Applications of Narayana's Formula

Karl H. Wehrhahn

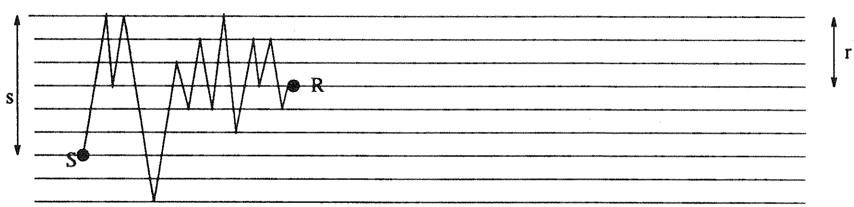
School of Mathematics and Statistics, University of Sydney,
Sydney, N.S.W., Australia 2006.

Abstract. We adapt a formula due to T.V. Narayana to count ray paths from a source to a receiver according to the number of times the paths change direction. In the simplest case we show that Narayana's formula leads to a hook-sum formula for $L(m,n)$ posets.

§1. Introduction and Motivation

This paper arose in an attempt to solve a problem of counting certain ray paths in seismology. The key ingredient of the solution is a formula first discovered by Narayana [Nara 55].

To describe our problem, we consider a profile of the earth with layers at equal intervals as shown in the diagram. Shock waves are generated by a source (**S**) at the sth layer and we assume they take one unit of time to traverse a layer.[1]



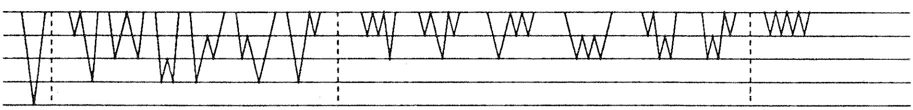
[1] In the terminology of seismology this means that all waves traversing the same number of layers are kinematic analogues [see CHP 89].

Each time a wave is reflected from the boundary of a layer, its contribution to the seismographic record is reduced. To interpret this record, and to construct synthetic seismograms, it is of interest to know the proportion of waves which arrive at the receiver (**R**) at the rth layer after having traversed $n+m$ layers, and having been reflected ℓ times where $\ell = 0, 1, 2, \ldots$. Here n is the number of layers traversed in a downward direction and m is the number traversed in an upward direction. We say that $n+m$ is the *length* of the wave.

It is clear that we can represent each wave of length $n+m$ as a string $X_1X_2\cdots X_{n+m}$ of D's (for *downward*) and U's (*upward*) which is subject to the condition that the difference between the number of U's and the number of D's is at most s for each substring of the form $X_1X_2\cdots X_i$. We need this *surface condition* to ensure that the wave does not disappear from the 0th layer. However we make no restriction on how deep the wave may penetrate the earth.

In the case $r = s = 0$ (receiver and source are at the surface of the earth) we have $n = m$ and strings satisfying the above condition are often called *balanced*. It is well-known that the total number of balanced strings of length $2n$ is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

For example, if $r = s = 0$ and $n = m = 4$, one wave is reflected once, six waves are reflected three times, six waves are reflected five times and one wave is reflected seven times, for a total of $C_4 = 14$, as illustrated in the following diagram.



More generally, let $T^{ni}_{mk}(s)$ be the number of strings of length $n+m$ which contain k pairs of the form DU and i pairs of the form UD (representing a wave which is reflected $i+k$ times) generated by a source s layers below the surface. We write $T^{ni}_{mk} = T^{ni}_{mk}(0)$.

Clearly, $|i-k| \leq 1$. In particular, if $r = s = 0$ and $n = m$ we have the identity,

$$C_n = T^{n0}_{n1} + T^{n1}_{n2} + \cdots + T^{nn-1}_{nn}.$$

By a formula of Narayana [Nara 55, Nara 59], $T^{nk-1}_{nk} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. In the above example, $T^{40}_{41} = 1$, $T^{41}_{42} = 6$, $T^{42}_{43} = 6$, $T^{43}_{44} = 1$ as expected.

