TREE-RAMSEY NUMBERS

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Abstract

We denote by $r(G_1, G_2)$ the ramsey number of two graphs G_1 and G_2. If T_{p+1} is a tree of order $p+1$ which is not a star, and if p is not a divisor of the positive integer $q-1$, then we shall show that $r(T_{p+1}, K_{1,q}) \leq p+q-1$, and we shall describe some trees and stars for which equality holds. Furthermore, we determine the ramsey numbers $r(T^*_{p+1}, T^*_{q+1})$ for $p, q \geq 4$, where T^*_n denotes a tree of order n with $\Delta(T^*_n) = n-2$.

1. Introduction

In this paper we consider finite, undirected, and simple graphs with the vertex set $V(G)$ and the edge set $E(G)$. We write $n(G) = |V(G)|$ for the order, $\bar{G}$ for the complement, and $d(x, G)$ for the degree of the vertex x of G. By $\delta(G)$ and $\Delta(G)$ we denote the minimum and maximum degree of G, respectively. For $A \subseteq V(G)$ let $G[A]$ be the subgraph induced by A. The set $N(x, G)$ consists of all vertices adjacent to the vertex x, and $N[x, G] = N(x, G) \cup \{x\}$. By $G \cup H$ we define the disjoint union of the graphs G and H. If p is a positive integer, then we use pG for the union of p copies of the graph G. We denote by K_n the complete graph of order n and by $K_{1,n}$ the star of order $n+1$. For a factorization of the complete graph K_n in two graphs F_1 and F_2, we write $K_n = F_1 \oplus F_2$. The ramsey number $r(G_1, G_2)$ of two graphs G_1 and G_2 is the least positive integer q such that for any factorization $K_q = F_1 \oplus F_2$, the graph G_i is a subgraph of F_i for at least one $i = 1, 2$.

If T_{p+1} is a tree of order $p+1$ which is not a star, and if p is not a divisor of the positve integer $q-1$, then we shall show in this paper that $r(T_{p+1}, K_{1,q}) \leq p+q-1$, and we shall give different examples where equality holds. Furthermore, we determine the ramsey numbers $r(T^*_{p+1}, T^*_{q+1})$ for $p, q \geq 4$, where T^*_n denotes a tree of order n with $\Delta(T^*_n) = n - 2$.

2. Preliminary Results

The next two theorems are very important for our research.

Theorem 2.1 (Kirkman [4] 1847, Reiß [6] 1859) The complete graph K_{2n} is 1-factorable.

Theorem 2.2 (Petersen [5] 1891) A graph G is 2-factorable if and only if G is $2p$-regular.

Lemma 2.1 Let T be any tree of order n, and let G be a graph with $\delta(G) \geq n - 1$. Then there exists is a subgraph T' of G which is isomorphic to T.

A proof of this well-known result can be found for example in the book of Chartrand and Lesniak [3, p. 72]. In the sequel, we also need an extension of Lemma 2.1. This extension is a consequence of the next lemma.

Lemma 2.2 Let G be a connected, non-complete graph of order $n(G) \geq p + 2$ with $\delta(G) \geq p \geq 3$. Furthermore, let T be a tree with $4 \leq n(T) \leq p + 1$ and $\Delta(T) \leq n(T) - 2$. If a is an arbitrary vertex of T, then there exists a tree $T_a \subseteq G$ which is isomorphic to T such that

$$N[a', G] \cap V(T_a) \neq V(T_a),$$

where $a' \in V(T_a)$ is the vertex isomorphic to a (if $f : V(T) \longrightarrow V(T_a)$ is an isomorphism with $f(a) = a'$, then we say that a' is isomorphic to a).

Proof. We proceed by induction on $n = n(T)$.
If $n = 4$, then T is a path of length 3. Since G is non-complete, there exist two vertices x and y in G of distance two. Using this observation, it is easy to see that Lemma 2.2 is valid for $n = 4$.
Now assume that $5 \leq n \leq p + 1$ and let a be an arbitrary vertex of T. Since T is not a star, we find an end vertex $v \neq a$ of T such that the tree $H = T - v$ is neither a star. Let u be adjacent to v in T. By the the induction hypothesis, there exists a tree $H_a \subseteq G$ which is isomorphic to H such that $N[a', G] \cap V(H_a) \neq V(H_a)$, where a' is the vertex isomorphic to a. Let $u' \in V(H_a)$ be isomorphic to u. Since $\delta(G) \geq p$,

we can find a vertex v' in G which is adjacent to u' in G such that $v' \notin V(H_a)$. Now the tree H_a together with the vertex v' and the edge $u'v'$ is isomorphic to T, and it has the desired properties. □

Lemma 2.3 Let G be a connected graph of order $n(G) \geq p+2$ with $\delta(G) \geq p \geq 2$. If T is a tree of order $n(T) \leq p+2$ and $\Delta(T) \leq p$, then there exists a subgraph T' of G which is isomorphic to T.

