 1, 2, \ldots, p$, do the following. If $|\mathcal{D}_i| = k$, then let $\mathcal{D}'_i = \mathcal{D}_i$. If $|\mathcal{D}_i| \geq k+1$, then let $\mathcal{D}'_i = U_{in} \cup \{u_i, v_i\}$, so $|\mathcal{D}'_i| = k+1$. Now let $\mathcal{D}' = \mathcal{D}'_1 \cup \mathcal{D}'_2 \cup \ldots \cup \mathcal{D}'_p$. Then $|\mathcal{D}'| = \sum_{i=1}^{p} |\mathcal{D}'_i| \leq \sum_{i=1}^{p} |\mathcal{D}_i| = |\mathcal{D}^*|$. We show next that $\mathcal{D}'$ is an (n,k)-dominating set of G^*.

Claim 5 *D' is an (n, k)-dominating set of G^*.*

Proof. If $|\mathcal{D}'_i| = k + 1$, then it is evident that every vertex of H_i is (n, k)-dominated by $\mathcal{D}'_i$, and therefore by $\mathcal{D}'$. If $|\mathcal{D}'_i| = k$, then $\mathcal{D}'_i = \mathcal{D}_i = U_{in} \cup \{u_i\}$ by Claim 3. Hence every vertex of H_i that is not on the $v_i - w_i$ path P_i is clearly (n, k)-dominated by $\mathcal{D}'_i$, and therefore by $\mathcal{D}'$. The only vertices of H_i whose (n, k)-domination are in doubt are those vertices on P_i. That these vertices are (n, k)-dominated by $\mathcal{D}'$ may be seen as follows. Consider the vertex w_i. Since $d(u_i, w_i) > n$, w_i is within distance n from only $k-1$ vertices of $\mathcal{D}_i$. However D^* is an (n, k)-dominating set of G^*, so there must exist a vertex $w \in D^* - D_i$ that is within distance n from w_i. Since $d(v_i, w_i) = n - 1$, and $w \notin V(H_i)$, it follows that $d(w, w_i) = n$. Thus w is a vertex of G that is adjacent to v_i, that is to say, $w = v_r$ for some r with $1 \leq r \leq p$ and $i \neq r$. Since $v_r \in D_r$, we have by Claim 4 that $|D_r| \geq k + 1$, and therefore that $D'_r = U_{rn} \cup \{u_r, v_r\}$. Hence we note that $U_{in} \cup \{v_r\} \subset D'$. Consequently, every vertex on P_i is within distance n from at least k vertices of D'. Hence if $|D'_i| = k$, then every vertex of H_i is (n, k)-dominated by D'. The result now follows. □

It follows from Claim 5 that D' is an (n, k)-dominating set of G^* with $|D'| \leq |D^*| \leq m^* = m + pk$. Now consider the set $D = D' \cap V(G)$. We claim that D is a dominating set of G. Suppose $v_i \in V(G) - D$. Then $v_i \notin D'_i$, so we must have $|D'_i| = k$. As in the proof of Claim 5, there exists a vertex v_r of G that belongs to D' which is adjacent to v_i. So $v_r \in D$ and v_i is adjacent to at least one vertex of $\mathcal{D}$. Hence D is a dominating set of G with $|D| = |D'| - \sum_{i=1}^{p} |U_{in} \cup \{u_i\}| \leq m^* + pk = m$. This completes the proof of Theorem 3. □

Finally, we demonstrate a polynomial time reduction of the problem CDM to our chordal (n, k)-domination problem.

Problem: Chordal (n, k)-Domination (CnkDM)
INSTANCE: A chordal graph G^* and integers $n, k \geq 2$ and m^*.
QUESTION: Is $\gamma_{n,k}(G^*) \leq m^*$?

