CLASSIFICATION OF TRIVALENT SYMMETRIC GRAPHS OF SMALL ORDER

Marston Conder and *Margaret Morton*

Department of Mathematics
University of Auckland
Private Bag 92019 Auckland
NEW ZEALAND

Abstract
A classification is given of all finite connected trivalent symmetric graphs on up to 240 vertices, based on an analysis of short relators in their automorphism groups.

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1. Introduction

Suppose G is a group of automorphisms of an undirected graph Γ. If G acts transitively on the vertices of Γ, and the stabilizer in G of a vertex v acts transitively on the vertices adjacent to v, then G is said to act *symmetrically* on Γ. In this case G acts transitively on the arcs (ordered edges) of Γ, and Γ is called a *symmetric* graph. Generalising for $s \geq 1$, an s-arc in Γ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices of Γ such that $\{v_{i-1}, v_i\}$ is an edge of Γ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$, and the graph Γ is said to be *s-arc-transitive* if its automorphism group acts transitively on the set of all s-arcs of Γ. In particular, a 1-arc-transitive graph is precisely a symmetric graph.

By a well known theorem of Tutte (see [**1**; §18]), a finite connected symmetric graph of degree 3 can be at most 5-arc-transitive, indeed its automorphism group G acts regularly on s-arcs for some $s \leq 5$. Tutte's theorem was extended by Djokovic and Miller [**10**], who classified finite connected trivalent symmetric graphs into seven types (according to the level of s-arc-transitivity and the existence of an involutory automorphism flipping an edge), and these types were later described in a unified way by Conder and Lorimer [**7**] in terms of generators and relations for their automorphism groups .

Conversely, given any group G containing a subgroup H and an element a such that $a^2 \in H$, we may construct a graph $\Gamma = \Gamma(G, H, a)$ on which G acts symmetrically, as follows: take as vertices of Γ the right cosets of H in G, and join two cosets Hx and Hy by an edge in Γ whenever $xy^{-1} \in HaH$. Defined in this way, Γ is an undirected graph on which the group G acts as a group of automorphisms under the action $g : Hx \to Hxg$ for each $g \in G$ and each coset Hx in G. The stabilizer in G of the vertex H is the subgroup H itself, and as this acts transitively on the set of neighbours of H (which are all of the form Hah for $h \in H$), it follows that Γ is symmetric.

The above construction (part of the folk-lore of algebraic graph theory) is given in more detail in [**11**]. The graph $\Gamma = \Gamma(G, H, a)$ is connected if and only if G is generated by HaH (or equivalently, by $H \cup \{a\}$), and is regular of degree d where $d = |H : H \cap a^{-1}Ha|$ is the number of right cosets of H contained in the double coset HaH. Similarly, other properties of Γ (such as its girth and diameter) depend on the choice of G, H and a, and in particular on relations satisfied in G by a and elements of H.

The purpose of this paper is to determine all connected trivalent symmetric graphs on up to 240 vertices, by analysis of all possible short relators in their automorphism groups. Our motivation comes from a question (in a personal communication) by Chris Godsil, Brendan McKay and Gordon Royle, related to Lovasz's conjecture on Hamilton paths and cycles in vertex-transitive graphs. As it turns out, all the graphs in our list have Hamilton paths, and all except the Petersen graph (on 10 vertices) and the Coxeter graph (on 28 vertices) have Hamilton cycles, thereby refuting a suggestion by a fourth person who shall remain nameless.

The complete list is described in Section 3, following some more background material and a description of our method in Section 2. Hamiltonicity of the graphs was checked using a computer program kindly supplied to us by Nick Wormald, and is not documented. Further details are available from the authors upon request.

Incidentally, our list may be compared with the "Foster census", a list of connected trivalent symmetric graphs on up to 512 vertices, compiled by hand by R. M. Foster over many years and recently published with a few additions (but no claim of completeness) in [**2**]. Although unaware of this until after we completed our calculations for those graphs on up to 240 vertices, our list matches the published list exactly. Hence, remarkably, Foster's hand-prepared list had only one omission.

