A Structural Method for Hamiltonian Graphs

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Abstract

In this paper, we shall introduce a special structure for graphs and show that a graph G is hamiltonian if and only if G has such a special structure. Using this result, we can prove a new weakened version of Fan's condition for hamiltonian graphs, which generalizes a recent result of Bedrossian, Chen and Schelp (1993).

1 Preliminaries and Main Results

We consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$ or just by V; the set of edges by $E(G)$ or just by E. We use $|G|$ as a symbol for the cardinality of $V(G)$. If H and S are subsets of $V(G)$ or subgraphs of G, we denote by $N_H(S)$ the set of vertices in H which are adjacent to some vertex in S, and set $d_H(S) = |N_H(S)|$. If $S = \{u\}$ and $H = G$, then let $N_G(u) = N(u)$ and set $d_G(u) = d(u)$. For $D \subseteq V(G)$, $G[D]$ denotes the subgraph of G induced by D. For basic graph-theoretic terminology, we refer the reader to [3].

Definition 1. Let H be a subgraph of G and $x, y \in V(G) \setminus V(H)$. $\{x, y\}$ is called *a pair of useful vertices of H* if $G[H \cup \{x, y\}]$ contains a hamiltonian path connecting x and y.

Definition 2. A graph G is call *L-decomposable* if G can be separated into $k+1$ pairwise disjoint subgraphs $G_0, G_1, \cdots, G_k$ such that the following four conditions are satisfied:

1) G_0 is complete.

2) For any $1 \le i \le k$, there exists a subset $S_i \subseteq N_{G_0}(G_i)$ with at least two vertices which contains a vertex x such that for every $y\ (\neq x) \in S_i$, $\{x, y\}$ is a pair of useful vertices of G_i.

3) For any three distinct S_i, S_j, S_l, we have $S_i \cap S_j \cap S_l = \emptyset$.

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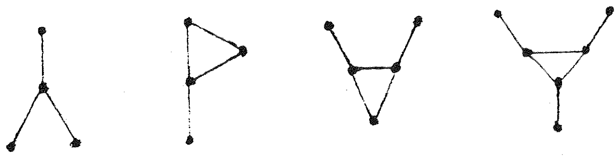
4) For any positive integer $r \leq k$, $|\bigcup_{1\leq j\leq r} S_{i_j}| = r$ if and only if $|V(G_0)| = k = r$.

If G is L-decomposable, then we say the partition $G_0, G_1, \cdots, G_k$ which satisfies the four conditions above *a L-decomposition of G*. In Section 2, we shall prove the following structural theorem.

Theorem 1. A graph G is hamiltonian if and only if G has a L-decomposition.

Theorem 1 has some applications. We shall give some examples here. In order to do this, we need some additional terminology and notations.

In Figure 1, we define four kinds of graphs, C–graph, F–graph, B–graph and N–graph.



C–graph F–graph B–graph N–graph

Figure 1.

Let S, T be two induced subgraphs of G with $\max\{|S|, |T|\} < |G|$. A graph G of order n is said to satisfy property $ST(n)$ if for any pair of vertices x and y at distance two in S or T, $\max\{d(x), d(y)\} \geq n/2$. If G contains no S as an induced subgraph, we call G S–free. If G contains neither S nor T as an induced subgraph, we call G ST–free.

The closure of a graph G denoted by $\overline{G}$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $|V(G)|$ untill no such pair remains. Let $V_0 = \{x : d(x) \geq n/2, x \in V(G)\}$.

The following result is due to Bondy and Chvátal.

Theorem 2[2]. A graph G is hamiltonian if and only if $\overline{G}$ is hamiltonian.

Now, using Theorems 1 and 2, we can easily prove the following two theorem known before.

Theorem 3[4]. Let G be a 2-connected graph of order n. If each pair of vertices x and y at distance 2 satisfies $\max\{d(x), d(y)\} \geq n/2$, then G is hamiltonian.

Theorem 4[1]. Let G be a 2-connected graph of order n. If G satisfies property $CF(n)$, then G is hamiltonian.

To prove Theorems 3 and 4, we assume, by contradiction, that G is a counterexample with as many as possible edges. By Theorem 2, $G[V_0]$ is a complete subgraph of G. Let G_0 be a induced complete subgraph of G with as many as possible vertices and $V_0 \subseteq V(G_0)$. Let $G_1, G_2, \cdots, G_k$ be the components of $G \setminus G_0$. We can easily verify that $G_0, G_1, \cdots, G_k$ is a L-decomposition of G under conditions of Theorem 3 or Theorem 4, which leads to a contradiction by Theorem 1.

In section 3, we shall prove the following more general theorem by using Theorems 1 and 2.

Theorem 5. Let G be a 2-connected graph of order n. If G satisfies property $CB(n)$, then G is hamiltonian.