Theorem 4 *Problem CnkDM is NP-complete.*

Proof. Clearly $CnkDM$ is in the class NP. We next show how a polynomial time algorithm for $CnkDM$ could be used to solve DCM in polynomial time. Given a graph G with vertex set $\{v_1, v_2, ..., v_p\}$ and size q and a positive integer m, we construct a graph G^* as follows. Consider a complete bipartite graph $K_{k,k-1}$ with partite sets U and W, where $|U| = k$. Add a new vertex w to this graph and join every two vertices of $W \cup \{w\}$ with an edge. Attach to every vertex of U a path of length n in such a way that the resulting paths $Q_1, Q_2, ..., Q_k$ (say) are pairwise disjoint. For $j = 1, 2, ..., k$, let u_i be the end-vertex of Q_i that does not belong to U. Finally, attach to w a $w - v$ path P of length $n - 1$. Let H denote the resulting graph. Further, let $H_1, H_2, ..., H_p$ be p (disjoint) copies of H. Let W_i be the name of the set in H_i corresponding to

W, and let $Q_{i1}, Q_{i2}, ..., Q_{ik}$ and P_i be the names of the paths of H_i corresponding to $Q_1, Q_2, ..., Q_k$ and P, respectively, in H. Further, let $u_{i1}, u_{i2}, ..., u_{ik}, v_i$ and w_i be the names of the vertices of H_i that are named $u_1, u_2, ..., u_k, v$ and w, respectively, in H. For each $i = 1, 2, ..., p$, identify the vertex v_i of G and the vertex v_i of H_i. Let G^* be the graph so constructed from G. We have $|V(G^*)| = |V(H)| \cdot p = [k(n+2)+n-1] \cdot p$ and $|E(G^*)| = q + |E(H)| \cdot p = q + [\frac{3}{2}k^2 + (n - \frac{3}{2})k + n - 1]p$, so G^* can be constructed from G in polynomial time. An example is presented in Figure 2 with $n = k = 3$, and where G is the 3-cycle v_1, v_2, v_3, v_1. Note that if G is chordal, then G^* is also chordal.

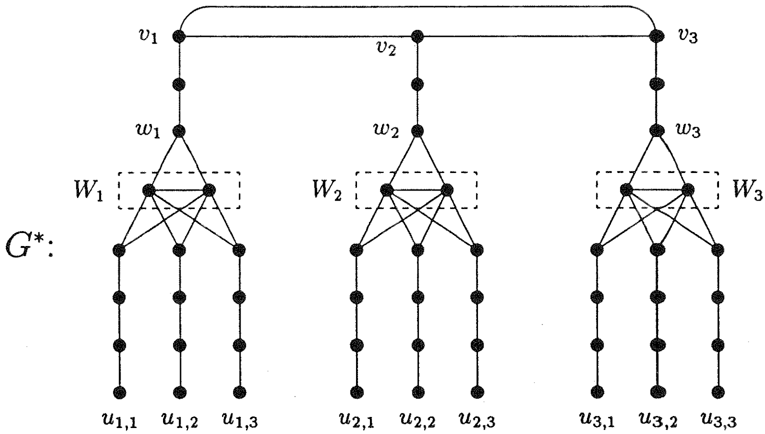

Figure 2: G is a (3,3) graph; G^* is a (51,63) graph.

We will show that G has a dominating set $\mathcal{D}$ with $|\mathcal{D}| \leq m$ if and only if G^* has an (n,k)-dominating set D^* with $|\mathcal{D}^*| \leq m^* = m + (2k-1)p$. Suppose first that G has a dominating set $\mathcal{D}$ with $|\mathcal{D}| \leq m$. Then it is evident that $\mathcal{D} \cup (W_1 \cup W_2 \cup ... \cup W_p) \cup \left(\bigcup_{i=1}^{p} \bigcup_{j=1}^{k} \{u_{ij}\} \right)$ is an (n,k)-dominating set of G^* of cardinality $|D| + (k-1)p + kp \leq m + (2k-1)p = m^*$.

Now let $\mathcal{D}^*$ be an (n,k)-dominating set of G^* with $|\mathcal{D}^*| \leq m^* = m + (2k-1)p$. Let $\mathcal{D}_i = \mathcal{D}^* \cap V(H_i)$. For each $j = 0, 1, 2, ..., 2n+1$, let U_{ij} be the set of all vertices of H_i at distance j from v_i. Note that $U_{in} = W_i$ and $U_{i,2n+1} = \{u_{i1}, u_{i2}, ..., u_{ik}\}$. Before proceeding further we prove five claims.

Claim 6 $|\mathcal{D}_i| \geq 2k - 1$ *for all* i.

