Existence of almost resolvable directed 5-cycle systems

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Abstract

A directed k-cycle system of order n is a pair (S, T), where S is an n-set and T is a collection of arc disjoint directed k-cycles that partition the complete directed graph K_n^*. An almost parallel class with deficiency x is a set of directed k-cycles which form a partition of $S \setminus \{x\}$. An almost resolvable directed k-cycle system is a directed k-cycle system in which the cycles can be partitioned into almost parallel classes. It is clear that $n \equiv 1 \pmod k$ is a necessary condition for the existence of such a system. It is well known that for k = 3 and 4 the necessary condition is also sufficient. In this paper, we shall introduce a special kind of skew Room frames and discuss their constructions. As an application, we show that an almost resolvable directed 5-cycle system of order n exists if and only if $n \equiv 1 \pmod 5$.

1. Introduction

A *directed k-cycle system* of order n is a pair (S, T), where S is an n-set and T is a collection of arc disjoint directed k-cycles that partition the complete directed graph K_n^*. An *almost parallel class with deficiency* x is a set of directed k-cycles which form a partition of $S \setminus \{x\}$. An *almost resolvable directed k-cycle system* of order n, denoted by ARDkCS(n), is a directed k-cycle system of order n in which the cycles can be partitioned into almost parallel classes. Simple counting shows that

$$n \equiv 1 \pmod k \qquad (1)$$

is a necessary condition for the existence of such a system. It has been shown that the necessary condition (1) is also sufficient in the case when k = 3 by Bennett and Sotteau [1] and in the case when k = 4 by Bennett and Zhang [2]. In this paper, we shall introduce a special kind of skew Room frames and discuss their constructions. As an application, we shall show that an almost resolvable directed 5-cycle system of order n exists if and only if $n \equiv 1 \pmod 5$. This complements the result of Heinrich, Lindner and Rodger [7] which completely settles the existence of almost resolvable (undirected) m-cycle systems for all odd m.

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For general background on Room frames and cycle systems, the reader is referred to the recent surveys by Dinitz and Stinson [5] and by Lindner and Rodger [8].

2. Strong skew Room frames and their application to ARDkCS

In this section we shall define a special class of skew Room frames called strong skew Room frames.These Room frames will be used to construct almost resolvable directed k-cycle systems for odd k.

Let S be a finite set, and let $\{ S_1, S_2, ..., S_n \}$ be a partition of S. An $\{ S_1, S_2, ..., S_n \}$-*Room frame* is an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

1. Every cell of F either is empty or contains an unordered pair of symbols of S.
2. The subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as *holes*).
3. Each symbol $x \notin S_i$ occurs once in row (or column) s, for any $s \in S_i$.
4. The pairs in F are those $\{ s, t \}$, where $(s, t) \in (S \times S) \setminus \cup_{1 \leq i \leq n} (S_i \times S_i)$.

As is usually done in the literature, we shall refer to a Room frame simply as a *frame*. The *type* of the frame is defined to be the multiset $\{ | S_i |: 1 \leq i \leq n \}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} t_2^{u_2} ... t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$. We briefly denote a frame of type $t_1^{u_1} t_2^{u_2} ... t_k^{u_k}$ by $RF(t_1^{u_1} t_2^{u_2} ... t_k^{u_k})$.

An $\{ S_1, S_2, ..., S_n \}$-Room frame F is called *skew* if , given any cell $(s, t) \in (S \times S) \setminus \cup_{1 \leq i \leq n} (S_i \times S_i)$, precisely one of (s, t) and (t, s) is empty. A skew $RF(t_1^{u_1} t_2^{u_2} ... t_k^{u_k})$ is denoted by $SRF(t_1^{u_1} t_2^{u_2} ... t_k^{u_k})$.

A skew Room frame F, based on S, is called *strong* if each unordered pair { x, y } in F can be replaced either by (x, y) or by (y, x) such that if an ordered pair (a, b) appears in row r, then r must appear in F as the second element in column a and as the first element in column b.

A strong skew Room frame of type T will be denoted by SSRF(T).

Example 2.1 Let S = { 0, 1, ..., 6 }, and let $S_i = \{ i \}$. An $SSRF(1^7)$ is shown in **Fig. 2.1**, where all the pairs are considered as ordered pairs. But the $SRF(1^7)$ in **Fig. 2.2** is not strong. For, if we take an ordered pair (1, 5) in row 0 and column 3, then the pair { 0, 2 } is forced to become an ordered pair (2, 0) in row 3. From the latter we further get an ordered pair (3, 4) in row 1, which contradicts the first pair. If we take the ordered pair (5, 1), the situation is similar.