In Section 2 we show how a generalization of Narayana's formula [Nara 79] can be used to find the numbers $T^{ni}_{mk}(s)$ in general. In Sections 3 and 4 we consider in more detail the case when $s = 0$. In this case it is well known [Nara 59] that the surface condition defines a partial order on the relevant sets of compositions. We show in Section 3 that the numbers T^{ni}_{mk} are sums of hook-lengths in the posets of compositions. These posets turn out to be $L(m,n)$ posets which form a subclass of the $sl(2,C)$ posets [Stan 80, Proc 82] characterized by the properties that they are graded, rank symmetric, rank unimodal and strongly Sperner. We are unable to find a proof of this correspondence in the literature hence in Section 4 we provide a simple proof.

§2. Reflecting Waves and Compositions

In this section we formulate the problem of finding $T^{ni}_{mk}(s)$ in terms of compositions. Recall that a *composition* of n into k parts is a sum $n_1 + n_2 + \cdots + n_k = n$ of positive integers n_i.

The first observation is that DU–strings with a source at level s come in four varieties according to how they begin and end, i.e., D–U strings, D–D strings, U–D strings and U–U strings.

(D–U Paths). We consider first strings of the form D–U. If such a string has k pairs of the form DU and i pairs of the form UD then $i = k - 1$ and the D's correspond to a composition of n into k parts, the U's correspond to a composition of m into k parts.

For example the string $DDDUDDUUDU$ has 3 DU pairs and 2 UD pairs yielding the compositions $3 + 2 + 1$ of 6 and $1 + 2 + 1$ of 4.

Because of the surface condition, not every composition of the m U's into k parts is compatible with a given composition of the n D's into k parts. Given compositions of n and m into k parts, say $n_1 + n_2 + \cdots + n_k = n$ and $m_1 + m_2 + \cdots + m_k = m$, the surface condition is equivalent to the system of inequalities:

$$
\begin{aligned}
m_1 - n_1 &\le s \\
m_1 + m_2 - (n_1 + n_2) &\le s \\
\vdots \qquad & \quad \vdots \\
m_1 + m_2 + \cdots + m_{k-1} - (n_1 + n_2 + \cdots + n_{k-1}) &\le s \\
m - n &\le s
\end{aligned}
$$

In the language of [Nara 1959], the composition $n_1 + n_2 + \cdots + n_k = n$ is said to *s–dominate* the composition $m_1 + m_2 + \cdots + m_k = m$. The total number of pairs of compositions $(n_1 + n_2 + \cdots + n_k = n, m_1 + m_2 + \cdots + m_k = m)$ in which $n_1 + n_2 + \cdots + n_k = n$ s–dominates $m_1 + m_2 + \cdots + m_k = m$, is shown by Narayana to be

$$DU_{mk}^{nk-1}(s) := \binom{n-1}{k-1}\binom{m-1}{k-1} - \binom{n-1+s}{k-2}\binom{m-1-s}{k}. \tag{1}$$

For example, if $n = 4$, $m = 3$, $k = 2$ and $s = 1$ then $DU_{32}^{41}(1) := \binom{3}{1}\binom{2}{1} - \binom{4}{0}\binom{1}{2} = 6$ which counts the six strings $DUDDDUU$, $DUUDDDU$, $DDUDDUU$, $DDUUDDU$, $DDDUDUU$ and $DDDUUDU$ corresponding to the pairs of compositions $(1+3, 1+2)$, $(1+3, 2+1)$, $(2+2, 1+2)$, $(2+2, 2+1)$, $(3+1, 1+2)$, $(3+1, 2+1)$ respectively.

Note that the $D-U$ paths are the only ones in which the number of UD pairs is exactly one less than the number of DU pairs. Hence

$$T_{mk}^{n,k-1}(s) = DU_{mk}^{n,k-1}(s).$$

(D–D Paths). If such a string has k pairs of the form DU and i pairs of the form UD then $i = k$ and the D's correspond to a composition of n into $k+1$ parts, the U's correspond to a composition of m into k parts. We can classify strings of this form according to the number D's at the end of the string. The number of such strings ending with j D's is just $DU_{mk}^{n-j,i-1} = DU_{mk}^{n-j,k-1}(s)$. Note that we must have $n - j \ge k$ and $n - j - m + s \ge 0$ or $j \le n - m + s$. Hence the total number of such strings is

$$
\begin{aligned}
DD^{nk}_{mk}(s) &:= \sum_{j=1}^{n-m+s} DU^{n-j,k-1}_{mk}(s) \\
&= \sum_{j=1}^{n-m+s} \left[\binom{n-j-1}{k-1}\binom{m-1}{k-1} - \binom{n-j-1+s}{k-2}\binom{m-1-s}{k} \right] \\
&= \binom{m-1}{k-1}\binom{n-1}{k} - \binom{m-1-s}{k}\binom{n-1+s}{k-1}, \qquad (2)
\end{aligned}
$$

where we have applied the summation formula for binomial coefficients twice.