Proof. Let J be the set of all integers j for which $u_{ij} \notin \mathcal{D}_i$. If $|J| \geq 1$, then for each $j \in J$, the vertex u_{ij} is within distance n from at least k vertices of $\mathcal{D}_i$ which must lie on the path Q_{ij}, so $|\mathcal{D}_i \cap V(Q_{ij})| \geq k$. Thus, if $|J| \geq 1$, then $|\mathcal{D}_i| \geq \sum_{j=1} |\mathcal{D}_i \cap V(Q_{ij})| = \sum_{j\in J} |\mathcal{D}_i \cap V(Q_{ij})| + \sum_{j\notin J} |\mathcal{D}_i \cap V(Q_{ij})| \geq k.|J| + (k - |J|) = (k-1)(|J|+1)+1 \geq 2k-1$. On the other hand, if $|J| = 0$, then $U_{i,2n+1} \subset \mathcal{D}_i$. Now consider the set U_{in}. If some vertex of U_{in} does not belong to $\mathcal{D}_i$, then since this vertex is at distance $n+1$ from each vertex of $U_{i,2n+1}$, we have $|\mathcal{D}_i - U_{i,2n+1}| \geq k$, so $|\mathcal{D}_i| \geq k + |U_{i,2n+1}| = 2k$. If every vertex of U_{in} belongs to $\mathcal{D}_i$, then $|\mathcal{D}_i| \geq |U_{in}| + |U_{i,2n+1}| = 2k - 1$.□

Claim 7 *If* $U_{i,2n+1} \not\subset \mathcal{D}_i$*, then* $|\mathcal{D}_i| \geq 2k$.

Proof. Let J be the set of all integers j for which $u_{ij} \notin \mathcal{D}_i$. Since $U_{i,2n+1} \not\subset \mathcal{D}_i$, we know that $|J| \geq 1$. If $|J| \geq 2$, then, as in the proof of Claim 6, $|\mathcal{D}_i| \geq (k-1)(|J| + 1) + 1 \geq 3k - 2 \geq 2k$ since $k \geq 2$. On the other hand, if $|J| = 1$, then, without loss of generality, we may assume that u_{i1} is the vertex in $U_{i,2n+1}$ that does not belong to $\mathcal{D}_i$, so $|\mathcal{D}_i \cap V(Q_{i1})| \geq k$. For $r = 1, 2, ..., k$, let u'_{ir} be the vertex that immediately precedes u_{ir} on the path Q_{ir}, and consider the set $U_{i,2n} - \{u'_{i1}\}$. If u'_{ir} is a vertex in this set that does not belong to $\mathcal{D}_i$, then it is evident that $|\mathcal{D}_i \cap (V(Q_{ir}) \cup U_{in})| \geq k$, so in this case $|\mathcal{D}_i| \geq |\mathcal{D}_i \cap V(Q_{i1})| + |\mathcal{D}_i \cap (V(Q_{ir}) \cup U_{in})| + |U_{i,2n+1} - \{u_{i1}, u_{ir}\}| \geq k + k + (k-2) = 3k - 2 \geq 2k$. If $U_{i,2n} - \{u'_{i1}\} \subset \mathcal{D}_i$, then $|\mathcal{D}_i| \geq |U_{i,2n+1} - \{u_{i1}\}| + |U_{i,2n} - \{u'_{i1}\}| + |\mathcal{D}_i \cap V(Q_{i1})| \geq (k-1) + (k-1) + k = 3k - 2 \geq 2k$. In both cases, $|\mathcal{D}_i| \geq 2k$.□

Claim 8 *If* $|\mathcal{D}_i| = 2k - 1$*, then* $\mathcal{D}_i = U_{in} \cup U_{i,2n+1}$.

Proof. Necessarily, $U_{i,2n+1} \subset \mathcal{D}_i$ by Claim 7. Now consider the set U_{in}. If some vertex of U_{in} does not belong to $\mathcal{D}_i$, then this vertex is within distance n from at least k vertices of $\mathcal{D}_i$, so $|\mathcal{D}_i - U_{i,2n+1}| \geq k$; that is to say, $|\mathcal{D}_i| \geq 2k$, which contradicts the fact that $|\mathcal{D}_i| = 2k - 1$. We deduce, therefore, that $U_{in} \cup U_{i,2n+1} \subseteq \mathcal{D}_i$. But $|U_{in} \cup U_{i,2n+1}| = 2k - 1$, implying that $U_{in} \cup U_{i,2n+1} = \mathcal{D}_i$.□

As an immediate consequence of Claim 8, we have the following result.

