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### Abstract

A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals. A Latin square is self-orthogonal if it is orthogonal to its transpose. In an earlier paper Danhof, Phillips and Wallis considered the question of the existence of self-orthogonal diagonal Latin squares of order 10. In this paper we shall present some constructions of self-orthogonal diagonal Latin squares and consequently consider the existence of self-orthogonal diagonal Latin squares.

## 1 Introduction

A Latin square of order  $n$  is an  $n \times n$  array such that every row and every column is a permutation of an  $n$ -set. A transversal in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A transversal Latin square is a Latin square whose main diagonal is a transversal. A diagonal Latin square is a transversal Latin square whose back diagonal also forms a transversal.

Two Latin squares of order  $v$  are orthogonal if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A Latin square is self-orthogonal if it is orthogonal to its transpose. Orthogonal (transversal) Latin squares of order  $v$  are denoted briefly by  $OLS(v)$  ( $OTLS(v)$ ). Self-orthogonal (diagonal) Latin squares of order  $v$  are denoted briefly by  $SOLS(v)$  ( $SODLS(v)$ ).

In an earlier paper [1] Danhof, Phillips and Wallis considered the question of the existence of self-orthogonal diagonal Latin squares of order 10. In this paper we shall present some constructions of self-orthogonal diagonal Latin squares and consequently consider the existence of self-orthogonal diagonal Latin squares. We shall prove

**Theorem 1.1** *A SODLS of order  $v$  exists for all positive integers  $v$ , with the exception of  $v \in \{2, 3, 6\}$  and the possible exception of  $v \in \{10, 14\}$ .*

For our purpose, we need the following construction which was devised by Wallis. A Latin square is symmetric if it is equal to its transpose. We denote by  $SOLSSOM(v)$  a self-orthogonal Latin square of order  $v$  with a symmetric orthogonal mate.

**Lemma 1.2.** ([7]) *If there exists a  $SOLSSOM(v)$ , then there exists a  $SODLS(v)$ .*

For  $SOLSSOM$  we have the known

**Lemma 1.3.** ([6, 9, 11]) *A  $SOLSSOM$  of order  $v$  exists for all positive integers  $v$ , with the exception of  $v \in \{2, 3, 6\}$  and the possible exception of  $v \in \{10, 14\} \cup E$ , where*

$$E = \{46, 54, 58, 62, 66, 70\}.$$

Now we need only consider the case  $v \in E$ . For our purpose, let  $OLS(v, n)$  denote  $OLS(v)$  with a sub- $OLS(n)$  missing. Usually we leave the size  $n$  hole in the lower right corner. Similarly we define  $IOTLS(v, n)$ ,  $ISOLS(v, n)$ ,  $ISODLS(v, n)$  and  $ISOLSSOM(v, n)$ . We also denote by  $ISOLS^*(v, n)$  an  $ISOLS(v, n)$  in which the elements in the cells  $\{(i, v - n + 1 - i) : 1 \leq i \leq v - n\}$  are distinct and different from the missing elements. Finally, we denote by  $SOLS(v, k)$  the  $SOLS(v)$  in which the cells  $\{(v - k + i, v - i + 1) : 1 \leq i \leq k\}$  are a transversal about the elements  $a_1, a_2, \dots, a_k$ .

For subsequent use, we construct the following examples.

**Example 1.4.** There exists a  $SOLS(9, 2)$ .

**Proof.** Let  $GF(9) = \{a_1 = 0, a_2 = 1, a_3, \dots, a_9\}$ . Define the  $9 \times 9$  square array  $A = (c_{ij})$  by

$$c_{ij} = \lambda a_i + (1 - \lambda) a_j$$

where  $\lambda \in GF(9) \setminus \{a_1, a_2\}$  and  $2\lambda \neq 1$ . It is easy to see that  $A$  is a  $SOLS(9, 2)$ .

**Example 1.5.** There exists a  $SOLS(11, 6)$ .

**Proof.** Define the  $11 \times 11$  square array  $A = (c_{ij})$  by

$$c_{ij} = 2i - j \pmod{11}$$

$1 \leq c_{ij} \leq 11$ . It is easy to see that  $A$  is a  $SOLS(11, 6)$ .

## 2 Main result

In this section we shall consider the case  $v \in E$ . For these values of  $v$  we use recursive constructions. These recursive constructions rely on the existence of other orthogonal arrays and on information regarding the location of transversals in certain Latin squares. To this end we need more notation.

Let  $A = (a_{ij})$  be a Latin square. We call two transversals disjoint if they have no cell in common. A transversal  $T$  is symmetric if  $(i, j) \in T$  if and only if  $(j, i) \in T$ . A pair of transversals  $T$  and  $S$  are symmetric if  $(i, j) \in T$  if and only if  $(j, i) \in S$ . Finally,  $t$  transversals will be called  $n$ -intersecting if the only elements they have in common are the elements in their missing subsquares of side  $n$ .