(U–U Paths). If such a string has k pairs of the form DU and i pairs of the form UD then $i = k$ and the D's correspond to a composition of n into k parts, the U's correspond to a composition of m into $k+1$ parts. We can classify strings of this form according to the number U's at the beginning of the string. The number of such strings beginning with j U's is just $DU^{n,k-1}_{m-j,k}(s-j)$. Hence the total number of such strings is

$$
\begin{aligned}
UU^{nk}_{mk}(s) &:= \sum_{j=1}^{s} DU^{n,k-1}_{m-j,k}(s-j) \\
&= \sum_{j=1}^{s} \left[\binom{n-1}{k-1}\binom{m-j-1}{k-1} - \binom{n-1+s-j}{k-2}\binom{m-j-1-s+j}{k} \right] \\
&= \binom{m-1}{k}\binom{n-1}{k-1} - \binom{m-1-s}{k}\binom{n-1+s}{k-1}. \qquad (3)
\end{aligned}
$$

(U–D Paths). If such a string has $k-1$ pairs of the form DU and i pairs of the form UD then $i = k$ and the D's correspond to a composition of n into k parts, the U's correspond to a composition of m into k parts. We can classify strings of this form according to the number of U's at the beginning of the string. The number of such strings beginning with j U's is just $DD^{n,k-1}_{m-j,k-1}(s-j)$. Hence the total number of such strings is

$$
\begin{aligned}
UD^{nk}_{m,k-1}(s) &:= \sum_{j=1}^{s} DD^{n,k-1}_{m-j,k-1}(s-j) \\
&= \sum_{j=1}^{s} \left[\binom{n-1}{k-1}\binom{m-j-1}{k-2} - \binom{m-j-s+j-1}{k-1}\binom{n-1+s-j}{k-2} \right] \\
&= \binom{m-1}{k-1}\binom{n-1}{k-1} - \binom{m-1-s}{k-1}\binom{n-1+s}{k-1}. \qquad (4)
\end{aligned}
$$

Conclusions.

(1) We can now find $T^{ni}_{mk}(s)$ for the three possibilities $i = k-1$, $i = k$ or $i = k+1$. Since

$$T^{ni}_{mk}(s) = DU^{ni}_{mk}(s) + DD^{ni}_{mk}(s) + UD^{ni}_{mk}(s) + UU^{ni}_{mk}(s),$$

we have,

$$\begin{aligned} T^{n,k-1}_{mk}(s) &= DU^{n,k-1}_{mk}(s) \\ T^{nk}_{mk}(s) &= DD^{nk}_{mk}(s) + UU^{nk}_{mk}(s) \\ T^{n,k+1}_{mk}(s) &= UD^{n,k+1}_{mk}(s) \end{aligned}$$

(2) The total number of ray paths with n D's and m U's with source at level s and reflected $2k$ times on the way to the receiver is the sum of (2) and (3), i.e.,

$$\begin{aligned} DD^{nk}_{mk}(s) + UU^{nk}_{mk}(s) = \binom{m-1}{k-1}\binom{n-1}{k} + \binom{m-1}{k}\binom{n-1}{k-1} \\ - 2\binom{m-1-s}{k}\binom{n-1+s}{k-1}. \end{aligned}$$

The total number of ray paths with n D's and m U's with source at level s and reflected $2k-1$ times on the way to the receiver is the sum of (1) and (4), i.e.,

$$\begin{aligned} DU^{n,k-1}_{mk}(s) + UD^{nk}_{m,k-1}(s) = 2\binom{m-1}{k-1}\binom{n-1}{k-1} - \binom{m-1-s}{k}\binom{n-1+s}{k-2} \\ - \binom{m-1-s}{k-1}\binom{n-1+s}{k-1}. \end{aligned}$$