Proof. If G is complete or $n(T) \leq p+1$, then the statement follows from Lemma 2.1. In the remaining case that G is not complete and $n(T) = p+2$, we prove the lemma by induction on p.
First, assume that $p = 2$. Then simple observations show that G contains a path of length 3.
Second, assume that $p \geq 3$. Then let v be an end vertex of T such that $H = T - v$ is not a star, and let a be adjacent to v in T. According to Lemma 2.2, there exists a tree $H_a \subseteq G$ which is isomorphic to H such that $N[a', G] \cap V(H_a) \neq V(H_a)$, where a' is the vertex isomorphic to a. Since $\delta(G) \geq p$, we can find a neighbour v' of a' with $v' \notin V(H_a)$. If we now join H_a and v' by the edge $a'v'$, then we obtain a tree $T' \subseteq G$, isomorphic to T. □

3. Main Results

Let H and G be two graphs. If there exists a subgraph H' of G which is isomorphic to H, then we say short that H is a subgraph of G, and we write $H \subseteq G$. In the following R_m^n means an m-regular graph of order n.

Our first result is an extension of the next theorem of Burr [1] from 1974.

Theorem (Burr [1] 1974) Let $p, q \geq 2$ be two integers. If T_{p+1} is a tree of order $p+1$, then $r(T_{p+1}, K_{1,q}) \leq p+q$. If there exists a positive integer t such that $q - 1 = tp$, then $r(T_{p+1}, K_{1,q}) = p+q$.

Theorem 3.1 Let $p, q \geq 2$ be two integers and T_{p+1} be a tree of order $p+1$ which is not a star. If p is not a divisor of $q-1$, then

$$r(T_{p+1}, K_{1,q}) \leq p+q-1\,.$$

If furthermore, p and q fulfil one of the following conditions, then equality holds.

i) If $q = 2$, then $r(T_{p+1}, K_{1,q}) = p+q-1 = p+1$.

ii) If $p = q \geq 3$, then $r(T_{p+1}, K_{1,q}) = p + q - 1 = 2p - 1$.

iii) If $q - 1 = kp + 1$ for an integer $k \geq 1$, then $r(T_{p+1}, K_{1,q}) = p + q - 1$.

iv) If $q - 1 = kp + s$ for an integer $k \geq 1$ with $2 \leq s \leq p - 1$, then $r(T_{p+1}, K_{1,q}) = p + q - 1$, if $k + s + 1 - p \geq 0$ or $\Delta(T_{p+1}) = p - 1$. (In particular, we have $r(T_{p+1}, K_{1,q}) = p + q - 1$, if $q - 1 = kp + p - 1$ or if $q - 1 = kp + p - 2$ $(p \geq 3)$.)

v) If $p > q \geq 3$ and $\Delta(T_{p+1}) = p - 1$, then $r(T_{p+1}, K_{1,q}) = p + q - 1$, if $p + q$ is even or if q is odd and p is even, and $r(T_{p+1}, K_{1,q}) = p + q - 2$, if p is odd and q is even.

Proof. Let G be any graph of order $p + q - 1$. If $K_{1,q}$ is not a subgraph of $\bar{G}$, then $\Delta(\bar{G}) \leq q - 1$ and hence $\delta(G) \geq p - 1$. From the hypothesis that p is not a divisor of $q - 1$, we conclude that there exists a component H of G with $n(H) \geq p + 1$. Since $\Delta(T_{p+1}) \leq p - 1$, it follows from Lemma 2.3 that $T_{p+1} \subseteq H \subseteq G$ and therefore $r(T_{p+1}, K_{1,q}) \leq p + q - 1$.

i) If $q = 2$, then the complete graph $G = K_p$ shows immediately the inequality $r(T_{p+1}, K_{1,q}) \geq p + 1$.

ii) If $p = q \geq 3$, then we obtain $r(T_{p+1}, K_{1,q}) \geq 2p - 1$ from $G = 2K_{p-1}$.

iii) If $q - 1 = kp + 1$, then the graph $G = (k + 1)K_p$ of order $p + q - 2$ yields $r(T_{p+1}, K_{1,q}) \geq p + q - 1$.

iv) If $q - 1 = kp + s$ with $2 \leq s \leq p - 1$ and $k + s + 1 - p \geq 0$, then there exists the graph

$$G = (p + 1 - s)K_{p-1} \cup (k + s + 1 - p)K_p$$

of order $n(G) = p + q - 2$. Since T_{p+1} is not a subgraph of G and $\Delta(\bar{G}) \leq q - 1$, we see that $r(T_{p+1}, K_{1,q}) \geq p + q - 1$. (In particular, for $s = p - 1$ or $s = p - 2$, the condition $k + s + 1 - p \geq 0$ is valid, and thus $r(T_{p+1}, K_{1,q}) = p + q - 1$ for $q - 1 = kp + p - 1$ or $q - 1 = kp + p - 2$.)