2. Background and method

As outlined in [**7**], if the group G acts symmetrically on the finite connected trivalent graph Γ, then G has to be a homomorphic image of one of seven finitely-presented groups, G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 or G_5, where:

(i) G_1 is the modular group, generated by two elements $h,\ a$ subject to the relations $h^3 = a^2 = 1$,

(ii) G_2^1 is the extended modular group, generated by three elements $h,\ a,\ p$ subject to the relations $h^3 = a^2 = p^2 = 1$, $apa = p$, and $php = h^{-1}$,

(iii) G_2^2 is generated by three elements $h,\ a,\ p$ subject to the relations $h^3 = p^2 = 1$, $a^2 = p$, and $php = h^{-1}$,

(iv) G_3 is generated by four elements $h,\ a,\ p,\ q$ subject to the relations $h^3 = a^2 = p^2 = q^2 = 1$, $qp = pq$, $h^{-1}ph = p$, $qhq = h^{-1}$, and $apa = q$,

(v) G_4^1 is generated by five elements $h,\ a,\ p,\ q,\ r$ subject to the relations $h^3 = a^2 = p^2 = q^2 = r^2 = 1$, $pq = qp$, $pr = rp$, $rq = pqr$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $apa = p$, and $aqa = r$,

(vi) G_4^2 is generated by five elements $h,\ a,\ p,\ q,\ r$ subject to the relations $h^3 = p^2 = q^2 = r^2 = 1$, $a^2 = p$, $pq = qp$, $pr = rp$, $rq = pqr$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, and $a^{-1}qa = r$,

(vii) G_5 is generated by six elements $h,\ a,\ p,\ q,\ r,\ s$ subject to the relations $h^3 = a^2 = p^2 = q^2 = r^2 = s^2 = 1$, $pq = qp$, $pr = rp$, $ps = sp$, $qr = rq$, $qs = sq$, $sr = pqrs$, $h^{-1}ph = p$, $h^{-1}qh = r$, $h^{-1}rh = pqr$, $shs = h^{-1}$, $apa = q$, and $ara = s$.

In fact the relations show G_2^2 and G_4^2 can be generated by just h and a, while G_3, G_4^1 and G_5 can be generated by h, a and p, since for example in G_4^2 we have $p = a^2$, $q = h^{-1}ph$, and $r = a^{-1}qa$. The given presentations, however, are very tidy, and have proved useful in the construction of many families of examples (see [5]–[8]), using the general method described in the Introduction.

Now suppose G is a homomorphic image of any one of these seven groups. Since taking a homomorphic image (or factor group) is equivalent to inserting further relations into the presentation, we may for notational convenience represent the images of the given generators using the same symbols as for the original group in each case.

In particular, if H is the subgroup generated by (the images of) all the given generators other than a, then G acts symmetrically on the connected graph $\Gamma = \Gamma(G, H, a)$, with the subgroup H stabilizing the vertex H. Except in degenerate situations (where H has index 1 or 2 in G), the given relations imply that the vertex H has three neighbours, namely Ha, Hah and Hah^2. Moreover, because of the given relations, every other vertex of Γ is also of the form Hw where $w = w(a, h)$ is a word in h, h^2 and a (with first letter a), and the vertex Hw is joined to each of Haw, $Hahw$ and Hah^2w in Γ. As a consequence, any circuit of length m in the graph Γ constructed from this group G is of the form $Hw - Hah^{\epsilon_1}w - \ldots - Hah^{\epsilon_m}ah^{\epsilon_{m-1}}\ldots ah^{\epsilon_2}ah^{\epsilon_1}w = Hw$, for some $\epsilon_i \in \{0,1,2\}$ (for $1 \le i \le m$), where necessarily $ah^{\epsilon_m}ah^{\epsilon_{m-1}}\ldots ah^{\epsilon_2}ah^{\epsilon_1} \in H$ and $\epsilon_i \neq 0$ for $2 \le i \le m$.

We may exploit these facts in order to classify all connected trivalent symmetric graphs on a small number of vertices. Since a cubic graph of girth $2k+1$ has at least $1+3+3\times 2+\ldots+3\times 2^{k-1} = 3\times 2^k - 2$ vertices, while one of girth $2k+2$ has at least $1+3+3\times 2+\ldots+3\times 2^{k-1}+2^k = 4\times 2^k - 2$ vertices, every cubic graph on 240 or fewer vertices has girth at most 13. In particular, any such graph contains a circuit of length at most 13, and its automorphism group can be obtained as a homomorphic image of one of the seven finitely-presented groups G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 or G_5, by inserting at least one additional relator of the form $xah^{\epsilon_m}ah^{\epsilon_{m-1}}\ldots ah^{\epsilon_1} = 1$, where $x \in H$ and $\epsilon_i \in \{0,1,2\}$ for $1 \le i \le m$, with $m \le 13$. Conversely, if G is any homomorphic image so obtained, then the corresponding graph $\Gamma(G,H,a)$ is connected, trivalent and symmetric, and contains a circuit of the appropriate length.