Claim 9 *If* $\mathcal{D}_i$ *contains some vertex of* P_i*, then* $|\mathcal{D}_i| \geq 2k$.

For each $i = 1, 2, ..., p$, do the following. If $|\mathcal{D}_i| = 2k - 1$, then let $\mathcal{D}'_i = \mathcal{D}_i$. If $|\mathcal{D}_i| \geq 2k$, then let $\mathcal{D}'_i = \{v_i\} \cup U_{in} \cup U_{i,2n+1}$, so $|\mathcal{D}'_i| = 2k$. Now let $\mathcal{D}' = \mathcal{D}'_1 \cup \mathcal{D}'_2 \cup ... \cup \mathcal{D}'_p$. Then $|\mathcal{D}'| = \sum_{i=1}^{p} |\mathcal{D}'_i| \leq \sum_{i=1}^{p} |\mathcal{D}_i| = |\mathcal{D}^*|$. We show next that $\mathcal{D}'$ is an (n, k)-dominating set of G^*.

Claim 10 *$\mathcal{D}'$ is an (n,k)-dominating set of G^*.*

Proof. If $|\mathcal{D}'_i| = 2k$, then it is evident that every vertex of H_i is (n,k)-dominated by $\mathcal{D}'_i$, and therefore by $\mathcal{D}'$. If $|\mathcal{D}'_i| = 2k-1$, then $\mathcal{D}'_i = \mathcal{D}_i = U_{in} \cup U_{i,2n+1}$ by Claim 8. Hence every vertex of H_i that is not on the $v_i - w_i$ path P_i is clearly (n,k)-dominated by $\mathcal{D}'_i$, and therefore by $\mathcal{D}'$. The only vertices of H_i whose (n,k)-domination are in doubt are those vertices on P_i. That these vertices are (n,k)-dominated by D', may be seen as follows. Consider the vertex w_i. Since w_i is at distance $n+2$ from each vertex of $U_{i,2n+1}$, the vertex w_i is within distance n from only $k-1$ vertices of $\mathcal{D}_i$. However $\mathcal{D}^*$ is an (n,k)-dominating set of G^*, so there must exist a vertex $w \in \mathcal{D}^* - \mathcal{D}_i$ that is within distance n from w_i. Since $d(v_i, w_i) = n-1$, and $w \notin V(H_i)$, it follows that $d(w, w_i) = n$. Thus w is a vertex of G that is adjacent to v_i, so $w = v_s$ for some s with $1 \leq s \leq p$ and $i \neq s$. Since $v_s \in \mathcal{D}_s$, we have by Claim 9 that $|\mathcal{D}_s| \geq 2k$, and therefore that $\mathcal{D}'_s = \{v_s\} \cup U_{sn} \cup U_{s,2n+1}$. Hence we note that $U_{in} \cup \{v_s\} \subset \mathcal{D}'$. Consequently, every vertex on P_i is within distance n from at least k vertices of $\mathcal{D}'$. Hence, if $|\mathcal{D}'_i| = 2k-1$, then every vertex of H_i is (n,k)-dominated by $\mathcal{D}'$. The result now follows. □

It follows from Claim 10 that $\mathcal{D}'$ is an (n,k)-dominating set of G^* with $|\mathcal{D}'| \leq |\mathcal{D}^*| \leq m^* = m + (2k-1)p$. Now consider the set $\mathcal{D} = \mathcal{D}' \cap V(G)$. We claim that $\mathcal{D}$ is a dominating set of G. Suppose $v_i \in V(G) - \mathcal{D}$. Then $v_i \notin \mathcal{D}'_i$, so we must have $|\mathcal{D}'_i| = 2k-1$. As in the proof of Claim 10, there exists a vertex v_s of G that belongs to $\mathcal{D}'$ and that is adjacent to v_i. So $v_s \in \mathcal{D}$ and v_i is adjacent to at least one vertex of $\mathcal{D}$. Hence $\mathcal{D}$ is a dominating set of G with $|\mathcal{D}| = |\mathcal{D}'| - \sum_{i=1}^{p} |U_{in} \cup U_{i,2n+1}| \leq m + (2k-1)p - (2k-1)p = m$. This completes the proof of Theorem 4. □

The following decision problem for the independence number is known to be NP-complete for general graphs (see [19]).

