

Transgression and the calculation of cocyclic matrices

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Abstract

It is conjectured that binary cocyclic matrices are a uniform source of Hadamard matrices. In testing this conjecture, it is useful to have a general method of calculating cocyclic matrices. We present such a method in this paper. The method draws on standard cohomology theory of finite groups. In particular we employ the Universal Coefficient Theorem, which expresses the second cohomology group explicitly as an internal direct sum of two subgroups. One subgroup arises as the image of a transgression homomorphism. The method reduces essentially to determination of (representative cocycles for) the image of transgression. There is a resultant description of a given cocyclic matrix as the Hadamard product of certain matrices. The factors in the product generally are not canonically determined, and this may be significant in the development of algorithms for calculating cocyclic Hadamard matrices. An example is given to illustrate the method.

1 Introduction

It is hoped that cocyclic matrices will provide a uniform approach to the Hadamard Conjecture. Specifically, de Launey and Horadam make the following *Cocyclic Hadamard Conjecture* in [1]:

For all $t \geq 1$, there is a $4t \times 4t$ cocyclic matrix that is Hadamard.

So far, no counterexamples to this conjecture are known, and there are Hadamard matrices of encouragingly many types which are known to be cocyclic. To make further progress in testing the Cocyclic Hadamard Conjecture, it is natural to examine more closely the abstract problem of calculating cocyclic matrices.

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de Launey and Horadam, in [1] and [4], describe calculation of cocyclic Hadamard matrices developed over abelian groups (see also [3]). As it stands, their method does not extend to the context of cocycles developed over groups which are not necessarily abelian. In this paper, we describe a general method which may be used in the latter context. In particular, we draw on standard results from the cohomology theory of finite groups to calculate *representative* cocyclic matrices. An important feature of the method is that its output is not canonically determined. It might be possible to exploit this feature in generating Hadamard matrices.

The requisite notation and standard cohomology theory are set up in Sections 2 and 3. This background enables us to define the main calculational tool employed, a homomorphism called transgression, in Section 4. By way of illustration, in Section 5 we use our method to determine all binary cocyclic matrices developed over the dihedral group of order 8. In Section 6, the non-canonical nature of the method is explored, with reference to this example.

2 Cohomological Preliminaries

Throughout, G will be a finite group and U a finite abelian group on which G acts trivially—that is, each element of G fixes each element of U . A (2-)cocycle from G to U is a set map $\psi: G \times G \rightarrow U$ such that $\psi(g, h)\psi(gh, k) = \psi(g, hk)\psi(h, k)$ for all $g, h, k \in G$. Binary cocycles (those mapping into $\mathbb{Z}_2 = \langle -1 \rangle$) are used in [3]. Here, we will preserve generality and replace \mathbb{Z}_2 with U as much as possible. All cocycles considered will be *normalised*: $\psi(1, 1) = 1$. The set of all 2-cocycles from G to U forms an abelian group $Z^2(G, U)$ under the obvious pointwise multiplication • defined by

$$(\psi_1 \bullet \psi_2)(g, h) = \psi_1(g, h)\psi_2(g, h).$$

Certainly one can always manufacture 2-cocycles. Given a set map $\phi: G \rightarrow U$ such that $\phi(1) = 1$, the assignment

$$\psi(g, h) = \phi(g)\phi(h)\phi(gh)^{-1}$$

defines a 2-cocycle ψ from G to U , called a (2-)coboundary (referred to as a *principal cocycle* in [3]). The set of all 2-coboundaries is a subgroup $B^2(G, U)$ of $Z^2(G, U)$. In many situations, it is necessary only to consider the cocycles up to equivalence modulo coboundaries. The abelian quotient group

$$H^2(G, U) = Z^2(G, U)/B^2(G, U)$$

is called the *second cohomology group of G with (trivial) coefficients in U* .

The second cohomology group is an object of considerable interest in several areas of algebra. We describe one well-known application to group extension problems. Each central extension of U by G gives rise to a 2-cocycle in the following way. Suppose that

$$1 \rightarrow U \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1$$

is a short exact sequence of groups, and choose a normalised transversal map $\sigma: G \rightarrow E$ for the central extension of U by G so specified: by this it is meant that $\pi\sigma$ is the identity on G , and $\sigma(1) = 1$. Then $\mu_\sigma: G \times G \rightarrow U$ defined by

$$\mu_\sigma(g, h) = \iota^{-1}(\sigma(g)\sigma(h)\sigma(gh)^{-1})$$

is a 2-cocycle from G to U . If σ' is another normalised transversal map then $\mu_{\sigma'}$ lies in the same cohomology class as μ_σ —indeed, each cocycle in the class of μ_σ arises in this way. Conversely, given $\mu \in Z^2(G, U)$, one may define a central extension of U by G which gives rise to a 2-cocycle in the same cohomology class as μ . There is consequently a one-to-one correspondence between the elements of $H^2(G, U)$ and the set of equivalence classes of central extensions of U by G (equivalence of extensions will not be defined). This characterisation of $H^2(G, U)$ is of obvious usefulness in group extension problems, where typically one is given G and U and required to list all possibilities for E (up to isomorphism, for instance).