(3) Let $R(s,r,n,m)$ be the total number of ray paths with source at level s, receiver at level r with n D's and m U's. Then

$$
\begin{aligned}
R(s,r,n,m) &= \sum_{k\geq 1} DU_{mk}^{n,k-1}(s) + \sum_{k\geq 1} DD_{mk}^{n,k}(s) + \sum_{k\geq 1} UD_{mk-1}^{n,k}(s) + \sum_{k\geq 1} UU_{mk(s)}^{n,k} \\
&= \sum_{k\geq 1}\left[\binom{n-1}{k-1}\binom{m-1}{k-1} - \binom{n-1+s}{k-2}\binom{m-1-s}{k}\right] \\
&= \sum_{k\geq 1}\left[\binom{n-1}{k}\binom{m-1}{k-1} - \binom{n-1+s}{k-1}\binom{m-1-s}{k}\right] \\
&= \sum_{k\geq 1}\left[\binom{n-1}{k-1}\binom{m-1}{k-1} - \binom{n-1+s}{k-1}\binom{m-1-s}{k-1}\right] \\
&= \sum_{k\geq 1}\left[\binom{n-1}{k-1}\binom{m-1}{k} - \binom{n-1+s}{k-1}\binom{m-1-s}{k}\right] \\
&= \binom{n+m}{n} - \binom{n+m}{n+s+1}
\end{aligned}
$$

Here we have used the identity

$$
\sum_{k\geq 0}\binom{n}{k}\binom{m}{k+\ell} = \binom{n+m}{n+\ell}
$$

several times and simplified.

For example in the case $s=r=0$ we have from $m=n+s-r$ that $n=m$ and we obtain the familiar result

$$
R(0,0,n,n) = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} = C_n,
$$

the n-Catalan number.

(4) The proportion of ray paths with source at level s, receiver at level r with n D's and m U's which are reflected $2k$ times is given by the ratio

$$
\left[DD_{mk}^{nk}(s) + UU_{mk}^{nk}(s)\right]/R(s,r,n,m).
$$

The proportion of ray paths with source at level s, receiver at level r with n D's and m U's which are reflected $2k-1$ times is given by the ratio

$$
\left[DU_{mk}^{n,k-1}(s) + UD_{m,k-1}^{nk}(s)\right]/R(s,r,n,m).
$$

§3. The Poset of Compositions

In the case $s = 0$, it is well known [Nara 59] that the set of compositions of n into k parts form a partially ordered set, which we denote by P_{nk}, with respect to dominance order. Recall [Stan 86, Chapter 3] that for a poset P, the principal order ideal $< x >$ of $x \in P$ is

$$< x > = \{y \in P | y \leq x\}.$$

The hook-length h_x of $x \in P$ is $h_x = | < x > |$.
Hence if $n = m$ we can interpret the $T_{nk}^{n,k-1}$ as the sum of the hook-lengths in the poset P_{nk}. Thus from (1),

$$T_{nk}^{n,k-1} = \sum_{x \in P_{nk}} h_x = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}.$$

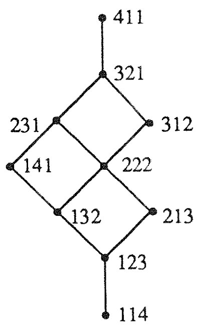
More generally, from (1) we have,

$$\begin{aligned} T_{mk}^{n,k-1} &= \sum_{x \in P_{nk}} \sum_{\substack{y \in P_{mk} \\ y \leq x}} 1 \\ &= \binom{m-1}{k-1}\binom{n-1}{k-1} - \binom{n-1}{k-2}\binom{m-1}{k} \qquad (5) \\ &= \frac{m + k(n-m)}{nm}\binom{n}{k-1}\binom{m}{k}. \end{aligned}$$

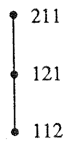
Example 1. Consider the case $n = 6$, $m = 4$. Then $T_{41}^{60} = 1$, $T_{42}^{61} = 12$, $T_{43}^{62} = 25$, $T_{44}^{63} = 10$. The total number of ray paths of the form $D - U$ is given by

$$\frac{(n - m + 2)m}{(n + m)(n + 1)}\binom{m+n}{m} = \frac{4 \cdot 4}{10 \cdot 7}\binom{10}{4} = 48,$$

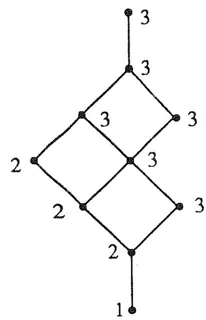
as expected. We can also verify this directly from the posets as shown. For $k = 3$ the entries on the right-hand copy of P_{63} are the number of elements of P_{43} dominated by the corresponding element of P_{63}. The total of these is $T_{43}^{62} = 25$.