	2 6	4 5		1 3		
		3 0	5 6		2 4	
			4 1	6 0		3 5
4 6				5 2	0 1	
	5 0				6 3	1 2
2 3		6 1				0 4
1 5	3 4		0 2			

Fig. 2.1 An SSRF(1^7)

			1 5		4 6	2 3
3 4				2 6		5 0
6 1	4 5				3 0	
	0 2	5 6				4 1
5 2		1 3	6 0			
	6 3		2 4	0 1		
		0 4		3 5	1 2	

Fig. 2.2 An SRF(1^7) which is not strong

In order to use strong skew Room frames to construct almost resolvable directed k-cycle systems for odd k we need two sequences as follows (see also [7]). For $0 \le i \le [k/2]$, define

$$e_i = (-1)^{i+1}[(i+1)/2] \pmod k,$$
$$d_i = [k/2] + 1 + (-1)^i [(i+1)/2] \pmod k.$$

Lemma 2.2 The sequences $(d_0, d_1, \ldots, d_{[k/2]})$ and $(e_0, e_1, \ldots, e_{[k/2]})$ satisfy the following properties:

(1) $\{ |d_i - d_{i-1}|_k \mid 1 \le i \le [k/2] \} = \{ i \mid 1 \le i \le [k/2] \}$,

(2) $d_i - d_{i-1} = e_{i-1} - e_i$ for $1 \le i \le [k/2]$,

(3) $d_{[k/2]} = e_{[k/2]}$,

(4) $\{ d_0, d_1, \ldots, d_{[k/2]}, e_0, e_1, \ldots, e_{[k/2]} \} = \{ i \mid 0 \le i \le k-1 \}$,

where $|i - j|_k$ is defined to be a positive integer x such that $x \le [k/2]$ and $x \equiv i - j \pmod k$ or $x \equiv j - i \pmod k$.

The following construction is a slightly revised version of The Skew Room Frame Construction in [7], adapted here for the directed case.

Construction 2.3 Suppose there exist an SSRF(h^u) and an ARDkCS($hk+1$) for odd k. Then there exists an ARDkCS($huk+1$).

Proof: Let the given SSRF(h^u) F be based on S with partition $\{ S_1, S_2, \ldots, S_u \}$. Let $K = \{ 0, 1, \ldots, k-1 \}$. We shall construct an ARDkCS($huk+1$) on $X = \{ \infty \} \cup (S \times K)$. In this construction, all additions are defined modulo k. Define a collection of directed k-cycles **C** as follows:

(1) for each S_i, $1 \le i \le u$, define an ARDkCS($hk+1$) on the set $\{ \infty \} \cup (S_i \times K)$ and place these directed k-cycles in **C**;

(2) for each pair (x, y) of row r and column c in F and for each j, $0 \le j \le k-1$, place in **C** two directed k-cycles:

$(x, y, r, d, j) = ((x, d_0+j), \ldots, (y, d_{2t}+j), (r, d_{2t+1}+j), (x, d_{2t}+j), \ldots, (y, d_0+j))$ and

$(y, x, c, e, j) = ((y, e_0+j), \ldots, (x, e_{2t}+j), (c, e_{2t+1}+j), (y, e_{2t}+j), \ldots, (x, e_0+j))$ if $k = 4t+3$,

or $(x, y, r, d, j) = ((x, d_0+j), \ldots, (x, d_{2t-1}+j), (r, d_{2t}+j), (y, d_{2t-1}+j), \ldots, (y, d_0+j))$ and

$(y, x, c, e, j) = ((y, e_0+j), \ldots, (y, e_{2t-1}+j), (c, e_{2t}+j), (x, e_{2t-1}+j), \ldots, (x, e_0+j))$ if $k = 4t+1$.

We need to show that (X, **C**) is a directed k-cycle system and also it is almost resolvable. We shall focus on the case when $k = 4t+3$, the case when $k = 4t+1$ can be proved similarly.

To see that (X, **C**) is a directed k-cycle system, we need only to show, by simple counting argument, that any arc of K_n^*, $n = huk+1$, is contained in at least one directed k-cycle of **C**. For any arc (α, β) of K_n^*, if $\alpha = \infty$ or $\beta = \infty$, then by (1) the arc appears in some directed k-cycle of C. If the first coordinates of α and β are in the same set S_i for some i, then by (1) the arc also appears in some directed k-cycle of C. Otherwise, we may suppose α = (a, p), β = (b, q) and a and b belong to different S_i. In the following case 1 and case 2, without loss of generality, we further suppose that the ordered pair (a, b) appears in row r and column c of F.

Case 1. When q = p, there is a unique j such that $p = e_0 + j$. Then there is a directed k-cycle (b, a, c, e, j) in C containing the given arc (α, β).

