ORTHOGONAL RESOLUTIONS OF TRIPLE SYSTEMS

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Abstract

Existence results concerning double and multiple orthogonal resolutions of triple systems are surveyed, and a number of open questions mentioned.

1 Introduction

A *triple system of order* v *and index* λ (briefly TS(v,λ)) is a pair $(V,\mathcal{B})$ where V is a v-set, and $\mathcal{B}$ is a collection of 3-subsets of V called *triples* such that each 2-subset of V is contained in exactly λ triples.

A *parallel class* of a TS(v,λ) $(V,\mathcal{B})$ is a subset of $\mathcal{B}$ which partitions the set V. A *resolution* of $(V,\mathcal{B})$ is a collection of parallel classes which partitions the set $\mathcal{B}$).

Let $(V,\mathcal{B})$ be a TS(v,λ). Two resolutions $\mathcal{R},\mathcal{S}$ of $(V,\mathcal{B})$ are *orthogonal* if $|R_i \cap S_j| \leq 1$ for any $R_i \in \mathcal{R}$ and any $S_j \in \mathcal{S}$. A TS(v,λ) admitting two orthogonal resolutions is said to be *orthogonally resolvable* or *doubly resolvable*. More generally, a set $\mathcal{R} = \{\mathcal{R}_1,\ldots,\mathcal{R}_s\}$ of s resolutions of a TS(v,λ) is a *d-orthogonal set of resolutions* if for any d-subset $\{i_1,\ldots,i_d\}$ of $\{1,\ldots,s\}$ and any d parallel classes $R_{i_1,j_1}, R_{i_2,j_2},\ldots,R_{i_d,j_d}$ where $R_{i_k,j_k} \in R_{i_k}$, we have $|R_{i_1,j_1} \cap R_{i_2,j_2} \cap \ldots \cap R_{i_d,j_d}| \leq 1$. A d-orthogonal set $\mathcal{R}$ of resolutions is *regular of strength* t $(2 \leq t \leq d)$ if $\mathcal{R}$ is a t-orthogonal set but contains no $(t-1)$-orthogonal set. Thus 2-orthogonal is the same as orthogonal (a 2-orthogonal set is necessarily regular of strength 2).

The motivation for studying orthogonal resolutions of TS(v,λ) (or, more generally, of designs) comes from their connection to the existence of various kinds of square and (multidimensional) cubical arrays.

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A d-orthogonal set of resolutions of a $\mathrm{TS}(v, \lambda)$ gives rise to an interesting multidimensional array. A *multidimensional Kirkman design* of dimension d, index λ and order v (briefly $\mathrm{MKD}(d, \lambda, v)$) is a d-dimensional array such that

1. every cell of the array is either empty or contains a 3-subset of a v-set V,
2. every element of V is contained in exactly one cell of any $(d-1)$-dimensional subarray,
3. the collection of 3-subsets in the nonempty cells is the collection of triples of a $\mathrm{TS}(v, \lambda)$ (called the *underlying triple system* of the MKD).

An $\mathrm{MKD}(d, \lambda, v)$ is *regular of strength* t $(2 \leq t \leq d)$ if its projection on any t dimensions is an $\mathrm{MKD}(t, \lambda, v)$ but its projection on any $t-1$ dimensions is never an MKD. The following theorem is obvious.

Theorem 1.1 *A regular MKD(d, λ, v) of strength t with underlying triple system $(V, \mathcal{B})$ exists if and only if there exists a regular d-orthogonal set of d resolutions of strength t of $(V, \mathcal{B})$.*

Proof. If $\{\mathcal{R}_1, \ldots, \mathcal{R}_d\}$ is a d-orthogonal set of resolutions of $(V, \mathcal{B})$ where $\mathcal{R}_i = \{R_{i,1}, \ldots, R_{i,s}\}$, place in the cell $(j_1, j_2, \ldots, j_d)$ of a d-dimensional array K the set $R_{1,j_1} \cap R_{2,j_2} \cap \ldots R_{d,j_d}$ which is either empty or is a triple of $\mathcal{B}$. Conversely, given an $\mathrm{MKD}(d, \lambda, v)$ K with an underlying triple system $(V, \mathcal{B})$, the triples of K in any $(d-1)$-dimensional hypercube obtained by fixing the coordinate i_c in the ith coordinate direction $(i \in \{1, \ldots, d\})$ yield a resolution $\mathcal{R}_i$, and $\{\mathcal{R}_1, \ldots, \mathcal{R}_d\}$ is a d-orthogonal set. □

Several special cases of MKDs are of interest; next we consider these in more detail.