Thus, we assume in the following that $q - 1 = kp + s$ with $2 \leq s \leq p - 3$ and $\Delta(T_{p+1}) = p - 1$.

If $p + q$ is even or q is odd and p is even, then according to Theorem 2.1 and Theorem 2.2, there exists the factorization

$$K_{p+q-2} = R_{p-2}^{p+q-2} \oplus R_{q-1}^{p+q-2},$$

which implies, together with the condition $\Delta(T_{p+1}) = p - 1$, the inequality $r(T_{p+1}, K_{1,q}) \geq p + q - 1$.

If q is even and p is odd, then we shall investigate the two cases depending on whether k is even or odd.

If k is odd, then it follows from $q = kp+s+1$ that s is even. Hence, by Theorem 2.1, there exists the graph

$$F = kK_p \cup R_{p-2}^{p+s-1}$$

of order $n(F) = p+q-2$. Then the factorization $K_{p+q-2} = F \oplus \bar{F}$ shows $r(T_{p+1}, K_{1,q}) \geq p+q-1$.
In the case that k is even, we conclude that $s = 2t+1$ is odd. If $p+t$ is even, then there exists

$$F_1 = (k-1)K_p \cup 2R_{p-2}^{p+t}$$

and if $p+t$ is odd, then there exists

$$F_2 = (k-1)K_p \cup R_{p-2}^{p+t-1} \cup R_{p-2}^{p+t+1}.$$

We observe that $n(F_1) = n(F_2) = p+q-2$, and the factorizations $K_{p+q-2} = F_i \oplus \bar{F}_i$ for $i = 1,2$, yield the desired result.

v) Now let $p > q \geq 3$ and $\Delta(T_{p+1}) = p-1$.
If $p+q$ is even or q is odd and p is even, then the inequality $r(T_{p+1}, K_{1,q}) \geq p+q-1$ follows from the above factorization $K_{p+q-2} = R_{p-2}^{p+q-2} \oplus R_{q-1}^{p+q-2}$.
In the case p odd and q even, let G be an arbitrary graph of order $p+q-2$. If $K_{1,q}$ is not a subgraph of $\bar{G}$, then we have $\Delta(\bar{G}) \leq q-1$ and hence $\delta(G) \geq p-2$, and thus G is connected. Since the integers $p+q-2$ and $p-2$ are both odd, we can find a vertex v in G with $|N(v,G)| \geq p-1$. Consequently, $T_{p+1} \subseteq G$, and we have proved $r(T_{p+1}, K_{1,q}) \leq p+q-2$.
Finally, the factorization

$$K_{p+q-3} = R_{p-3}^{p+q-3} \oplus R_{q-1}^{p+q-3}$$

shows the opposite inequality $r(T_{p+1}, K_{1,q}) \geq p+q-2$. □

For the special case that the trees are stars, Burr and Roberts [2] determined the ramsey numbers exactly.

Theorem (Burr, Roberts [2] 1973) Let $p, q \geq 2$ be two integers. Then

$$r(K_{1,p}, K_{1,q}) = \begin{cases} p+q-1, & \text{if } p \text{ and } q \text{ are both even,} \\ p+q, & \text{otherwise.} \end{cases}$$

It is our aim now to determine the ramsey numbers of two trees T_1 and T_2 which fulfil the property $\Delta(T_i) = n(T_i) - 2$ for $i = 1,2$.

Theorem 3.2 Let $p, q \geq 4$ be two integers. Then

$$r(T_{p+1}^*, T_{q+1}^*) = \begin{cases} p+q-1, & \text{if } q-2 = tp \text{ or } p-2 = tq, \\ p+q-3, & \text{if } p \text{ is odd and } q = p, \\ p+q-2, & \text{otherwise.} \end{cases}$$

Proof. Let G be a graph of order $p+q-1$ and assume that T^*_{q+1} is not a subgraph of $\bar{G}$.
If $\Delta(\bar{G}) \leq q-2$, then $\delta(G) \geq p$, and we deduce from Lemma 2.1 that $T^*_{p+1} \subseteq G$.
If $\Delta(\bar{G}) \geq q-1$, then let $v \in V(G)$ such that $d(v,\bar{G}) = \Delta(\bar{G})$. We choose a vertex set $A \subseteq N(v,\bar{G})$ with $|A| = q-1$, and we define $B = V(G) - (A \cup \{v\})$. We have $|B| = p-1$ and all edges between A and B are necessarily elements of $E(G)$. This implies $T^*_{p+1} \subseteq G$, and so we have proved $r(T^*_{p+1}, T^*_{q+1}) \leq p+q-1$.
Let without loss of generality $q-2 = tp$. Then the graph $G = (t+1)K_p$ shows $r(T^*_{p+1}, T^*_{q+1}) \geq p+q-1$.