Case 2. When $|q - p|_k \notin \{ 0, |d_{2t+1} - d_{2t}|_k \}$, by property (1) in Lemma 2.2 there is a unique i such that $|q - p|_k = |d_i - d_{i-1}|_k$. By property (2) in Lemma 2.2 we have $|q - p|_k = |e_i - e_{i-1}|_k$. If i is even, then there is a unique j such that the directed k-cycle (b, a, c, e, j) in C contains the given arc ((a, p), (b, q)), where $j = p - e_{i-1}$ or $p - e_i$ according to $q - p = e_i - e_{i-1}$ or $e_{i-1} - e_i$, respectively. If i is odd, then there is a unique j such that the directed k-cycle (a, b, r, d, j) in C contains the given arc ((a, p), (b, q)), where $j = p - d_{i-1}$ or $p - d_i$ according to $q - p = d_i - d_{i-1}$ or $d_{i-1} - d_i$, respectively.

Case 3. When $|q - p|_k = |d_{2t+1} - d_{2t}|_k$, by property (2) in Lemma 2.2 we have $|q - p|_k = |e_{2t+1} - e_{2t}|_k$. Since a and b belong to different S_i, a must appear in row b and b must appear in row a of F. By definition of strong skew Room frame F, we have the following two subcases to consider.

Subcase 3.1 Suppose $q - p = d_{2t+1} - d_{2t} = e_{2t} - e_{2t+1}$. If a appears in row b as the second element, we may let (x, a) appears in row b and column c of F. Then, there is a directed k-cycle (x, a, b, d, j) in C containing the given arc ((a, p), (b, q)), where $j = q - d_{2t+1}$. If a appears in row b as the first element, since F is strong , b must appear in column a of F as the second element. Let (x, b) appears in row r and column a of F. Then, there is a directed k-cycle (b, x, a, e, j) in C containing the given arc ((a, p), (b, q)), where $j = q - e_{2t}$.

Subcase 3.2 Suppose $q - p = e_{2t+1} - e_{2t} = d_{2t} - d_{2t+1}$. If b appears in row a as the second element, since F is strong , a must appear in column b of F as the first element. Let (a, y) appears in row r and column b of F. Then, there is a directed k-cycle (y, a, b, e, j) in C containing the given arc ((a, p), (b, q)), where $j = q - e_{2t+1}$. If b appears in row a as the first element, we may let (b, y) appears in row a and column c of F. Then, there is a directed k-cycle (b, y, a, d, j) in C containing the given arc ((a, p), (b, q)), where $j = q - d_{2t}$.

We have proved that (X, C) is a directed k-cycle system and we shall now show that it is almost resolvable. For each set $H \in \{ S_1, S_2, ..., S_u \}$, denote by $\pi(\infty, H)$ the almost parallel class that has deficiency ∞ and by $\pi((x, j), H)$ the almost parallel class with deficiency (x, j) in the resolution of ARDkCS(hk + 1) on the set $\{ \infty \} \cup (H \times K)$.

For each $w \in \{ \infty \} \cup (S \times K)$ define the almost parallel class $\pi(w)$ with deficiency w as follows:

(1) $\pi(\infty) = \cup_{1\le i\le u}\pi(\infty, S_i)$; and

(2) for each $(x, j) \in S \times K$ with $x \in S_i$,

$$\pi((x, j)) = \pi((x, j), S_i) \cup \{(a, b, r, d, j) \mid \text{all } (a, b) \text{ in column } x \text{ of } F\} \cup \{(b, a, c, e, j) \mid \text{all } (a, b) \text{ in row } x \text{ of } F\}.$$ □

Corollary 2.4 Suppose there is an SSRF(1^u). Then there exists and ARDkCS(uk + 1) for odd k.

Proof: From [3, Theorem 3], there exists a directed k-cycle system of order k + 1 for any odd k, which is also an ARDkCS(k + 1).Then the conclusion follows from Construction 2.3. □

Corollary 2.5 Suppose there is an SSRF(2^u). Then there exists and ARDkCS(2uk + 1) for odd $k \ge 3$.

Proof: We construct an ARDkCS(2k + 1) on Z_{2k+1}. Let k = 2t + 1 (so $t \ge 1$). Let

$$c = (-1, 2, -3, \ldots, (-1)^t t, (-1)^t(t+1), (-1)^{t+1}(t+2), \ldots, (-1)^t t, (-1)^{2t}(2t+1)),$$

where each component of c is reduced modulo 2k + 1. Let - c and c + i be formed by replacing each component c_j (for $1 \le j \le k$) of c by - c_j (mod 2k + 1) and $c_j + i$ (mod 2k + 1), respectively. Then c and - c form an almost parallel class with deficiency 0 and $C = \{ c + i, -c + i \mid 0 \le i \le 2k \}$ is an ARDkCS(2k + 1). Then the conclusion follows from Construction 2.3. □

3. Constructions of strong skew Room frames

In this section, we shall discuss constructions of strong skew Room frames. We mainly use the direct constructions for Room frames.