2 Generalized Room squares of degree 3

A regular MKD with $d = 2$, $\lambda = v - 2$ (and, of course, $t = 2$) whose underlying triple system is the simple $\mathrm{TS}(v, v-2)$ (that is, whose triples are all 3-subsets of a v-set, or, which is the same, the complete 3-uniform v-hypergraph), can be thought of as a straightforward generalization of a Room square, and is therefore called a *generalized Room square* of degree 3 (and order v, briefly $\mathrm{GRS}(v)$). It is a square array of side $\binom{v-1}{2}$ by $\binom{v-1}{2}$); each row (column) of a $\mathrm{GRS}(v)$ contains $\frac{v}{3}$ nonempty cells, and the triples in the nonempty cells of each row (column) form a parallel class. By Theorem 1.1, a $\mathrm{GRS}(v)$ exists if and only there exists a pair of orthogonal 1-factorizations of the complete 3-uniform v-hypergraph.

A first example of a GRS was given by Baker [1] for v=9. This is the smallest possible nontrivial GRS, as clearly GRS(6) cannot exist. We present a different example of a GRS(9) (cf. [21]).

Example 2.1 *Two orthogonal 1-factorizations $\mathcal{F}$, $\mathcal{G}$ of the complete 3-uniform hypergraph ${K_9}^3$ on $VZ_8 \cup \{\infty\}$ are as follows: the base resolutions of $\mathcal{F}$ are*

$$
\begin{aligned}
F_1 &= \{\{0,1,\infty\},\{2,3,6\},\{4,5,7\}\},\\
F_2 &= \{\{0,2,\infty\},\{1,3,5\},\{4,6,7\}\},\\
F_3 &= \{\{0,3,\infty\},\{2,5,6\},\{1,4,7\}\},\\
F_4 &= \{\{0,4,\infty\},\{1,2,3\},\{5,6,7\}\},
\end{aligned}
$$

and the base resolutions of $\mathcal{G}$ *are*

$$
\begin{aligned}
G_1 &= \{\{0,1,\infty\},\{3,4,7\},\{2,5,6\}\},\\
G_2 &= \{\{0,2,\infty\},\{3,4,6\},\{1,5,7\}\},\\
G_3 &= \{\{0,3,\infty\},\{4,5,6\},\{1,2,7\}\},\\
G_4 &= \{\{0,4,\infty\},\{1,3,6\},\{2,5,7\}\}.
\end{aligned}
$$

The corresponding GRS is of side 28.

Only finitely many GRS have been constructed by a direct method similar to that of Example 2.1. The orders of these include v=12, 18, 24, 30, 42 (see [25], and v=15 (see [21]). In Example 2.2 we present an example of a GRS of order v=21; no example of this order has been known previously.

Example 2.2 *As in Example 2.1, we present only the base 1-factorizations of two orthogonal 1-factorizations of* K_{21}^3 *on* $Z_{20} \cup \{\infty\}$:

First 1-factorization:

$\{\{\infty,0,1\},\{2,3,8\},\{4,5,7\},\{6,9,10\},\{11,12,15\}, \{13,18,19\},\{14,16,17\}\}$
$\{\{\infty,17,19\},\{0,2,10\},\{3,9,13\},\{1,4,11\},\{5,14,15\}, \{6,7,8\},\{12,16,18\}\}$
$\{\{\infty,0,3\},\{4,6,9\},\{1,12,15\},\{7,10,18\},\{5,13,16\}, \{14,17,19\},\{2,8,11\}\}$
$\{\{\infty,1,17\},\{0,8,10\},\{2,9,12\},\{4,16,19\},\{5,11,18\}, \{6,14,15\},\{3,7,13\}\}$
$\{\{\infty,0,15\},\{4,8,9\},\{7,12,16\},\{3,6,10\},\{2,5,18\}, \{1,13,19\},\{11,14,17\}\}$
$\{\{\infty,13,19\},\{1,6,11\},\{2,7,14\},\{3,10,15\},\{8,9,18\}, \{0,4,12\},\{5,16,17\}\}$
$\{\{\infty,0,7\},\{1,14,16\},\{8,9,15\},\{2,3,10\},\{4,5,17\}, \{6,11,13\},\{12,18,19\}\}$
$\{\{\infty,6,14\},\{0,2,11\},\{1,5,10\},\{8,13,16\},\{4,12,18\}, \{3,17,19\},\{7,9,15\}\}$
$\{\{\infty,0,9\},\{7,12,18\},\{3,5,16\},\{1,10,14\},\{8,15,19\}, \{2,11,17\},\{4,6,13\}\}$
$\{\{\infty,0,10\},\{1,5,9\},\{11,15,19\},\{2,3,7\},\{12,13,17\}, \{4,6,8\},\{14,16,18\}\}$

Second 1-factorization:

$\{\{\infty,10,11\},\{0,7,15\},\{3,9,14\},\{1,2,17\},\{5,8,16\}, \{4,18,19\},\{6,12,13\}\}$
$\{\{\infty,8,10\},\{0,6,12\},\{3,13,16\},\{7,15,18\},\{9,17,19\}, \{1,2,14\},\{4,5,11\}\}$
$\{\{\infty,10,13\},\{0,1,5\},\{2,7,9\},\{3,6,11\},\{4,8,15\}, \{12,14,17\},\{16,18,19\}\}$
$\{\{\infty,3,7\},\{0,4,8\},\{1,6,11\},\{5,13,14\},\{10,12,16\}, \{9,15,17\},\{2,18,19\}\}$
$\{\{\infty,14,19\},\{0,6,10\},\{7,9,17\},\{1,8,11\},\{2,4,13\}, \{3,5,16\},\{12,15,18\}\}$
$\{\{\infty,12,18\},\{0,9,10\},\{1,5,11\},\{7,8,17\},\{2,4,6\}, \{3,16,19\},\{13,14,15\}\}$
$\{\{\infty,10,17\},\{0,5,9\},\{1,3,18\},\{7,14,19\},\{12,15,16\}, \{6,8,13\},\{2,4,11\}\}$
$\{\{\infty,2,14\},\{0,6,13\},\{7,11,19\},\{8,10,16\},\{1,3,17\}, \{5,9,12\},\{4,15,18\}\}$
$\{\{\infty,10,19\},\{0,3,15\},\{1,6,7\},\{9,13,18\},\{4,5,12\}, \{2,11,16\},\{8,14,17\}\}$
$\{\{\infty,8,18\},\{1,12,13\},\{2,3,11\},\{6,10,19\},\{0,9,16\}, \{4,5,7\},\{14,15,17\}\}$

However, the known recursive constructions (see below) yield GRSs for infinitely many orders.

Two Steiner systems S(3,5,v) $(V,\mathcal{B}_1),(V,\mathcal{B}_2)$ are *orthogonal* if

(i) $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$, and

(ii) whenever Q, Q' are two blocks of $\mathcal{B}_1$ such that $Q \cap Q' = P$, $|P|=2$, and R, R' are such that $(Q \setminus P) \cup R \in \mathcal{B}_2$, $(Q' \setminus P) \cup R' \in \mathcal{B}_2$ then $R \neq R'$.

The existence of orthogonal STSs implies the existence of (ordinary) Room squares (see, e.g., [6]). For GRSs, we have the following analogue:

Theorem 2.1 *If there exist two orthogonal Steiner systems $S(3,5,v)$ then there exists a $GRS(v+1)$.*

Proof. Label the rows and columns of a $\binom{v}{2}$ by $\binom{v}{2}$ array with 2-subsets of a v-set V. If $(V, \mathcal{B}_1), (V, \mathcal{B}_2)$ are two orthogonal Steiner systems $S(3,5,v)$, place the triple $\{a,b,c\}$ in the cell in row labelled $\{x,y\}$ and column labelled $\{u,v\}$ if $\{a,b,c,x,y\} \in \mathcal{B}_1$ and $\{a,b,c,u,v\} \in \mathcal{B}_2$. Place the triple $\{a,b,\infty\}$ (where ∞ is the additional element) in the "diagonal" cell in row labelled $\{a,b\}$ and column labelled $\{a,b\}$. □

Unfortunately, the only order v for which a pair of orthogonal $S(3,5,v)$ is presently known to exist, is $v=17$ (see [31]), and GRS(18) is known already.