Suppose $\psi \in Z^2(G, U)$. A *cocyclic matrix associated to ψ , developed over G* , is a $|G| \times |G|$ matrix of the form

$$(\psi(g, h))_{g, h \in G}.$$

This is a natural way of explicitly displaying the action of ψ . We wish to concentrate on the problem of determining, for given G and U , a full set of cocyclic matrices developed over G . The solution of this problem may be broken down into two steps. The first is to calculate a full set of representative cocyclic matrices. The second is to calculate all coboundary matrices. Each cocyclic matrix is the Hadamard (coordinatewise) product of two matrices, one obtained in each step. In this paper, we provide a method of carrying out the first step. The second step probably should be turned over to a computer, given the number of 2-coboundaries involved: this number grows approximately exponentially as the order of G .

It may be shown that the theory presented in [3] (drawn from [1] and [4]) is a specialisation of that to be discussed here. An advantage of the broader, algebraic approach is that it allows us to utilise the considerable machinery established over the years in the study of cohomology groups. On the other hand, traditional cohomological calculations, especially with reference to group extension problems, involve determination of the isomorphism type or order of cohomology groups, rather than the explicit calculation of cocycles.

3 The Universal Coefficient Theorem

Our method of calculating representative cocyclic matrices is motivated by the well-known *Universal Coefficient Theorem*. We make some definitions preparatory to stating this result.

The famous *Schur multiplier* of G , $H_2(G)$, is usually defined to be $H^2(G, \mathbb{C}^*)$, where \mathbb{C}^* denotes the multiplicative group of the field \mathbb{C} of complex numbers. However, in the sequel we will use an equivalent definition of $H_2(G)$ made via the choice of a presentation for G . By $\text{Hom}(H_2(G), U)$ we denote the set of all homomorphisms

between the finite abelian groups $H_2(G)$ and U ; this is itself an abelian group under pointwise multiplication of maps.

We now wish to isolate a certain subgroup K of $H^2(G, U)$, but it is not fruitful to define K precisely. (Matrices associated to representative cocycles chosen for the elements of K are symmetric, and identified as such in [3].) Instead, for illustrative purposes, a brief description of these matrices will be given in the case $U = \mathbb{Z}_2$.

Denote by G' the derived subgroup of G . Since G/G' is a finite abelian group, it has an (essentially unique) primary invariant decomposition as a direct product of cyclic groups of prime-power order. Suppose the 2-torsion subgroup of this direct product is

$$\mathbb{Z}_{2^{e_1}} \times \cdots \times \mathbb{Z}_{2^{e_n}},$$

and set $m = \sum_{i=1}^n e_i$. For each i , $1 \leq i \leq n$, select either the $2^{e_i} \times 2^{e_i}$ back negacyclic matrix or $J_{2^{e_i}}$ (the $2^{e_i} \times 2^{e_i}$ all 1s matrix), and then form the Kronecker product of the n matrices so chosen together with J_r , where $r = |G|/2^m$. For example, if $G/G' \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ then $r = 3|G'|$ and the set of all such matrices is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & A \\ 1 & 1 & A & A \\ 1 & A & A & A \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & B \end{pmatrix} \otimes J_r,$$

where $A, B \in \{-1, 1\}$. Returning to the case of general G , it may be seen easily that each matrix formed in the stated way is in fact cocyclic over G , and the cohomology classes of the associated cocycles are pairwise distinct. These cohomology classes are the elements of the subgroup K of $H^2(G, \mathbb{Z}_2)$. For general U , K is defined similarly via the primary invariant decomposition of G/G' . A fundamental property of K is that it is complemented in $H^2(G, U)$, as the following theorem states.