Let G be an additive abelian group of order g, and let H be a subgroup of order h of G, where g - h is even. A *frame starter* in G\H is a set of unordered pairs $S = \{\{ s_i, t_i \} : 1 \le i \le (g - h)/2 \}$ such that the following two properties are satisfied:

1. $\{ s_i : 1 \le i \le (g - h)/2 \} \cup \{ t_i : 1 \le i \le (g - h)/2 \} = G\backslash H$.
2. $\{ \pm(s_i - t_i) : 1 \le i \le (g - h)/2 \} = G\backslash H$.

The *type* of the frame is defined to be $h^{g/h}$. When H = {0}, a frame starter is simply called a *starter*. A frame starter $S = \{\{ s_i, t_i \} : 1 \le i \le (g - h)/2 \}$ in G\H is called *skew* if $\{ \pm(s_i + t_i) : 1 \le i \le (g - h)/2 \} = G\backslash H$.

From a skew frame starter, one can construct a skew Room frame easily. For each element b in G and each pair { s, t } in a skew starter S, place in cell (b, b + s + t) the pair { b + s, b + t } to form a skew Room frame F indexed by G. We can further prove that the skew Room frame is also strong.

Lemma 3.1 If there is a skew frame starter of type h^u, then there exists an SSRF(h^u).

Proof: To prove the above-defined skew Room frame F is strong, we need to show that b must appear in column b + s as the second element and in column b + t as the first element in F, where the pair { b + s, b + t } in F is considered as the ordered pair (b + s, b + t). In fact, F contains (b + s - t, b) in cell (b - t, b + s) and (b, b + t - s) in cell (b - s, b + t). This completes the proof. □

For example, the SSRF(1^7) in **Fig. 2.1** is constructed from a skew frame starter S = {{ 2, 6 }, { 4, 5 }, { 1, 3 } } in $Z_7\backslash\{0\}$. The following known skew frame starters will be useful.

Lemma 3.2 ([10]) Let n be a prime power such that $n = 2^k t + 1$, where $t > 1$ is odd. Then there is a skew frame starter of type (1^n).

Lemma 3.3 ([9]) There is a skew frame starter of type (1^n) for $n = 16k^2 + 1$, where k is any positive integer.

Lemma 3.4 ([6], [12]) If $q \equiv 1 \pmod 4$ is a prime power and $n \geq 1$, then there is a skew frame starter in $(GF(q) \times (Z_2)^n) \backslash (\{0\} \times (Z_2)^n)$.

Lemma 3.5 There are skew frame starters of type (4^4) and type (1^{35}).

Proof: The first skew frame starter S can be found in [11, Lemma 5.1]. In $(Z_4 \times Z_4) \backslash \{ (0, 0), (0, 2), (2, 0), (2, 2) \}$, S = { {(3, 2), (1, 1)}, {(3, 0), (3, 1)}, {(2, 1), (3, 3)}, {(0, 3), (1, 3)}, {(1, 0), (2, 3)}, {(1, 2), (0, 1)} }. The second is shown below, where the starter is in $Z_{35}\backslash\{0\}$. S = { {1, 2}, {3, 5}, {4, 7}, {6, 10}, {8, 15}, {9, 21}, {11, 25}, {12, 29}, {13, 24}, {14, 30}, {16, 26}, {17, 22}, {18, 31}, {19, 34}, {20, 28}, {23, 32}, {27, 33} }. □

For some group G there is no skew starter in G as pointed out in the following.

Lemma 3.6 ([13]) Suppose that G is an abelian group of order $n \equiv 3 \pmod 6$ in which the 3-Sylow subgroup is cyclic. Then there is no skew starter in G.

For example, $n = 15$ is such an order. But, we can use starter-adder construction to find an SSRF(1^{15}). If $S = \{\{ s_i, t_i \} : 1 \leq i \leq (g - h)/2 \}$ is a frame starter in G\H, then a set $A = \{\{ a_i \} : 1 \leq i \leq (g - h)/2 \}$ is defined to be an *adder* for S if the elements in A are distinct in G\H, and the set $S + A = \{\{ s_i + a_i , t_i + a_i \} : 1 \leq i \leq (g - h)/2 \}$ is again a frame starter. An adder is said to be *skew* if for any $a \in A$, $-a$ is not in A. It is well known that the existence of a frame starter and a skew adder implies the existence of a skew Room frame with the same type. The frame F, indexed by G, will contain in cell $(b, b - a_i)$ the pair $\{ b + s_i , b + t_i \}$ for any $b \in G$.