2 The Proof of Theorem 1

If G is a hamiltonian graph, let $C = c_1c_2\cdots c_nc_1$ be a hamiltonian cycle of G. Set $G_0 = G[\{c_1, c_2\}]$ and $G_1 = G[\{c_3, \cdots, c_n\}]$. Then G_0, G_1 satisfy the four conditions of Definition 2. Thus G has a L-decomposition.

Conversely, let $G_0, G_1, \cdots, G_k$ be a L-decomposition of G. By Definition 2, G_0 is a complete subgraph with $|G_0| \geq 2$ and for any $1 \leq i \leq k$, there exists some $S_i \subseteq N_{G_0}(G_i)$ which satisfies the conditions 2)–4) of Definition 2. By condition 2), S_i contains a vertex x_i such that for any $y \in S_i \setminus \{x_i\}$, $\{x_i, y\}$ is a pair of useful vertices of G_i for all $1 \leq i \leq k$. Using the following Claim we will give a structural proof of the sufficiency.

Claim. G_0 contains either a cycle $C = u_{i_1}u_{i_2}\cdots u_{i_k}u_{i_1}$ with $|V(C)| = |G_0|$ (when $|G_0| = 2$, C is just an edge.) such that

$$\{u_{i_j}, u_{i_{j+1}}\} = \{x_{i_j}, y_{i_j}\},\ j = 1, \cdots, k, \quad j \bmod k \qquad (*)$$

or q pairwise disjoint paths $P_i = u_{i_1}u_{i_2}\cdots u_{i_{r_i+1}}$, $i = 1, 2, \cdots q$

$$\{u_{i_j}, u_{i_{j+1}}\} = \{x_{i_j}, y_{i_j}\},\ j = 1, 2, \cdots, r_i, \qquad (**)$$

and

$$u_{i_1}, \cdots, u_{i_{r_i+1}} \notin \bigcup_{j \notin \{i_1, \cdots, i_{r_i}\}} S_j \qquad (***)$$

where $y_{i_j} \in S_{i_j} \setminus \{x_{i_j}\}$.

In fact, let $P = u_{i_1}\cdots u_{i_{r_i+1}}$ be a longest path satisfying the equation (**). Then $u_{i_2}, \cdots, u_{i_{r_i}} \notin \bigcup_{j \notin \{i_1, \cdots, i_{r_i}\}} S_j$ by condition 3). If $u_{i_1}, u_{i_{r_i+1}} \notin \bigcup_{j \notin \{i_1, \cdots, i_{r_i}\}} S_j$, then P is desired. Otherwise, there exists a subset, say $S_{i_{r_i+1}}$, such that $\{u_{i_1}, u_{i_{r_i+1}}\} \cap S_{i_{r_i+1}} \neq \emptyset$. By the maximality of P and $|S_{i_{r_i+1}}| \geq 2$, we have that $S_{i_{r_i+1}} = \{u_{i_1}, u_{i_{r_i+1}}\}$. Since $|\bigcup_{1 \leq j \leq r} S_j| \geq r$ for any $r \leq k$, we need only to consider the following two cases.

Case 1. $|\bigcup_{1 \leq j \leq r_i+1} S_j| = r_i + 1$.

Then $|V(G_0)| = k = r_i + 1$ by condition 4). Thus $C = u_{i_1}\cdots u_{i_{r_i+1}}u_{i_1}$ is a cycle of G_0 with $|V(C)| = |G_0|$ satisfying (*).

Case 2. $|\bigcup_{1 \leq j \leq r_i+1} S_j| > r_i + 1$.

By condition 3), there is a $l \in \{1, \cdots, r_i\}$ such that $|S_{i_l}| \geq 3$. We assume without loss of generality that $\{x_{i_l}, y_{i_l}, z_{i_l}\} \subseteq S_{i_l}$ satisfying $x_{i_l} = u_{i_l}$, $y_{i_l} = u_{i_l+1}$ and $z_{i_l} \notin$

$V(P)$. Then we can construct a new path $P' = z_{i_t} u_{i_t} u_{i_t - 1} \cdots u_{i_1} u_{i_{r_i+1}} u_{r_i} \cdots u_{i_t+1}$ which is longer than P and satisfies (**) when the subscripts are rewritten. This contradiction completes the proof of the Claim.

Now, from the Claim above, if G_0 contains a cycle C with $V(C) = |G_0|$ satisfying (*), then it is easy to check that G is hamiltonian. Otherwise, by the Claim above, G_0 contains q pairwise disjoint paths $P_i = u_{i_1} u_{i_2} \cdots u_{i_{r_i+1}}$, $i = 1, 2, \cdots q$ which satisfy both (**) and (***), and we have $\sum_{i=1}^{q} r_i = k$. Since G_0 is a complete subgraph of G, we can easily check that G has a hamiltonian cycle.