The same principle was used by Miller in [**11**] to classify connected trivalent symmetric graphs of girth at most six, and by Morton in [**12**] to determine all 4- or 5-arc-transitive trivalent connected graphs of girth at most 13 (other than those obtainable by inserting the relator $(ha)^{12} = 1$ into the presentation for G_4^1). We extend the results of both these papers here.

As in [**12**], the possibilities for additional relators may be systematically enumerated, and coset enumeration used to help determine the order of the resulting finitely-presented group in each case. Note that for any additional relator of the form $xah^{\epsilon_m}ah^{\epsilon_{m-1}}\ldots ah^{\epsilon_1}$ (as above), there are only a small number of possibilities for the element $x \in H$, since the vertex-stabilizer H has order $3\times 2^{s-1}$ when the group G acts regularly on s-arcs. Thus for example, in the extreme case of G_5, we know $|H| = 48$, and x has to be of the form $p^\alpha q^\beta r^\gamma s^\delta h^\epsilon$ with $\alpha,\beta,\gamma,\delta \in \{0,1\}$ and $\epsilon \in \{0,1,2\}$.

For the modular group G_1, part of the task was completed—in the context of a different problem—by Conder in [**4**]. In that paper, as here, the process is mostly straightforward, but difficulties arise in those cases where the insertion of a single relator produces either an infinite group, or a finite group of large or incalculable order. (The example of G_4^1 with relation $(ha)^{12} = 1$ inserted is a case in point.)

The latter problem, however, is easily settled using the same general principle as described above: if the insertion of an additional relator produces a group or graph larger than required, or even infinite, then at least one further additional relator of small length (corresponding to some other small circuit) has to be inserted. In each such case the possibilities for second and subsequent additional relators need to be enumerated in turn.

Although this may mean considering relators of the form $xah^{\epsilon_m}ah^{\epsilon_{m-1}}\ldots ah^{\epsilon_1}$ for $m > 13$, there are still only finitely many possibilities at each level, and indeed the number of them that require checking declines as the search proceeds, many being equivalent to possibilities considered previously. As long as all possibilities are checked at each level, all possible graphs (on up to the required number of vertices) are guaranteed to be found.

This is analogous to the principle used in the "low index subgroups" procedure for finding subgroups of small index in a finitely-presented group (see [**9**]), and may be programmed on a computer. The main difference is that the procedure in [**9**] involves the insertion of additional subgroup generators, whereas our problem requires the insertion of additional relators. Of course our approach essentially requires an enumeration of all *normal* subgroups of small index in each of the given finitely-presented groups; more about this connection will be described in a subsequent paper.

3. Results

The results of our computations are summarised below, with the graphs classified according to the seven types described earlier. In each case we give the order of the graph, followed by the additional relator(s) which produce the automorphism group of the graph when inserted into the presentation for the corresponding group (G_1, G_2^1, etc.). For brevity we use the symbols u and v to denote the products ha and h^2a respectively. Thus, for example, the graphs obtained from the group G_1 with relator u^6 adjoined (along with others) correspond to some of the family of graphs of girth 6 described by Miller in [**11**].

The graphs appear in the order in which they were found, and all duplications have been eliminated by computation of automorphism groups and testing for graph isomorphism, with the help of the CAYLEY system [**3**].

Note that no graphs of types (iii) or (vi) appear: all graphs found from quotients of G_2^2 turned out to be 3-arc-transitive, and similarly all those found from G_4^2 turned out to be 5-arc-transitive. Nevertheless, graphs of these two types do exist, as was shown in [**7**].

Also note that although the upper bound of 13 on the girth of the graphs is used to determine possible additional relators, none of the graphs listed below actually *has* girth 13, for in each case the first additional relator has length at most 12 in terms of u and v.