A frame starter and a skew adder (S, A) is called *strong* if $\{ - s_i - a_i : 1 \leq i \leq (g - h)/2 \} = \{ t_j : 1 \leq j \leq (g - h)/2 \}$. Since S and S + A are both frame starters, it is equivalent to $\{ - t_i - a_i : 1 \leq i \leq (g - h)/2 \} = \{ s_j : 1 \leq j \leq (g - h)/2 \}$.

Lemma 3.7 If there is a frame starter and a skew adder (S, A) which is strong , then there is a strong skew Room frame with the same type.

Proof: Since the skew frame F contains the ordered pair $(b + s_i , b + t_i)$ in cell $(b, b - a_i)$ for any $b \in G$, we know that $(b , b + t_i - s_i)$ appears in cell $(b - s_i , b - a_i - s_i)$. Since (S, A) is strong, there is an integer j such that $b - a_i - s_i = b + t_j$. That is, b appears in column $b + t_j$ as the first element in F. Similarly, b appears in column $b - a_i - t_i = b + s_j$, for some j, as the second element in F. When i and j run through 1 to (g - h)/2, we know that b appears in column $b + s_i$ as the second element and in column $b + t_i$ as the first element in F. Therefore, the skew frame F is strong. The proof is complete. □

Lemma 3.8 There exists an SSRF(1^{15}).

Proof: Let $G = Z_{15}$ and $H = \{0\}$. Take S = { (1, 4), (10, 6), (12, 13), (11, 3), (2, 8), (5, 7), (9, 14) } and A = { 1, 13, 12, 11, 10, 6, 7 }. It is readily checked that the frame starter and skew adder (S, A) is strong. The conclusion then follows from Lemma 3.7. □

Lemma 3.9 There exists an SSRF(2^8).

Proof: Let $G = Z_{16}$ and H = { 0, 8 }. Take S = { (2, 5), (7, 3), (9, 11), (15, 14), (10, 4), (12, 1), (6, 13) } and A = { 1, 14, 3, 12, 5, 6, 7 }. It is readily checked that the frame starter and skew adder (S, A) is strong. The conclusion then follows from Lemma 3.7. □

4. Existence of almost resolvable directed 5-cycle systems

In this section, we shall solve the existence of almost resolvable directed 5-cycle systems. We start with group divisible directed k-cycle systems.

A *group divisible directed k-cycle system* (GDDkCS) is a triple (S, **G**, **T**), where **G** is a partition of the set S and **T** is a collection of arc disjoint directed k-cycles that partition the complete directed multipartite graph on S with partition **G**. The *group type* of the GDDkCS is the multiset { |G| : G ∈ **G** }. An *almost parallel class with deficiency* G for G ∈ **G** is a set of directed k-cycles which form a partition of S\G. An *almost resolvable group divisible directed k-cycle system* (ARGDDkCS) is a GDDkCS in which the cycles can be partitioned into almost parallel classes such that for each group G ∈ **G** there are exactly |G| almost parallel classes with deficiency G.

We wish to remark that an ARGDDkCS of type 1^n is just an ARDkCS(n). By Construction 2.3 we can get an ARGDDkCS from an SSRF.

Lemma 4.1 If there exists an SSRF(h^u), then for any odd integer $k \geq 3$, there exists an ARGDDkCS of type $(hk)^u$.

To get an ARDkCS from an ARGDDkCS and some ARDkCS we have the following obvious filling-in-holes construction.

Lemma 4.2 If there exists an ARGDDkCS (S, **G**, **T**) and if for any G in **G** there exists an ARDkCS(|G| + 1), then there exists an ARDkCS(|S| + 1).

We shall further use group divisible designs to construct ARGDDkCS and ARDkCS. A *group divisible design* (or GDD), is a triple (X, **G**, **B**) which satisfies the following properties:

(1) **G** is a partition of X into subsets called *groups*,

(2) **B** is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in a unique block.

The *group type* of the GDD is the multiset { |G| : G ∈ **G** }. A TD(k, n) is a GDD of group type n^k and block size k. It is well known that the existence of a TD(k, n) is equivalent to the existence of k - 2 mutually orthogonal Latin squares (MOLS) of order n and also to the existence of resolvable TD(k-1, n). For more about TD and MOLS the reader is referred to Beth, Jungnickel and Lenz [4]. We have the following two weighting construction.