Therefore, Theorem 1 is true. ◇

Theorem 1 has the following consequence.

Corollary 1. Let G_0 be a complete subgraph of G with $|G_0| \geq 2$. If G_0 contains a pair of useful vertices of each component of $G \setminus G_0$ and $G[N(G_0)]$ is C–free, then G is hamiltonian.

Proof. Let $G_1, \cdots, G_k$ be all the components of $G \setminus G_0$ and set $G^* = G[N(G_0)]$. By Theorem 1, it is sufficient to show that $G_0, G_1, \cdots, G_k$ is a L-decomposition of G.

By the hypothesis, we can choose $S_i \subseteq V_{G_0}(G_i)$ such that S_i satisfies 2) of Definition 2 and $|S_i|$ is as large as possible. Since G^* is C–free, 3) of Definition 2 is satisfied. Thus we only need to show that 4) of Definition 2 is also satisfied.

In fact, let $r \leq k$ be any positive integer. Since G^* is C-free, we have $|V(G_0)| \geq k \geq r$. If $|V(G_0)| = k = r$, then $|\bigcup_{1 \leq j \leq r} S_{i_j}| = r$. Conversely, if $|\bigcup_{1 \leq j \leq r} \boldsymbol{S_{i_j}}| = r$, then $|S_{i_j}| = 2$ $(j = 1, 2, \cdots, r)$ and each vertex $x \in \bigcup_{1 \leq j \leq r} S_{i_j}$ is a common vertex of some two pairs of useful vertices. Let $x \in S_{i_s} \cap S_{i_t}$ and $y \in N_{G_{i_s}}(x)$, $z \in N_{G_{i_t}}(x)$. When $|G_0| > r$, then there exists some $w \in V(G_0) \setminus (\bigcup_{1 \leq j \leq r} S_{i_j})$. Since G^* is C-free, we have $wy \in E$ or $wz \in E$. Therefore, either $S_{i_s} \cup \{w\}$ or $S_{i_t} \cup \{w\}$ still satisfies 2) of Definition 2, which contrary to the choice of S_{i_s} or S_{i_t}. Thus $|V(G_0)| = k = r$. This completes the proof of Corollary 1. ◇

3 The Proof of Theorem 5

In order to prove Theorem 5, we need the following theorem.

Theorem 6[5]. If G is 3-connected and CN–free, then for any distinct vertices x, y of G, there exists a hamiltonian path connecting x and y.

Now, set $V_0 = \{x \in V(G) : d(x) \geq n/2\}$. By Theorem 2, we may assume that $G[V_0]$ is a complete subgraph of G if $V_0 \neq \emptyset$. Let G_0 be a complete subgraph of G such that $V_0 \subseteq V(G_0)$ and $|V(G_0)|$ is as large as possible. Let $G_1, \cdots, G_k$ be all the components of $G \setminus G_0$. Then by the property $CB(n)$, $G[N(G_0)]$ is C–free and G_s is CB–free for any $1 \leq s \leq k$. By Corollary 1, we need only to show that G_0 contains a pair of useful vertices of G_s for $1 \leq s \leq k$.

Assume that there is a component G_s of $G \setminus G_0$ such that G_0 does not contain any pair of useful vertices of G_s. Let S be a minimal cut vertex set of G_s and $v \in S$. Then by the assumption and Theorem 6, $|S| \leq 2$. Since G_s is C–free, $G_s \setminus S$ has only two components H_1, H_2. Let $H = G[V(H_1) \cup V(H_2) \cup \{v\}]$ and $S_{-i} = \{u \in V(H_1) : d_H(u,v) = i\}$ and $S_i = \{u \in V(H_2) : d_H(u,v) = i\}$ for $i \geq 0$. Denote $m := \max\{i : S_i \neq \emptyset\}$ and $n := \max\{i : S_{-i} \neq \emptyset\}$. Clearly, we have $V(G_s) = S \bigcup (\bigcup_{i=-n}^{m} S_i)$, and $G[S_i \cup S_j]$ is complete if and only if $|i-j| = 1$ since G_s is CB-free.

If $|S| = 1$, then there exist some $x \in S_m$ and $y \in S_{-n}$ such that neither x nor y is a cut vertex of G_s and $N_{G_0}(x) \neq \emptyset$ and $N_{G_0}(y) \neq \emptyset$, since G is 2-connected. Because of the structure of G_s, there exists a path P connecting x and y in G_s with $V(P) = V(G_s)$. Thus by the assumption, $N_{G_0}(x) = N_{G_0}(y)$ and $|N_{G_0}(x)| = 1$, which contrary to the fact that $G[N(G_0)]$ is C–free.