Case (i): G_1 : 1-arc-regular graphs

26	$u^6, \quad u^2vuvuv^2$
42	$u^6, \quad u^2vuvuvuv^2$
38	$u^6, \quad u^2v^2u^2vuv^2$
62	$u^6, \quad u^2vuvuvuvuv^2$
56	$u^6, \quad u^2v^2u^2vuvuv^2$
86	$u^6, \quad u^2vuvuvuvuvuv^2$
78	$u^6, \quad u^2vuv^2u^2vuvuv^2$
74	$u^6, \quad u^2v^2uvu^2vuvuv^2$
114	$u^6, \quad u^2vuvuvuvuvuvuv^2$
104	$u^6, \quad u^2vuv^2u^2vuvuvuv^2$
98	$u^6, \quad u^2v^2uvu^2vuvuvuv^2$
146	$u^6, \quad u^2vuvuvuvuvuvuvuv^2$
134	$u^6, \quad u^2vuvuv^2u^2vuvuvuv^2$
126	$u^6, \quad u^2vuv^2uvu^2vuvuvuv^2$
122	$u^6, \quad u^2v^2uvuvu^2vuvuvuv^2$
182	$u^6, \quad u^2vuvuvuvuvuvuvuvuv^2$
168	$u^6, \quad u^2vuvuv^2u^2vuvuvuvuv^2$
158	$u^6, \quad u^2vuv^2uvu^2vuvuvuvuv^2$
152	$u^6, \quad u^2v^2uvuvu^2vuvuvuvuv^2$
222	$u^6, \quad u^2vuvuvuvuvuvuvuvuvuv^2$
206	$u^6, \quad u^2vuvuvuv^2u^2vuvuvuvuv^2$
194	$u^6, \quad u^2vuvuv^2uvu^2vuvuvuvuv^2$
186	$u^6, \quad u^2vuv^2uvuvu^2vuvuvuvuv^2$
182	$u^6, \quad u^2v^2uvuvuvu^2vuvuvuvuv^2$
234	$u^6, \quad u^2vuvuv^2uvu^2vuvuvuvuvuv^2$
224	$u^6, \quad u^2vuv^2uvuvu^2vuvuvuvuvuv^2$
218	$u^6, \quad u^2v^2uvuvuvu^2vuvuvuvuvuv^2$
144	$u^8, \quad u^2vu^2vuv^2uv^2$
112	$uv^4uv^4, \quad u^2v^2u^2vuvuv^2$
208	$uv^4uv^4, \quad u^2vuv^2u^2vuvuvuv^2$
168	$u^{12}, \quad uv^3uv^3uv^3, \quad u^3v^2uvu^2v^3$
162	$u^4v^2u^2v^4$

Case (ii): G_2^1 : 2-arc-regular graphs (with an edge-flip of order 2)

4	u^3
8	u^4
20	u^5
16	u^3v^3
24	u^6, uvu^2vuv^2
32	u^6, $uvuvuvuv$
50	u^6, $uvuvuvuvuv$
54	u^6, $uvuvu^2vuvuv^2$
72	u^6, $uvuvuvuvuvuv$
98	u^6, $uvuvuvuvuvuvuv$
96	u^6, $uvuvuvu^2vuvuvuv^2$
128	u^6, $uvuvuvuvuvuvuvuv$
162	u^6, $uvuvuvuvuvuvuvuvuv$
150	u^6, $uvuvuvuvu^2vuvuvuvuv^2$
200	u^6, $uvuvuvuvuvuvuvuvuvuv$
216	u^6, $uvuvuvuvuvu^2vuvuvuvuvuv^2$
56	u^7, $uvuvuvuv$
84	u^7, $pvuvuvuvuv$
182	u^7, $pvuvuvuvuvuvuv$, $pv^2u^2v^2u^2v^2u^2v^2$
168	u^7, $uvu^2vu^2vuv^2uv^2$
48	u^4v^4
64	u^8, uvu^3vuv^3
120	u^8, $pv^2u^2v^2u^2v^2$
240	u^8, $uvuvuvuvuv$, $u^2v^3u^3vuv^2uv^3$
112	u^8, $uv^2u^3v^2uv^3$
168	u^8, $u^2v^2u^2v^2u^2v^2$, $pv^3uv^2uv^2uv^3$
108	u^9, $u^2vu^2v^2uv^2$
168	u^9, avu^8v, $uvuv^3uvuv^3$
60	$uv^2uv^2uv^2$, u^5v^5
240	$uv^2uv^2uv^2$, u^{15}
240	u^5v^5
120	u^5v^5, $uvuvuvuvuv$
220	$uvuvuvuvuv$, u^{11}, $u^3v^3u^3v^3$, $uvu^2vuv^3uv^3$
220	$uvuvuvuvuv$, u^{12}, $pv^4u^4v^4$
144	uv^4uv^4, $u^2vuvuvuvu^2v^3$
192	uvu^3vuv^3, u^8v^8