Construction 4.3 Suppose (S, G, **B**) is a GDD and let w : X $\rightarrow \mathbf{Z}^+ \cup \{ 0 \}$. Suppose there exists an ARGDDkCS of type { w(x) : x $\in$ B } for every B $\in$ **B**. Then there exists an ARGDDkCS of type $\{ \Sigma_{x \in G} w(x) : G \in \mathbf{G} \}$.

Proof: The resultant system will be based on $S^* = \cup_{x \in S} S_x$, where for x $\in$ S, S_x are pairwise disjoint and $|S_x| = w(x)$. The new partition of S* will be $\mathbf{G}^* = \{ \cup_{x \in G} S_x : G \in \mathbf{G} \}$. Suppose $\mathbf{A}_B$ is the set of cycles for an ARGDDkCS of type { w(x) : x $\in$ B }, B $\in$ **B**. Then, $\mathbf{B}^* = \cup_{B \in \mathbf{B}} \mathbf{A}_B$ is the set of cycles for the ARGDDkCS. For any x in certain G, let $\mathbf{B}_x$ consists of all blocks in **B** containing x. Let P(x, B, j) denote the j-th almost parallel class with deficiency S_x, $1 \le j \le w(x)$, in the ARGDDkCS of type { w(x) : x $\in$ B } for B $\in \mathbf{B}_x$. Then P(x, j) = $\cup_B$ P(x, B, j), where B runs over $\mathbf{B}_x$, is an almost parallel class with deficiency $\cup_{x \in G} S_x$. For each G $\in$ **G**, there are all together $\Sigma_{x \in G} w(x)$ almost parallel classes with deficiency $\cup_{x \in G} S_x$. This completes the proof. □

Construction 4.4 Suppose (S, G, T) is an ARGDDkCS of type T. If there exists a resolvable TD(3, m), then there exists an ARGDDkCS of type mT = { mt : t $\in$ T }.

Proof: Let M = { 1, 2, ..., m } and let the TD(3, m) be based on M x {1, 2, 3 } having three groups M x { j }, $1 \le j \le 3$. For each cycle c = ($c_1, c_2, ..., c_k$) in **T** and each block B = { (x, 1), (y, 2), (z, 3) } in the TD(3, m), define a directed k-cycle c x B = ((c_1, x), (c_2, y), ..., (c_{k-2}, x), (c_{k-1}, y), (c_k, z)). All these cycles will form the set of cycles for the resultant ARGDDkCS, which will be based on the set S x M having the partition { G x M : G $\in$ **G** }. Let P(G, j) be the j-th almost parallel class of the given ARGDDkCS with deficiency G, $1 \le j \le |G|$. Let Q(i) be the i-th parallel class of the resolvable TD(3, m). Denote P(G, j, i) = { c x B : c $\in$ P(G, j), B $\in$ Q(i) }. Then P(G, j, i) is an almost parallel class with deficiency G x M and there are m|G| such almost parallel classes. This completes the proof. □

We are now in a position to show the existence of an ARD5CS(n). First, from the proof of Corollary 2.4 and Corollary 2.5 we have ARD5CS(n) for n = 6 and 11.

Lemma 4.5 For any odd integer $k \ge 3$, there are ARDkCS(k +1) and ARDkCS(2k +1).

Lemma 4.6 For any prime power $q \equiv 1 \pmod{k}$, there exists an ARDkCS(q).

Proof: Let x be a primitive element of GF(q). Let $y = x^d$, d = (q - 1)/k. Denote B(i, g) = ($x^i y^1 + g, x^i y^2 + g, ..., x^i y^k + g$) and **B** = { B(i, g) : $1 \le i \le d$ and g $\in$ GF(q) }. Then, (GF(q),

B) is the desired ARDkCS(q) where for each $g \in GF(q)$, $\{ B(i, g) : 1 \le i \le d \}$ is the almost parallel class with deficiency g. □

Lemma 4.7 There exist ARGDD5CS of type 5^u for u = 5, 6, 7 and 9.

Proof: There is an ARDkCS(6) from Lemma 4.5, which is also an ARGDD5CS of type 1^6. Applying Construction 4.4 with k = 5 we obtain an ARGDD5CS of type 5^6. An SSRF(1^7) exists from Lemma 3.2, which leads to an ARGDD5CS of type 5^7 by Lemma 4.1. An ARGDD5CS of type 5^5 (S, G, T) is shown below, where $S = Z_{25}$ and $\mathbf{G} = \{ \{ 0, 5, 10, 15, 20 \} + i : 0 \le i \le 4 \}$. **T** is generated modulo 25 by the following initial directed cycles: (1, 3, 17, 21, 13), (2, 14, 11, 7, 16), (4, 12, 19, 22, 23), (6, 24, 18, 9, 8), which form an almost parallel class with deficiency 0. For type 5^9, take $S = Z_{45}$ and $\mathbf{G} = \{ \{ 0, 9, 18, 27, 36 \} + i : 0 \le i \le 8 \}$. The initial directed cycles are:

(1, 3, 13, 5, 8), (2, 32, 10, 11, 26), (4, 23, 37, 41, 21), (6, 38, 15, 39, 44),
(7, 19, 25, 33, 17), (12, 29, 40, 35, 24), (14, 34, 31, 30, 43), (16, 42, 28, 22, 20).