For our purpose, we need *ISOLSSOM*(12, 2) from a starter-adder type construction. This idea has been described by several authors (see, for example, [5, 10]). The plan is to construct an *ISOLS*( $v, n$ )  $A$  and a symmetric orthogonal mate  $B$  from their first row (given by  $\underline{e}_c = (e_c(1, 1), e_c(1, 2), \dots, e_c(1, v - n))$  and  $\underline{f}_c = (e_c(1, v - n + 1), e_c(1, v - n + 2), \dots, e_c(1, v))$ ) and from the last  $n$  entries of the first column (given by  $\underline{g}_c = (e_c(v - n + 1, 1), e_c(v - n + 2, 1), \dots, e_c(v, 1))$ ), where  $C = A$  or  $B$ . The entries of the array are  $\{1, 2, \dots, v - n\} \cup X$ , where  $X = \{x_1, x_2, \dots, x_n\}$ . The array is constructed modulo  $v - n$ , where the  $x_i$  act as "infinity" elements according to the following rules.

$$(a) \begin{cases} e_c(s + 1, t + 1) = e_c(s, t) & \text{if } e_c(s, t) = x_i \\ e_c(s + 1, t + 1) \equiv e_c(s, t) + 1 \pmod{v - n} & \text{otherwise} \end{cases} \quad 1 \leq s, t \leq v - n;$$

$$(b) \begin{cases} e_c(s + 1, v - n + t) \equiv e_c(s, v - n + t) + 1 \\ e_c(v - n + t, s + 1) \equiv e_c(v - n + t, s) + 1 \end{cases} \pmod{v - n} \quad 1 \leq t \leq n, 1 \leq s \leq v - n.$$

Note that in case (a) all cell labels are determined modulo  $v - n$ , but in case (b) this applies only to the row and column labels respectively.

It is not difficult to determine the conditions that  $\underline{e}_c, \underline{f}_c, \underline{g}_c, C = A$  or  $B$ , must satisfy, but we shall not concern ourselves with that. Simple calculations verify that they work. For example, an *ISOLS*(14, 1) can be constructed from

$$\begin{aligned} \underline{e}_A &= (1, 9, 4, 13, 10, 3, 6, 11, 7, 12, 2, 5, x_1) \\ \underline{f}_A &= (8) \qquad \qquad \qquad \underline{g}_A = (9), \end{aligned}$$

an *ISOLSSOM*(12, 2) can be constructed from

$$\begin{aligned} \underline{e}_A &= (1, 8, 2, 9, 6, 3, x_1, 10, x_2, 4) \\ \underline{f}_A &= (5, 7) \qquad \qquad \qquad \underline{g}_A = (4, 9) \\ \underline{e}_B &= (x_1, 1, 6, 8, 7, x_2, 3, 5, 4, 10) \\ \underline{f}_B &= (2, 9) \qquad \qquad \qquad \underline{g}_B = (2, 9). \end{aligned}$$

We then have

**Example 2.1.** There exists an *ISOLSSOM*(12, 2) and a *SOLS*(14, 6).

**Proof.** It is easy to see that a *SOLS*(14, 6) comes from the above *ISOLS*(14, 1).

Now we want to construct a special *ISODLS*(12, 2)  $A$  and a special *ISOLS\**(12, 2)  $B$  from this *ISOLSSOM*(12, 2).

**Lemma 2.2.** *A is an ISOLS(12, 2) with two pairs of symmetric transversals and a symmetric transversal, all six transversals (including the main diagonal) being 2-intersecting and pairwise disjoint.*

**Proof.** It is easy to see that the elements in the cells  $\{(i, i + j) : 1 \leq i \leq 10\}$  ( $j = 1, 3, 5, 7, 9$ ) are a transversal about the elements  $1, 2, \dots, 10$ , in which the cells  $\{(i, i + 5) : 1 \leq i \leq 10\}$  are a symmetric transversal, the cells  $\{(i, i + 1) : 1 \leq i \leq 10\}$  and  $\{(i, i + 9) : 1 \leq i \leq 10\}$  and the cells  $\{(i, i + 3) : 1 \leq i \leq 10\}$  and  $\{(i, i + 7) : 1 \leq i \leq 10\}$  are two pairs of symmetric transversals.

**Lemma 2.3.** *For the ISOLS(12, 2) A, any permutation, when applied simultaneously to the rows and columns of A, produces an ISOLS(12, 2) with two pairs of symmetric transversals and a symmetric transversal, all six transversals (including the main diagonal) being 2-intersecting and pairwise disjoint.*

**Lemma 2.4.** *For the incomplete symmetric Latin square B, there exists a permutation which, when applied simultaneously to the rows and columns of B, produces an incomplete Latin square with constant back diagonal.*

**Proof.** 
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & x_1 & x_2 \\ 1 & 8 & 9 & 7 & 5 & 6 & 4 & 2 & 3 & 10 & x_1 & x_2 \end{pmatrix}.$$

**Lemma 2.5.** *For the incomplete symmetric Latin square B, there exists a permutation which, when applied simultaneously to the rows and columns of B, produces an incomplete Latin square with constant value  $x_2$  in cells  $\{(i, 11 - i) : 1 \leq i \leq 10\}$ .*

**Proof.** 
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & x_1 & x_2 \\ 1 & 2 & 3 & 4 & 5 & 10 & 9 & 8 & 7 & 6 & x_1 & x_2 \end{pmatrix}.$$

By combining Lemma 2.3 and Lemma 2.4, we have

**Example 2.6.** There exists an *ISODLS*(12, 2) with two pairs of symmetric transversals and a symmetric transversal, all six transversals (including the main diagonal) being 2-intersecting and pairwise disjoint.