Now let G be a graph of order $p+q-2$ with $q-2 \neq tp$ and $p-2 \neq tq$, and in addition assume that T^*_{q+1} is not a subgraph of $\bar{G}$.
If $\Delta(\bar{G}) \leq q-2$, then $\delta(G) \geq p-1$, and hence there is a component H of G with $n(H) \geq p+1$. In view of Lemma 2.3, we conclude $T^*_{p+1} \subseteq H \subseteq G$.
If $\Delta(\bar{G}) \geq q-1$, then let $v \in V(G)$ with $d(v,\bar{G}) = \Delta(\bar{G})$. We choose a vertex set $A \subseteq N(v,\bar{G})$ with $|A| = q-1$, and we define $B = V(G) - (A \cup \{v\})$. We have $|B| = p-2$ and all edges between A and B are elements of $E(G)$. If there are two vertices in A which are adjacent in G, then $T^*_{p+1} \subseteq G$ is immediate. So, we assume now that $\bar{G}[A] = K_{q-1}$. Consequently, all vertices of B are adjacent to v in G.
If $q \geq p-1$, then it is a simple matter to obtain $T^*_{p+1} \subseteq G$. Therefore, all that remains is the case $p = q+s$ with $s \geq 3$. If we define $H_1 = \bar{G}[B]$ and $H_2 = G[B]$, then it is not difficult to see that $T^*_{p+1} \subseteq G$ or $\Delta(H_2) \leq s-2$. From $\Delta(H_2) \leq s-2$, we deduce $\delta(H_1) \geq p-3-(s-2) = q-1$. Because $p-2 \neq tq$, we thus obtain, using Lemma 2.3, the contradiction $T^*_{q+1} \subseteq \bar{G}$. Since we have checked all the possibilities, we have proved $r(T^*_{p+1}, T^*_{q+1}) \leq p+q-2$ for this case.
If p and q are not both odd, then according to Theorem 2.1 and Theorem 2.2, there exists the factorization

$$K_{p+q-3} = R^{p+q-3}_{p-2} \oplus R^{p+q-3}_{q-2}.$$

If p and q are odd, and without loss of generality $q \geq p+4$, then we define $G = K_p \cup R^{q-3}_{p-2}$ and $K_{p+q-3} = G \oplus \bar{G}$. These two factorizations yield the desired equality $r(T^*_{p+1}, T^*_{q+1}) = p+q-2$ for the discussed cases.

Finally, let $p = q$ be odd, and let G be a graph of order $p+q-3$. Furthermore, we assume that T^*_{q+1} is not a subgraph of $\bar{G}$.
If $\Delta(\bar{G}) \leq q-3$, then $\delta(G) \geq p-1$ and G is connected. In view of Lemma 2.3, we conclude $T^*_{p+1} \subseteq G$.
If $\Delta(\bar{G}) = q-2$, then $\delta(G) \geq p-2$, $\Delta(G) \geq p-1$, and G is connected. Now let $b \in V(G)$ with $d(b,G) = \Delta(G)$. We choose $A \subseteq N(b,G)$ such that $|A| = p-1$, and we define $B = V(G) - (A \cup \{b\})$. So, it follows $|B| = q-3 \geq 2$ and all edges between A and B are contained in $\bar{G}$. But now it is easy to see that $T^*_{p+1} = T^*_{q+1} \subseteq \bar{G}$.

If $\Delta(\bar{G}) \geq q-1$, then let v be a vertex with $d(v,\bar{G}) = \Delta(\bar{G})$. We choose $A \subseteq N(v,\bar{G})$ with $|A| = q-1$ and we define $B = V(G) - (A \cup \{v\})$. Hence, we see that $|B| = p-3 \geq 2$ and all edges between A and B are necessarily in G. This implies $T^*_{q+1} = T^*_{p+1} \subseteq G$, and we obtain $r(T^*_{p+1}, B^*_{q+1}) \leq p+q-3$ for this case.
If $p = q$ is odd, then the factorization

$$K_{p+q-4} = R^{p+q-4}_{p-2} \oplus R^{p+q-4}_{q-3}$$

shows the desired equality, and the theorem is proved. □

In connection with Lemma 2.3, we like to formulate the following conjecture.

Conjecture Let G be a connected graph of order $n(G) \geq p+3$ with $\delta(G) \geq p \geq 3$. If T is a tree of order $n(T) \leq p+3$ and $\Delta(T) \leq p-1$, then $T \subseteq G$.

We note that there exist examples which show that this conjecture is not valid for $\Delta(T) = p$ in general.

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