If $|S| = 2$, let $v' \in S$ and $v' \neq v$. Since G_s is 2-connected, $N(v') \cap S_i \neq \emptyset$ for some $1 \leq i \leq m$ and $N(v') \cap S_{-j} \neq \emptyset$ for some $1 \leq j \leq n$. Let $i_0 = \max\{i : N(v') \cap S_i \neq \emptyset\}$ and $j_0 = \max\{j : N(v') \cap S_{-j}\}$. By the hypothesis of Theorem 5, we may assume that there exists some t with $0 \leq t \leq m$ such that $N_{G_0}(S_t) \neq \emptyset$.

Since G_s is 2-connected, we have

(a) $|S_i| \geq 2$ for any $m-1 \geq i \geq i_0$ and $|S_{-j}| \geq 2$ for any $n-1 \geq j \geq j_0$.

By (a) and the structure of G_s, we have

(b) If $|S_m| \geq 2$, then for any two distinct vertices x and y in S_m, there exists a path P in G_s connecting x and y with $V(P) = V(G_s)$.

(c) For any $x \in S_{i-1}$ and $y \in S_i$ $(1 \leq i \leq m)$, there exists a path in G_s connecting x and y with $V(P) = V(G_s)$.

Since $|N_{G_0}(G_s)| \geq 2$. By the assumption, (c) and the hypothesis of Theorem 5, we have

(d) $n + m \geq 3$.

Now, we distinguish the following two cases.

Case 1. $0 \leq t < m$, that is there exists some $x \in S_t$ and $y \in V(G_0)$ such that $xy \in E$.

Then by the hypothesis of Theorem 5 and $1 \leq t < m$, there exists a vertex $z \in S_{t-1}$ or $z \in S_{t+1}$ such that $yz \in E$. By the assumption and (c), for any $y' \in V(G_0) \setminus \{y\}$ and $w \in S_{t-1} \cup S_{t+1}$, $y'w \notin E$. Thus we can find a vertex set $F = \{x, y, z, y', w\}$ such that $G[F]$ is a B–graph and does not satisfy the condition of Theorem 5, a contradiction.

Case 2. For any $0 \leq i \leq m-1$, $N_{G_0}(S_i) = \emptyset$, that is $t = m$.

Symmetrically, we may assume that for any $0 \leq j \leq n-1$, $N_{G_0}(S_{-j}) = \emptyset$.

If $N_{G_0}(v') \neq \emptyset$, let $y \in V(G_0)$ such that $v'y \in E$. Then by the hypothesis of Theorem 5, we have $y \in N_{G_0}(S_{i_0})$ or $y \in N_{G_0}(S_{-j_0})$. Thus $i_0 = m$ or $j_0 = n$.

Without loss of generality, let $y \in N_{G_0}(S_{i_0})$. When $y \notin N_{G_0}(S_{-j_0})$, then by the hypothsis of Theorem 5, there exists a vertex $y' \in V(G_0) \setminus \{y\}$ such that $v'y' \in E$ or $y' \in N_{G_0}(S_m)$ or $y' \in N_{G_0}(S_{-j_0})$ whenever $j_0 = n$. By the structure of G_s, we can derive that $\{y, y'\}$ is a pair of useful vertices of G_s, contrary to the assumption. When $y \in N_{G_0}(S_{i_0}) \cap N_{G_0}(S_{-j_0})$, that is $i_0 = m$ and $j_0 = n$. Since G is 2-connected, there exists a vertex $y' \in V(G_0)$ such that $y' \in N_{G_0}(S_m) \cup N_{G_0}(S_{-n})$ or $v'y' \in E$. Also by the structure of G_s, we can derive that $\{y, y'\}$ is a pair of useful vertices of G_s, contrary to the assumption. Hence in rest proof we suppose that $N_{G_0}(v') = \emptyset$.

Since G is 2-connected, there exist $x \neq x' \in S_m \cup S_{-n}$ and $y \neq y' \in V(G_0)$ such that $xy \in E$ and $x'y' \in E$. By the assumption and (b), $\{x, x'\} \not\subseteq S_m$ and $\{x, x'\} \not\subseteq S_{-n}$. Let $x \in S_m$ and $x' \in S_{-n}$. By (d), let $m \geq 2$, then $S_m \in N(y)$ by the hypothesis of Theorem 5.

If $i_0 = m$, then $S_{m-1} \subseteq N(v')$ by the hypothesis of Theorem 5. Thus by the structure of G_s, we can derive that $\{y, y'\}$ is a pair of useful vertices of G_s, contrary to the assumption. If $i_0 < m$, then $S_{i_0-1} \subseteq N(v')$. Thus we can also get a contradiction as before.

Therefore, Theorem 5 is true.

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