P_{63}



P_{43}



§4. Relationship to L(m,n) Posets

The purpose of this section is to give a simple proof that the posets P_{nk} are isomorphic to a certain subclass of the $sl(2,\mathbb{C})$ (cf. [Stan 80]) posets which we now define.

Definition. The poset $L(m,n)$ consists of n-tuples $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ satisfying

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq m$$

ordered by the condition $\mathbf{a} \leq \mathbf{b}$ if and only if

$$a_1 \leq b_1, \ a_2 \leq b_2, \ldots, a_n \leq b_n.$$

Note that we can represent $(a_1, a_2, \ldots, a_n)$ by a left-justified array of boxes in which the ith column has a_{n-i+1} boxes. Then the elements of $L(m,n)$ can be interpreted as Young diagrams which fit into an $m \times n$ grid. We have $\mathbf{a} \leq \mathbf{b}$ provided the Young diagram of $\mathbf{a}$ is contained in the Young diagram of $\mathbf{b}$.

Now let $Q(m,n)$ be the poset whose elements are all n-tuples $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ satisfying

$$0 < c_1 < c_2 < \cdots < c_n \leq m+n$$

ordered by the condition $\mathbf{c} \leq \mathbf{d}$ if and only if

$$c_1 \leq d_1, \ c_2 \leq d_2, \ldots, c_n \leq d_n.$$

The mapping $\mathbf{a} \mapsto \mathbf{c}$ defined by $c_i = a_i + i$ clearly defines an isomorphism between $L(m,n)$ and $Q(m,n)$.

Theorem.

$$P_{nk} \simeq Q(n-k, k-1) \simeq L(n-k, k-1) \simeq L(k-1, n-k).$$

Proof. The second isomorphism has been described above. The third is determined by taking transposes of Young diagrams. It suffices to find an isomorphism between P_{nk} and $Q(n-k, k-1)$.

First note that the poset $Q(n-k, k-1)$ consists of all $(k-1)$-tuples $\mathbf{b} = (b_1, b_2, \ldots, b_{k-1})$ satisfying the condition

$$0 < b_1 < b_2 < \cdots < b_{k-1} \leq n-1. \tag{6}$$

The number of such sequences is clearly $\binom{n-1}{k-1}$ so at least we have

$$|P_{nk}| = |Q(n-k, k-1)|.$$

From (6) we can obtain a composition

$$c_1 + c_2 + \cdots + c_k = n$$

of n into k parts by

$$\begin{aligned} c_1 &= b_1 \\ c_2 &= b_2 - b_1 \\ \vdots \quad & \quad \vdots \\ c_{k-1} &= b_{k-1} - b_{k-2} \\ c_k &= n - b_{k-1} \end{aligned}$$

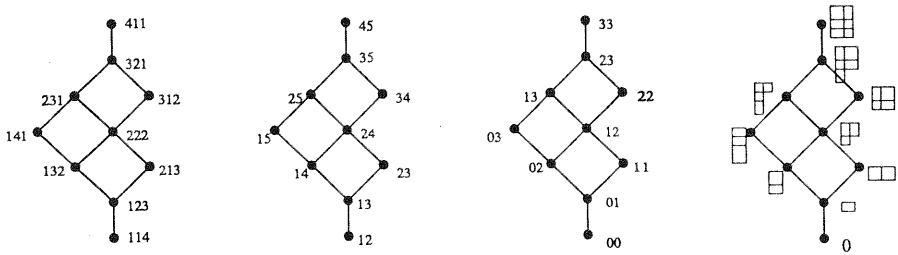
Conversely, given a composition of n into k parts, $c_1 + c_2 + \cdots + c_k = n$ we obtain a strictly increasing sequence $0 < b_1 < b_2 < \cdots < b_{k-1} \leq n-1$ of $k-1$ terms, with each $b_i \leq n-1$, by

$$\begin{aligned} b_1 &= c_1 \\ b_2 &= c_1 + c_2 \\ \vdots \quad & \quad \vdots \\ b_{k-1} &= c_1 + c_2 + \cdots + c_{k-1} \end{aligned}$$

These correspondences are clearly inverse to one another and preserve order.

Q.E.D

Example 2. If $n = 6,\ m = 4$ we have the following correspondences.



P_{63} Q(3,2) L(3,2) L(3,2)

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