Theorem 3.1 (Universal Coefficient Theorem) *There is an embedding*

$$\tau: \text{Hom}(H_2(G), U) \hookrightarrow H^2(G, U)$$

such that $H^2(G, U) = K \oplus \text{im}(\tau)$. □

Representatives for the elements of $\text{im}(\tau)$ are called *commutator cocycles* in [3]. For further discussion of the Universal Coefficient Theorem, see [2], pp.179-180.

Theorem 3.1 provides the motivation for our method. Since there is a well-understood method of calculating representative cocyclic matrices for the elements of K , the overall problem reduces essentially to the subproblem of calculating $\text{im}(\tau)$. In the next section, a method of solving this subproblem is given.

4 Transgression

Our goal in this section is to exhibit a homomorphism of $\text{Hom}(H_2(G), U)$ into $H^2(G, U)$, called *transgression*, which takes the role of τ in Theorem 3.1. There is a

transgression map in the theory of the Lyndon-Hochschild-Serre spectral sequence—see [7], pp.332-335. It is probable that that transgression coincides with ours in the present context. However, the use of transgression (as defined in our terms) to calculate cocycles explicitly appears to be a new technique.

Although it will not be proved here, transgression is an injective homomorphism whose image complements K in $H^2(G, U)$. The essential uniqueness of the primary invariant decomposition of G/G' means that K is canonically defined. However, this is not true of transgression: in general, there is more than one transgression, and concomitantly more than one complement of K in $H^2(G, U)$. This idea will be explored in more detail in Section 6.

Our definition of transgression requires a particular explicit description of $H_2(G)$. This is afforded by *Hopf's formula*, which is stated in terms of a presentation of G : say $G \cong F/R$, where F is a finitely generated free group and R some normal subgroup. If F_1 and F_2 are subgroups of F then $[F_1, F_2]$ denotes the subgroup $\langle [x, y] \mid x \in F_1, y \in F_2 \rangle$, where $[x, y] = x^{-1}y^{-1}xy$. As usual, $[F, F]$ is denoted F' . The following results go back to Schur (1907).

Theorem 4.1 (see [6], p.50, 2.4.6) *With the notation above,*

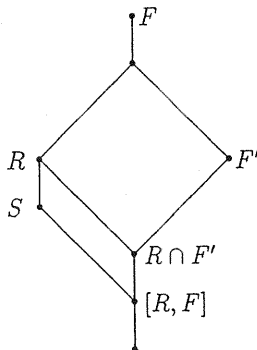
- (i) $R/[R, F]$ is a finitely generated abelian group,
- (ii) the torsion subgroup of $R/[R, F]$ is $(R \cap F')/[R, F]$,
- (iii) $(R \cap F')/[R, F] \cong H_2(G)$. □

The isomorphism in Theorem 4.1 (iii) is referred to as *Hopf's formula*.

Since $R/[R, F]$ is a finitely generated abelian group, it splits over its torsion subgroup $(R \cap F')/[R, F]$. A complement of $(R \cap F')/[R, F]$ in $R/[R, F]$, not necessarily unique, is torsion-free. We state these important facts as a separate result.

Theorem 4.2 *There is a complement $S/[R, F]$ of $(R \cap F')/[R, F]$ in $R/[R, F]$ (in general, more than one choice of S is possible) and $S/[R, F]$ is a free abelian group of the same rank as F .* □

Thus, unlike Hopf's formula, the isomorphism type of $S/[R, F]$ depends on the presentation of G . The following Hasse diagram may help to picture the situation.



The quotient F/S is called a *Schur cover*, or *covering group*, of G . As suggested by the diagram, we obtain the next result by application of the appropriate natural isomorphism theorem for groups.

Proposition 4.3 $R/S \cong H_2(G)$. □

Now we have all the ingredients needed to define transgression. Identifying F/R with G and R/S with $H_2(G)$ by Proposition 4.3, we seek in the first instance a map from $\text{Hom}(R/S, U)$ to $Z^2(F/R, U)$. But if $f \in \text{Hom}(R/S, U)$, then for any $\mu \in Z^2(F/R, R/S)$ it is obvious that $f \circ \mu \in Z^2(F/R, U)$. Since F/S is a central extension of R/S by F/R , it gives rise to a 2-cocycle from F/R to R/S as discussed in Section 2, and any such cocycle can be chosen as μ . More explicitly, let $\sigma: F/R \rightarrow F/S$ be a normalised transversal map for

$$1 \rightarrow R/S \xrightarrow{\iota} F/S \xrightarrow{\pi} F/R \rightarrow 1,$$

where ι is inclusion of subgroups and π is the canonical projection homomorphism. A 2-cocycle μ_σ arising from this central extension is defined by

$$\mu_\sigma(x_1R, x_2R) = \sigma(x_1R)\sigma(x_2R)\sigma(x_1x_2R)^{-1}.$$

Then (omitting proofs) it may be shown that the assignment

$$f \mapsto f \circ \mu_\sigma$$

induces an embedding

$$\tau_S: \text{Hom}(R/S, U) \hookrightarrow H^2(F/R, U)$$

such that $H^2(F/R, U) = K \oplus \text{im}(\tau_S)$. Notice that the definition of τ_S is dependent on the choice of S but independent of the choice of σ . Each homomorphism τ_S will be called transgression.