216	$u^2vu^2v^2uv^2$, $uvuvuvuvuvuv$
128	$u^2vu^2v^2uv^2$, u^6v^6
216	u^6v^6, $uv^3uv^3uv^3$, $uvuvuvuvuvuv$
168	$pv^2uvu^2v^6$
192	$u^2vu^3v^2uv^3$, $uvuvuvuvuvuv$
224	$u^2vu^2v^3uv^3$

Case (iii): G_2^2 : 2-arc-regular graphs (with no edge-flip of order 2)

None of order ≤ 240

Case (iv): G_3 : 3-arc-regular graphs

6	u^2v^2
10	u^5
18	u^6
20	$pqv^2u^2v^2$
28	$puv^2u^2v^2$
112	u^8
56	u^8, $pqv^3uv^2uv^3$
40	uv^3uv^3
192	$u^2v^2u^2v^2$
96	$u^2v^2u^2v^2$, u^{12}
80	u^5v^5
110	$qv^2uv^3uv^3$
220	$pqv^2u^2v^2u^2v^2$, u^{12}, $pqv^3uvuv^2uv^5$
162	u^6v^6
224	$u^2vuvu^2v^5$
182	qv^4uvuv^5

Case (v): G_4^1 : 4-arc-regular graphs (with an edge-flip of order 2)

14	u^6
102	u^9
204	pqu^3vuvuv^5

Case (vi): G_4^2 : 2-arc-regular graphs (with no edge-flip of order 2)

None of order ≤ 240

Case (vii): G_5 : 5-arc-regular graphs

30	pqu^4v^4
90	u^{10}
234	u^{12}, $qrsv^2u^2vu^4v^4$.

4. Summary and final comments

Our results may be summarised by the following table, whose columns indicate the maximum level of s-arc-transitivity:

$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
	4			
		6		
	8			
		10		
			14	
	16			
		18		
	20	20		
	24			
26				
		28		
				30
	32			
38				
		40		
42				
	48			
	50			
	54			
56	56	56		
	60			
62				
	64			
	72			
74				
78				
		80		
	84			
86				
				90
	96	96		

98	98			
			102	
104				
	108			
		110		
112	112	112		
114				
	120 (two)			
122				
126				
	128 (two)			
134				
144	144			
146				
	150			
152				
158				
162	162	162		
168 (two)	168 (four)			
182 (two)	182	182		
186				
	192 (two)	192		
194				
	200			
			204	
206				
208				
	216 (three)			
218				
	220 (two)	220		
222				
224	224	224		
234				234
	240 (three)			
$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$

The graphs on $4, 6, 8, 10, 14$ and 18 vertices are respectively K_4 (the tetrahedron), $K_{3,3}$, the cube, Petersen's graph, Heawood's graph and the Pappus graph. The two graphs on 20 vertices are the dodecahedron (2-arc-transitive) and the graph of the Desargues congfiguration (3-arc-transitive). Other graphs of interest include Coxeter's graph (3-arc-transitive on 28 vertices), Tutte's 8-cage (5-arc-transitive on 30 vertices), a double cover of the dodecahedron (3-arc-transitive on 40 vertices), a triple cover of Tutte's 8-cage (5-arc-transitive on 90 vertices), the Sextet graph $S(17)$ and a double cover of this graph (4-arc-transitive on 102 and 204 vertices respectively), and Wong's graph (5-arc-transitive on 234 vertices).

As stated in the Introduction, all but two of the graphs listed have Hamilton cycles, and the exceptions (Petersen's graph and Coxeter's graph) are well known to have Hamilton paths but no Hamilton cycles.

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