Problem: Independence (ID)
INSTANCE: A graph G and an integer m.
QUESTION: Is $\beta(G) \geq m$?

We will demonstrate a polynomial reduction of this problem to the bipartite (n,k)-independence problem with n even.

Problem: Bipartite (n,k)-Independence (BnkID)
INSTANCE: A bipartite graph G^* and integers $n, k \geq 2$ and m^*.
QUESTION: Is $\beta_{n,k}(G^*) \geq m^*$?

Theorem 5 *Problem BnkID with n even is NP-complete.*

Proof. Clearly *BnkID* is in the class *NP*. In what follows let $n = 2r$ where r is a positive integer. We show how a polynomial time algorithm for *BnkID* could be used to solve *ID* in polynomial time. Given a graph $G = (V, E)$ of order p and size q, and a positive integer m, we construct a graph $G^* = (V^*, E^*)$ as follows: Replace each edge uv of G by the tree T_{uv} shown in Figure 3, and attach to each vertex of G $k-1$ paths of length 1. Let S_{uv} denote the set of k end-vertices in T_{uv} at distance $n+1$ from u and v, and for each $v \in V$, let S_v denote the set of $k-1$ end-vertices of G^* adjacent to v. Then $|V^*| = kp + \left(\frac{3n}{2} + k - 1\right)q$ and $|E^*| = (k-1)p + \left(\frac{3n}{2} + k\right)q$, so G^* can be constructed from G in polynomial time. Since every cycle in G^* is of even length, we note that G^* is bipartite.

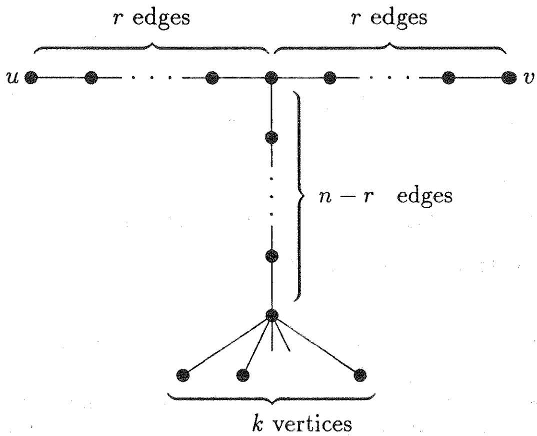

Figure 3: The tree T_{uv}.

We will show that the problem of determining the independence number of G can be transformed to the problem of determining the (n,k)-independence number of the bipartite graph G^*. We will prove that $\beta_{n,k}(G^*) = \beta(G) + (k-1)p + kq$.

Let S be a maximum independent set in G and consider the set $S^* = S \cup \bigcup_{uv \in E} (S_u \cup S_v \cup S_{uv})$. For any $x \in V^*$ and $y \in S_{uv}$, $d_{G^*}(x,y) \leq n$ implies x is a vertex of $T_{uv} - \{u,v\}$, so each vertex of S_{uv} is within distance n from exactly $k-1$ other vertices of S^*, namely the $k-1$ other vertices in S_{uv}. For any $u, v \in S$, $uv \notin E$ so $d_{G^*}(u,v) \geq 4r > n$. Thus each $u \in S$ is within distance n from exactly $k-1$ other vertices of S^*, namely the $k-1$ vertices in S_u. For any $x \in S_u, x$ is within distance n from at most $k-1$ other vertices of S^*, namely the $k-2$ other vertices of S_u and, if $u \in S$, then also u. Thus S^* is an (n,k)-independent set of G^*, whence $\beta_{n,k}(G^*) \geq |S^*| = \beta(G) + (k-1)p + kq$.