5 Example

Transgression has been used to calculate a full set of representative cocyclic matrices developed over G

- abelian
- metacyclic; that is, an extension of one cyclic group by another (for example, quaternion and dihedral groups are metacyclic).

The isomorphism type of $H_2(G)$ is well-known if G is abelian or metacyclic, and Hopf's form is easily calculated in those cases. For abelian groups, the description of cocyclic matrices produced by the method may be reconciled with that given in [1], [3] and [4].

We will now undertake an explicit calculation of transgression for $U = \mathbb{Z}_2$ and G a specific metacyclic group, namely the dihedral group of order 8. In this case, G is presented as the quotient F/R , where F is free on two generators a, b and R is the normal subgroup of F generated by a^4, b^2 and $b^{-1}aba$. Note that G is an extension of its normal cyclic subgroup $\langle a \rangle \cong \mathbb{Z}_4$ by $\langle b \rangle \cong \mathbb{Z}_2$. It is known that $H_2(G) \cong \mathbb{Z}_2$ in this case (see [6], p.98, 2.11.3).

We begin by determining $(R \cap F')/[R, F]$ and choosing a Schur complement. This is done in Proposition 5.2 below, which depends on the next lemma.

Lemma 5.1 $[a^2, b][R, F] = (a^4)^{-1}(b^{-1}aba)^2[R, F]$.

Proof. Since $b^{-1}aba[R, F] \in R/[R, F]$ is central in $F/[R, F]$, we see that, modulo $[R, F]$,

$$\begin{aligned} [a^2, b] &\equiv a^{-2}b^{-1}aba \cdot a^{-1}b^{-1}ab \\ &\equiv a^{-3}b^{-1}aba \cdot a^{-1}b^{-1}aba \\ &\equiv a^{-4}(b^{-1}aba)^2, \end{aligned}$$

as required. □

Proposition 5.2

$$R/[R, F] = \langle [a^2, b][R, F] \rangle \times \langle b^2[R, F], b^{-1}aba[R, F] \rangle; \quad (1)$$

the first factor is $(R \cap F')/[R, F] \cong \mathbb{Z}_2$ and the second is a Schur complement $S/[R, F]$.

Proof. Certainly $[a^2, b] \in R$, and by Lemma 5.1, R is generated modulo $[R, F]$ by $[a^2, b], b^2$ and $b^{-1}aba$. We now show that $[a^2, b][R, F]$ has trivial square. Modulo $[R, F]$,

$$\begin{aligned} [a^2, b]^2 &\equiv a^{-2}b^{-1}a^2ba^{-2} \cdot b^{-1}a^2b \\ &\equiv a^{-2}b^{-1}a^2b \cdot b^{-1}a^2ba^{-2} \\ &\equiv a^{-2}b^{-1} \cdot a^4 \cdot ba^{-2} \\ &\equiv a^{-2}b^{-1} \cdot ba^{-2}a^4 \\ &\equiv 1, \end{aligned}$$

using the fact that $a^4[R, F]$ and $[b, a^{-2}][R, F]$ lie in the central subgroup $R/[R, F]$ of $F/[R, F]$.

Since $(R \cap F')/[R, F] \cong \mathbb{Z}_2$, we have

$$R/[R, F] \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$$

by Theorem 4.2. Thus, $R/[R, F]$ can be generated by no fewer than three of its elements. But then by Lemma 5.1, $[a^2, b] \notin [R, F]$: otherwise, $R/[R, F]$ would be generated by $a^4[R, F]$ and $b^{-1}aba[R, F]$. By the same reasoning, $\langle b^2[R, F], b^{-1}aba[R, F] \rangle$ is genuinely a complement of $\langle [a^2, b][R, F] \rangle$ in $R/[R, F]$. □

Since the Schur multiplier is small, calculation of $\text{im}(\tau_S)$ for this example should not be difficult.