These cycles form an almost parallel class with deficiency 0. □

Lemma 4.8 There exists an ARD5CS of order 21.

Proof: Let $S = Z_{20} \cup \{ \infty \}$. Four directed cycles (0, 4, 8, 12, 16) + i for $0 \le i \le 3$ form an almost parallel class with deficiency ∞. For any $g \in Z_{20}$, the following four cycles form an almost parallel class with deficiency g: (2, 7, 10, 17, 8) + g , (3, 13, 19, 18, 11) + g , (4, 6, 14, 12, 9) + g , (15, 16, 5, 1, ∞) + g . Let **T** denote the set of all these cycles. Then, (S, T) is the desired ARD5CS(21).

Lemma 4.9 For any integer v, $1 \le v \le 9$, there exists an ARD5CS(5v + 1).

Proof: For v = 1, 2 and 4, an ARD5CS(5v + 1) exists by Lemmas 4.5 and 4.8. For v = 5, 6, 7 and 9, an ARD5CS(5v + 1) exists by Lemmas 4.7 and 4.2. Finally, Lemma 4.6 takes care of the cases v = 3 and 8. □

Lemma 4.10 For any integer v, $10 \le v \le 24$, there exists an ARD5CS(5v + 1).

Proof: We shall deal with these cases in **Table 4.1**. □

v	5v + 1	Authority	Ingredients
10	51	Corollary 2.5	SSRF(2^5), Lemma 3.4
11	56	Corollary 2.4	SSRF(1^{11}), Lemma 3.2
12	61	Lemma 4.6	
13	66	Corollary 2.4	SSRF(1^{13}), Lemma 3.2
14	71	Lemma 4.6	
15	76	Corollary 2.4	SSRF(1^{15}), Lemma 3.8
16	81	Lemma 4.6	
17	86	Corollary 2.4	SSRF(1^{17}), Lemma 3.3
18	91	Corollary 2.5	SSRF(2^9), Lemma 3.4
19	96	Corollary 2.4	SSRF(1^{19}), Lemma 3.2
20	101	Lemma 4.6	
21	106	Lemma 4.2	Apply Construction 4.4 with $T = 5^7$ and m = 3 to get an ARGDD5CS of type 15^7
22	111	Lemma 4.2	Apply Construction 4.4 with $T = 1^{11}$ and m = 10 to get an ARGDD5CS of type 10^{11}
23	116	Corollary 2.4	SSRF(1^{23}), Lemma 3.2
24	121	Lemma 4.6	

Table 4.1

Lemma 4.11 For any integer v, $25 \le v \le 30$, there exists an ARD5CS(5v + 1).

Proof: Start with a TD(6, 5), which exists from [4], and give weight 5 to each point of the TD except 5 - a points in some group, for which we give weight 0 each. Applying Construction 4.3 with ARGDD5CS of type 5^5 and 5^6, we obtain an ARGDD5CS of type $25^5(5a)^1$for $0 \le a \le 5$. The conclusion then follows from Lemma 4.2 and Lemma 4.9. □

Lemma 4.12 For any integer v, $31 \le v \le 34$, there exists an ARD5CS(5v + 1).

Proof: We shall deal with these cases in **Table 4.2**. □

v	5v + 1	Authority	Ingredients
31	156	Corollary 2.4	SSRF(1^{31}), Lemma 3.2
32	161	Lemma 4.2	Apply Construction 4.4 with $T = 1^{16}$ and m = 10 to get an ARGDD5CS of type 10^{16}
33	166	Lemma 4.2	Apply Construction 4.4 with $T = 5^{11}$ and m = 3 to get an ARGDD5CS of type 15^{11}
34	171	Corollary 2.5	SSRF(2^{17}), Lemma 3.4

Table 4.2

Lemma 4.13 Suppose there exist a TD(7, t) and an ARD5CS(5v + 1) for v = t, a and b, where $0 \le a, b \le t$. Then, there exists an ARD5CS(5(5t + a + b) + 1).