By combining Lemma 2.3 and Lemma 2.5, we have

**Example 2.7.** There exists an *ISOLS\**(12, 2) with two pairs of symmetric transversals, all five transversals (including the main diagonal) being 2-intersecting and pairwise disjoint.

We also need the following:

**Lemma 2.8.** ([4, 5]) *If  $v \geq 3n + 1$  and  $v \neq 6$ , then there exists an ISOLS( $v, n$ ) except possibly for  $(v, n) = (6m + 2, 2m)$ .*

In the remainder of this section we shall prove our main result. We shall assume that the reader is familiar with the various methods of constructing an *ISOLS*( $v, n$ ) starting with an *ISOLS*( $s, t$ ) (see, for example, [3, 5]). We shall also assume that the reader is familiar with the various techniques of constructing an *ODLS*( $v$ ) from an

$OLS(v)$  by permuting rows and columns (see, for example, [2,8]). So we state the following constructions without proof.

**Lemma 2.9.** *If there exist an  $ISODLS(v, n)$  ( $v$  even) with  $t$  pairs of symmetric transversals and a symmetric transversal, all  $2t + 2$  transversals (including the main diagonal) being  $n$ -intersecting and pairwise disjoint, an  $OTLS(a)$ , an  $IOTLS(a + b_i, b_i)$ ,  $i = 0, 1, \dots, t$ , an  $ISOLS(a + b, b)$ , and a  $SOLS(na + k, k)$ , where  $k = b + b_0 + 2(b_1 + \dots + b_t)$  and  $k$  is even, then there is a  $SODLS(va + k)$ .*

**Lemma 2.10.** *If there exist an  $ISOLS^*(v, n)$  ( $v - n$  even) with  $t$  pairs of symmetric transversals, all  $2t + 1$  transversals (including the main diagonal) being  $n$ -intersecting and pairwise disjoint, an  $OTLS(a)$ , an  $IOTLS(a + b_i, b_i)$ ,  $i = 1, 2, \dots, t$ , an  $ISOLS(a + b, b)$ , and a  $SODLS(na + k)$ , where  $k = b + 2(b_1 + \dots + b_t)$  and  $na + k$  is even, then there is a  $SODLS(va + k)$ .*

From Lemma 2.9 we have the following Lemmas:

**Lemma 2.11.** *If there exists a  $SOLS(a)$ , an  $ISOLS(a + 2, 2)$  and a  $SOLS(a + k, k)$ , then there is a  $SODLS(8a + k)$ , where  $k$  is even and  $k \leq 6$ .*

**Proof.** Apply Lemma 2.9 to an  $ISODLS(8, 1)$  with a pair of symmetric transversals, all three transversals (including the main diagonal) being 1-intersecting and pairwise disjoint.

**Lemma 2.12.** *If there exist a  $SOLS(A)$ , and  $ISOLS(a + 1, 1)$ , a  $SOLS(2a + k, k)$ , then there is a  $SODLS(12a + k)$ , where  $k$  is even and  $k \leq 6$ .*

**Proof.** Apply Lemma 2.9 to the  $ISODLS(12, 2)$  in Example 2.6.

We then have

**Lemma 2.13** *There exists a  $SODLS(v)$  for  $v \in \{46, 54, 58\}$ .*

**Proof.**  $v \in \{46, 58\}$  comes from Lemma 2.11 with  $a, k \in \{(5, 6), (7, 2)\}$ .  $v = 54$  comes from Lemma 2.12 with  $(a, k) = (4, 6)$ . The conditions  $SOLS(a + k, k)$  and  $SOLS(2a + k, k)$  come from Examples 1.4, 1.5 and 2.1, others from Lemma 2.8.

From Lemma 2.10. we have

**Lemma 2.14.** *There exists a  $SODLS(v)$  for  $v \in \{62, 66, 70\}$ .*

**Proof.** Apply Lemma 2.10 with  $(v, n) = (12, 2)$ ,  $t = b = 2$ ,  $b_i = 0$  or  $2$ ,  $a = 5$ , and  $k \in \{2, 6, 10\}$ . The condition  $ISOLS^*(12, 2)$  comes from Example 2.7,  $SODLS(nh + k)$  from Lemmas 1.2, 1.3 and 2.8.

By combining Lemma 2.13 and Lemma 2.14 we then have

**Theorem 2.15.** *There exists a  $SODLS(v)$  for  $v \in E$ .*

By combining Lemmas 1.2 and 1.3 and Theorem 2.15, we obtain Theorem 1.1.

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