For the choice of S made in Proposition 5.2 we have

$$R/S = \langle [a^2, b]S \rangle = \langle a^4 S \rangle$$

and the following relations hold in F/S :

$$(bS)^2 = S, \quad (aS)^{bS} = a^{-1}S.$$

Next, we need to fix a normalised transversal function $\sigma: F/R \rightarrow F/S$. An obvious candidate is

$$\sigma: a^i b^j R \mapsto a^i b^j S \quad (2)$$

under the restrictions $0 \leq i \leq 3$ and $0 \leq j \leq 1$.

Now we are in a position to calculate a representative cocyclic matrix for each cohomology class in $\text{im}(\tau_S)$, where τ_S is defined according to the particular choice of S made in Proposition 5.2. Recall that $\mu_\sigma \in Z^2(F/R, R/S)$ is defined by

$$\mu_\sigma(a^i b^j R, a^k b^l R) = \sigma(a^i b^j R) \sigma(a^k b^l R) \sigma(a^{i+(-1)^j k} b^{j+l} R)^{-1}$$

for $0 \leq i, k \leq 3$ and $0 \leq j, l \leq 1$. The single nonzero element f of $\text{Hom}(R/S, \mathbb{Z}_2)$ is defined by

$$f: [a^2, b]S \mapsto -1.$$

We wish to calculate a cocyclic matrix associated to the representative $f \circ \mu_\sigma$ of $\tau_S(f)$. From this point onward, the exposition will be split up into consideration of the separate cases $j = 0$ and $j = 1$.

First suppose that $j = 0$. Then

$$\mu_\sigma(a^i R, a^k b^l R) = a^{i+k-\overline{i+k}} S,$$

where overlining denotes reduction mod 4. The exponent $i+k-\overline{i+k}$ is either 0 or 4, depending on whether $0 \leq i+k \leq 3$ or $i+k > 3$, respectively. Therefore,

$$(f \circ \mu_\sigma)(a^i R, a^k b^l R) = \begin{cases} 1 & \text{if } i+k \leq 3 \\ -1 & \text{otherwise.} \end{cases} \quad (3)$$

For $j = 1$,

$$\begin{aligned} \mu_\sigma(a^i b R, a^k b^l R) &= a^{i-k} b^{l+1} S \cdot \sigma(a^{i-k} b^{l+1} R)^{-1} \\ &= a^{i-k-\overline{i-k}} S, \end{aligned}$$

where we have used $(aS)^{bS} = a^{-1}S$ in the first line and $(bS)^2 = S$ in the second. Consequently,

$$(f \circ \mu_\sigma)(a^i b R, a^k b^l R) = \begin{cases} 1 & \text{if } i \geq k \\ -1 & \text{otherwise.} \end{cases} \quad (4)$$

This bipartite definition prompts us to index the rows and columns of a cocyclic matrix associated to $f \circ \mu_\sigma$ as $1, a, a^2, a^3, b, ab, a^2b, a^3b$. The matrix is then constructed

as a 2×2 block matrix. The two 4×4 blocks in the first row, constructed according to the rule (3), are back negacyclic. The two blocks in the second row, constructed according to (4), have the form of a back negacyclic matrix whose rows have been written in reverse order. Explicitly, this cocyclic matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \quad (5)$$

This is visibly not a Hadamard matrix: it has repeated rows.

To complete our analysis of this example, we give the general form of a binary cocyclic matrix developed over D_8 . This matrix, denoted $\mathcal{M}(A, B, C, D, E, X, Y, Z)$, where the letters A, \dots, Z take values ± 1 , is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & AX & B & ABXZ & C & DX & E & CDEXZ \\ 1 & B & Z & BZ & ACD & ADE & ACDZ & ADEZ \\ 1 & ABXZ & BZ & AXZ & ABCDE & ABCXZ & ABDZ & ABEXZ \\ 1 & CDEZ & ACDZ & ABCZ & Y & ABCYZ & ACDYZ & CDEYZ \\ 1 & CX & ADEZ & ABDXZ & CY & XY & ABDYZ & ADEXYZ \\ 1 & D & ACD & ABEZ & ACDY & DY & Y & ABEYZ \\ 1 & EX & ADE & ABCDEX & ABCDEY & ADEXY & EY & XY \end{pmatrix}. \quad (6)$$

That is, a general binary cocyclic matrix developed over D_8 is a Hadamard product

$$\mathcal{M}(A, B, C, D, E, 1, 1, 1) \bullet \mathcal{M}(1, 1, 1, 1, 1, X, Y, 1) \bullet \mathcal{M}(1, 1, 1, 1, 1, 1, 1, Z).$$

The first matrix in this product is associated to a coboundary, the second to a representative of an element of K (obtained by the method discussed before Theorem 3.1), and the third to a representative of an element of $\text{im}(\tau_5)$. A Hadamard matrix can be read off (6); for instance, $\mathcal{M}(-1, 1, -1, -1, -1, 1, -1, -1)$ is Hadamard:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}. \quad (7)$$

There are other cocyclic Hadamard matrices developed over D_8 : a computer search reveals 32 in all, whose associated cocycles fall into three distinct cohomology classes.