Proof: Delete t - a points from one group and t - b points from another group of the TD. Give weight 5 to each point of the resultant GDD. Applying Construction 4.3 gives an ARGDD5CS of type $(5t)^5(5a)^1(5b)^1$. The input ARGDD5CS of types 5^5, 5^6 and 5^7 are all from Lemma 4.7. Further apply Lemma 4.2, we get the desired ARD5CS(5(5t + a + b) + 1). □

Lemma 4.14 For any integer $v \ge 35$, there exists an ARD5CS(5v + 1).

Proof: We shall prove this Lemma by induction using Lemma 4.13. For any $v \ge 35$, we may write $v = 5t + a + b$ such that $0 \le a, b \le t$ and a TD(7, t) exists. For example, if $v \ge 265$, we may write $v = 5t + a + b$ such that t is odd ≥ 53 and $0 \le a, b \le 5$. A TD(7, t) exists from [4]. Other values of $v = 5t + a + b$ are given in **Table 4.3**, where a TD(7, t) exists from [4].

$35 \le v \le 44$,	t = 7,	$0 \le a, b \le 5$,
$45 \le v \le 54$,	t = 9,	$0 \le a, b \le 5$,
$55 \le v \le 64$,	t = 11,	$0 \le a, b \le 5$,
$65 \le v \le 84$,	t = 13,	$0 \le a, b \le 10$,
$85 \le v \le 114$,	t = 17,	$0 \le a, b \le 15$,
$115 \le v \le 144$,	t = 23,	$0 \le a, b \le 15$,
$145 \le v \le 184$,	t = 29,	$0 \le a, b \le 20$,
$185 \le v \le 204$,	t = 37,	$0 \le a, b \le 10$,
$205 \le v \le 264$,	t = 41,	$0 \le a, b \le 30$.

Table 4.3

By induction hypothesis, an ARD5CS(5t + 1) exists. Since an ARD5CS(5v + 1) exists for $v \leq 34$, by Lemma 4.13 there exists an ARD5CS(5v + 1) for $v \geq 35$. □

Combining Lemmas 4.9 - 4.12 and 4.14 we obtain the main theorem of this paper.

Theorem 4.15 There exists an ARD5CS(n) if and only if $n \equiv 1 \pmod 5$ and $n \geq 6$.

5. Concluding remarks

The existence problem for ARDkCS(n) has been solved for k = 3, 4 in [1], [2] and for k = 5 in this paper. But, for general k the problem is still open. The new concept of strong skew Room frames and their constructions, introduced and discussed in Sections 2 and 3, are useful in dealing with such a general problem. Especially, the existence problems for SSRF(1^n) and SSRF(2^n) are very much desirable. However, both of them are again open problems.

References

[1] F. E. Bennett and D. Sotteau, Almost resolvable decompositions of K_n^*, J. Combin. Theory (B) 30 (1981) 228-232.

[2] F. E. Bennett and X. Zhang, Resolvable Mendelsohn designs with block size 4, Aequationes Math. 40 (1990) 248-260.

[3] J. C. Bermond and V. Faber, Decomposition of the complete directed graph into k-circuits, J. Combin. Theory (B) 21 (1976) 146-155.

[4] T. Beth, D. Jungnickel and H. Lenz, Design Theory, Bibliographisches Institut, Zurich, 1985.

[5] J. H. Dinitz and D. R. Stinson, Room squares and related designs, in: Contemporary Design Theory, Wiley, 1992, 137-204.

[6] J. H. Dinitz and D. R. Stinson, The construction and uses of frames, Ars Combin. 10 (1980) 31-54.

[7] K. Heinrich, C. C. Lindner and C. A. Rodger, Almost resolvable decompositions of $2K_n$ into cycles of odd length, J. Combin. Theory (A) 49 (1988) 218-232.

[8] C. C. Lindner and C. A. Rodger, Decomposition into cycles II: cycle systems, in: Contemporary Design Theory, Wiley, 1992, 325-369.

[9] S. Lins and P. J. Schellenberg, The existence of skew strong starters in Z_{16k^2+1} : A simpler proof, Ars Combin. 11 (1981) 123-129.

[10] R. C. Mullin and E. Nemeth, An existence theorem for Room squares, Canad. Math. Bull. 12 (1969) 493-497.

[11] D. R. Stinson, Some constructions for frames, Room squares, and subsquares, Ars Combin. 12 (1981) 229-267.

[12] D. R. Stinson and W. D. Wallis, Some designs used in constructing skew Room squares, Ann. Discrete Math. 8 (1980) 171-175.

[13] W. D. Wallis and R. C. Mullin, Recent advances in complementary and skew Room squares, Congr. Numer. 8 (1973) 521-532.

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