Conversely, suppose S^* is a maximum (n,k)-independent set in G^*. Let S^*_{uv} be the set of vertices of $T_{uv} - \{u,v\}$ that belong to S^*. Since every two vertices of $T_{uv} - \{u,v\}$ are within distance n from each other, it is evident that $|S^*_{uv}| \leq k$. Since $(S^* - S^*_{uv}) \cup S_{uv}$ is an (n,k)-independent set of G^* of cardinality at least $|S^*|$, we may assume, without loss of generality, that $S^*_{uv} = S_{uv}$. Further, let $S^*_u = S^* \cap (S_u \cup \{u\})$. If $|S^*_u| < k-1$, then $(S^* - S^*_u) \cup S_u$ is an (n,k)-independent set of

G^* of cardinality exceeding that of S^*, which is impossible. Hence we know that $|S^*_u| \geq k-1$. If $|S^*_u| = k-1$, then, since $(S^* - S^*_u) \cup S_u$ is an (n,k)-dominating set of G^* of cardinality $|S^*|$, we may assume, without loss of generality, that S^* contains the $k-1$ vertices of S_u. So either $|S^*_u| = k-1$, in which case $S^*_u = S_u$, or $|S^*_u| = k$, in which case $S^*_u = S_u \cup \{u\}$. Now consider the set $S = S^* \cap V$. We claim that S is an independent set of G. If $u, v \in S$ and $uv \in E$, then $d_{G^*}(u,v) = 2r = n$, so u is within distance n from at least k vertices of S^*, namely the k vertices in $S_u \cup \{v\}$, which contradicts the (n,k)-independence of S^*. Hence S is an independent set of G, so $\beta(G) \geq |S| = |S^*| - |\bigcup_{uv \in E} (S_u \cup S_v \cup S_w)| = \beta_{n,k}(G^*) - (k-1)p - kq$. This, together with the earlier observation that $\beta(G) \leq \beta_{n,k}(G^*) - (k-1)p - kq$, implies that $\beta_{n,k}(G^*) = \beta(G) + (k-1)p + kq$. This completes the proof of the theorem. □

3 Results concerning (n,k)-domination and (n,k)-independence.

It is well-known that any maximal independent set is a dominating set; therefore $\gamma_{1,1} \leq \beta_{1,1}$. Fink and Jacobson [15] proved that $\gamma_{1,2} \leq \beta_{1,2}$ and conjectured that for any graph G and for all positive integers k, $\gamma_{1,k} \leq \beta_{1,k}$. This conjecture was proven by Favaron [13]. Here we prove that, for any graph G, and for all positive integers n and k, $\gamma_{n,k} \leq \beta_{n,k}$. To do this, we shall prove the following stronger property: In every graph, and for all positive integers n and k, there exist some subsets of vertices which are both (n,k)-independent and (n,k)–dominating. This result generalizes that of Favaron [13].

Theorem 6 *For any graph G and positive integers n and k, every (n,k)-independent set $\mathcal{D}$ such that $k|\mathcal{D}| - m_n(\mathcal{D})$ is a maximum is an (n,k)-dominating set of G.*

Proof. Let $\mathcal{D}$ be an (n,k)-independent set such that $k|\mathcal{D}| - m_n(\mathcal{D})$ is maximum. We show that $\mathcal{D}$ is an (n,k)-dominating set of G. If this is not the case, then there exists a vertex v of $V(G) - \mathcal{D}$ which is not (n,k)-dominated by $\mathcal{D}$. Let B be the set of vertices of $\mathcal{D}$ within distance n from v, so $N_n(v) \cap \mathcal{D} = B$. Then $0 \leq |B| < k$. Further, let A be the set of all vertices a in B such that $\deg_n(a, \mathcal{D}) = k-1$, and let S be a maximal $(n,1)$-independent set of A. The set $C = (\mathcal{D} - S) \cup \{v\}$ is still (n,k)-independent. Indeed $\deg_n(v, C) = |B| - |S| \leq |B| < k$; $\deg_n(x, C) \leq \deg_n(x, \mathcal{D}) < k$ for any x in $\mathcal{D} - B$; $\deg_n(b, C) \leq \deg_n(b, \mathcal{D}) + 1 < (k-1) + 1 = k$ for any b in $B - A$; $\deg_n(a, C) \leq \deg_n(a, \mathcal{D}) = k-1$ for any a in $A - S$ because every vertex of $A - S$ is within distance n from at least one vertex in S (the $(n,1)$-independent set S being maximal in A). Furthermore, $|C| = |\mathcal{D}| - |S| + 1$ and $m_n(C) = m_n(\mathcal{D}) - (k-1)|S| + |B| - |S| = m_n(\mathcal{D}) - k|S| + |B|$. Thus $k|C| - m_n(C) = k|\mathcal{D}| - k|S| + k - m_n(\mathcal{D}) + k|S| - |B| = k|\mathcal{D}| - m_n(\mathcal{D}) + k - |B| > k|\mathcal{D}| - m_n(\mathcal{D})$, in contradiction with the hypothesis on $\mathcal{D}$. Therefore $\mathcal{D}$ is an (n,k)-dominating set of G.□