6 The non-canonical nature of the method

We reiterate that our method is not canonical, primarily in the sense that it is dependent on a not necessarily unique choice of Schur complement. It should also be observed that the image of transgression is specified by a set of representative cocycles, and these in turn are defined according to the choice of a particular transversal map. In this section, we indicate the freedom of choice in our method by repeating the calculation of the previous section for a different Schur complement and transversal map.

First, fix S as in the previous section, but replace σ as defined in (2) with the normalised transversal map $\sigma': F/R \rightarrow F/S$ defined by

$$\sigma': b^i a^j R \mapsto b^i a^j S$$

for $0 \leq i \leq 1, 0 \leq j \leq 3$. It is natural then to index rows and columns of a cocyclic matrix associated to the representative $f \circ \mu_\sigma$ of $\tau_S(f)$ as $1, a, a^2, a^3, b, ba, ba^2, ba^3$. Mimicking the reasoning of the previous section, we construct our matrix as a 2×2 block matrix, where the two 4×4 blocks in the first column are back negacyclic, and the blocks in the second column have the form of a back negacyclic matrix whose columns have been written in reverse order:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

To reconcile this indexing of rows and columns with the previous one, we need to interchange columns 6 and 8 and interchange rows 6 and 8. The resulting matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}.$$

In [5], this is the given representative cocyclic matrix associated to the generator of $\text{Hom}(H_2(D_8), \mathbb{Z}_2)$. It arises as the Hadamard product of (5) with the coboundary matrix $\mathcal{M}(1, 1, -1, 1, 1, 1, 1, 1)$.

Next, we choose a different Schur complement. For the fixed presentation of G , there are precisely $|\text{Hom}(R/(R \cap F'), (R \cap F')/[R, F])| = 4$ distinct possibilities for $S/[R, F]$. As inspection of (1) reveals, one of these is

$$S/[R, F] = \langle b^2[a^2, b][R, F], b^{-1}aba[R, F] \rangle.$$

We proceed to calculate $\text{im}(\tau_S)$ for this choice of S .

Lemma 5.1 is independent of the choice of S . We still have $(aS)^{bS} = a^{-1}S$, but now $(bS)^2 = [a^2, b]S$, so that bS is an element of F/S of order 4. Certainly (2) is still a valid normalised transversal function. Denote by f the nonzero homomorphism from R/S to \mathbb{Z}_2 . Although (3) defines the action of $f \circ \mu_\sigma$ on pairs $(a^i R, a^k b^l R)$ for $0 \leq i, k \leq 3, 0 \leq l \leq 1$, and (4) remains unchanged for $0 \leq i, k \leq 3$ and $l = 0$, the latter definition must be modified appropriately when $l = 1$. For

$$\begin{aligned} \mu_\sigma(a^i b R, a^k b R) &= a^{i-k} b^2 S \cdot a^{-i-k} S \\ &= a^{i-k-i-k} \cdot [a^2, b] S \end{aligned}$$

and thus

$$(f \circ \mu_\sigma)(a^i b R, a^k b R) = \begin{cases} 1 & \text{if } i < k \\ -1 & \text{otherwise.} \end{cases}$$

A cocyclic matrix associated to a representative of $\tau_S(f)$ is therefore

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}, \tag{8}$$

with the usual indexing of rows and columns.

In fact, (8) is $\mathcal{M}(1, 1, 1, 1, 1, 1, -1, -1)$. This means that the cocycles associated to (8) and (7) lie in the same cohomology class in $H^2(D_8, \mathbb{Z}_2)$. Hence, by a judicious choice of transversal map, and with the current choice of S , calculation of transgression alone would produce a cocyclic Hadamard matrix without the need to calculate K or any coboundaries. This is desirable from the point of view of ensuring economy of the method. Consequently, further investigation of the non-canonical nature of the method, particularly its effect on generation of Hadamard matrices, should be carried out.

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