Corollary 1 *For any graph G, and for all positive integers n and k, $\gamma_{n,k}(G) \leq \beta_{n,k}(G)$.*

Proof. Let $\mathcal{D}$ be an (n,k)-independent set and an (n,k)-dominating set of G (such a set exists by Theorem 6). Then $\gamma_{n,k}(G) \leq |\mathcal{D}| \leq \beta_{n,k}(G)$.□

4 Bounds on $\gamma_{n,k}$

The following result yields a lower bound on the difference between $\gamma_{n,k}$ and $\gamma_{n,1}$ for $k \geq 2$. The idea of the proof is the same as that of Fink and Jacobson's theorem [16], which follows from our theorem by replacing n by 1.

Theorem 7 *If G is a graph with $\Delta(G) \geq k \geq 2$, then $\gamma_{n,k}(G) \geq \gamma_{n,1}(G) + k - 2$.*

Proof. Let $\mathcal{D}$ be a minimum (n,k)-dominating set of G. Since $\Delta(G) \geq k$, we note that $V(G) - \mathcal{D} \neq \emptyset$. Let $u \in V(G) - \mathcal{D}$, and let $v_1, v_2, ..., v_k$ be distinct members of $\mathcal{D}$ that are within distance n from u. Since $\mathcal{D}$ is an (n,k)-dominating set of G, each vertex in $V(G)-\mathcal{D}$ is within distance n from at least one member of $\mathcal{D}-\{v_2, v_3, ..., v_k\}$. It follows that the set $\mathcal{D}^* = (\mathcal{D} - \{v_2, v_3, ..., v_k\}) \cup \{u\}$ is an $(n,1)$-dominating set in G. Hence $\gamma_{n,1}(G) \leq |\mathcal{D}^*| = \gamma_{n,k}(G) - (k-1) + 1$, so that $\gamma_{n,k}(G) \geq \gamma_{n,1}(G) + k - 2$.□

The following lower bound on $\gamma_{n,k}$ generalizes the well-known bound $\gamma \geq \frac{p}{\Delta+1}$.

Theorem 8 *If G is a graph with p vertices and maximum n-degree Δ_n, then $\gamma_{n,k}(G) \geq kp/(\Delta_n + k)$.*

Proof. Let $\mathcal{D}$ be a minimum (n,k)-dominating set of G. Let $S = V(G) - \mathcal{D}$ and let N denote the number of pairs (u,v) with $u \in \mathcal{D}, v \in S$ and $d(u,v) \leq n$. Then, since the n-degree of each vertex in $\mathcal{D}$ is at most Δ_n, we have $N \leq \Delta_n \cdot |\mathcal{D}| = \Delta_n \cdot \gamma_{n,k}(G)$. Also, since each vertex in S is within distance n from at least k vertices of $\mathcal{D}$, we have $N \geq k \cdot |S| = k \cdot (p - \gamma_{n,k}(G))$. It follows that $k \cdot [p - \gamma_{n,k}(G)] \leq \Delta_n \cdot \gamma_{n,k}(G)$ whence $\gamma_{n,k}(G) \geq kp/(\Delta_n + k)$.□

That the bound given in Theorem 8 is sharp may be seen by considering the graph G obtained from a complete bipartite graph $K_{k,k}$ by subdividing each edge $n-1$ times. Then $\Delta_n = nk, p = (n+1)k$ and $\gamma_{n,k}(G) = k = kp/(\Delta_n + k)$.

We close with the following:

Conjecture 1 *If G is a graph with p vertices and minimum n-degree at least $n+k-1$, then $\gamma_{n,k}(G) \leq \frac{kp}{n+k}$.*

The conjecture is true for $k = 1$ and all integers $n \geq 1$ as proven by Oellermann, Henning and Swart [20]. The conjecture is also true for $n = 1$ and all integers $k \geq 1$ as proven by Cockayne, Gamble and Shepherd [12].

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