But the next theorem (cf. [8]), which is a generalization of a result due to Stiffler and Blake [25], yields several infinite classes of GRSs.

Theorem 2.2 *If there exists a 3-wise balanced design $S(3,K,v)$ and for each $k \in K$ there exists a $GRS(k+1)$ then there exists a $GRS(v+1)$.*

Corollary 2.3 *[25] If there exists a $GRS(k+1)$ and a Steiner system $S(3,k,v)$ then there exists a $GRS(v+1)$.*

In particular, there exists a GRS(v) for $v = q^a+2$, $q7$, 13, 16, since $S(3, q+1, q^a+1)$ is known to exist for every prime power q.

We also have the following multiplicative recursive construction (see [21]).

Theorem 2.4 *If a $GRS(v)$ exists and v is even then there exists a $GRS(3v)$.*

However, the spectrum for GRS remains to be determined. The smallest order v for which the existence of a GRS(v) is undecided, is $v=27$.

3 Kirkman squares

A *Kirkman square* is a (regular) MKD$(2,1,v)$ (briefly a KS(v)). Its underlying design is an STS(v), and its existence is equivalent to the existence of a pair of orthogonal resolutions of this underlying STS(v) (cf. Theorem 1.1); the latter is said to be *doubly resolvable* or *orthogonally resolvable.*

The first example of a doubly resolvable STS was obtained by Mathon and Vanstone [18], [19] who used PG(3,3) to construct a KS(27). Trivially, there exists no KS(9), and an exhaustive check of the four nonisomorphic resolvable KTS(15) and their resolutions reveals that there exists no KS(15). The existence of a KS(21) is in doubt; thus the example of Mathon and Vanstone yields the smallest order for which a KS is known to exist.

The main tool in direct constructions of Kirkman squares is the starter-adder method (see [9], [30], [23], [26]). Kirkman squares constructed by this method have the property that the automorphism group of the two orthogonal resolutions of the underlying STS is transitive on the parallel classes. Thus the order v of such a square must satisfy $v \equiv 3 \pmod{12}$. If $\mathcal{R} = \{R_1, \dots, R_r\}$ and $\mathcal{S} = \{S_1, \dots, S_r\}$ are two orthogonal resolutions of $(V, \mathcal{B})$ where $V = Z_{\frac{v-1}{2}} \times \{0,1\} \cup \{\infty\}$, and $\alpha = (0_0 1_0 \dots \frac{v-1}{2}_0)$ $(0_1 1_1 \dots \frac{v-1}{2}_1)(\infty)$, $\alpha R_i = R_{i+1}, \alpha S_i = S_{i+1}$, and if, say, $R_1 = \{B_1, \dots, B_r\}$, $S_1 = \{C_1, \dots, C_r\}$, and $C_i = B_i + a_i$ for $i = 1, \dots, r$ then $(B_1, \dots, B_r)$ is called the *starter* and $(a_1, \dots, a_r)$ is called the *adder*, and the coresponding KS is said to be a *starter-adder* KS. The orthogonality of $\mathcal{R}$ and $\mathcal{S}$ ensures that $a_i \neq a_j$ for $i \neq j$.

While enumerating all KTS(27) with a cyclic automorphism of order 13 (thus transitive on parallel classes), Janko and Van Trung [9] found that there are exactly 3 nonisomorphic starter-adder KS(27). One of these is displayed below in Example 3.1.

Example 3.1 *A Kirkman square of order 27*

B_1	B_2	B_3	B_4	B_5
$\infty 0_0 0_1$	$2_0 9_0 12_1$	$3_0 7_0 11_1$	$5_0 10_0 6_1$	$4_0 3_1 9_1$
$a_1 = 0$	$a_2 = 1$	$a_3 = 5$	$a_4 = 3$	$a_5 = 6$
B_6	B_7	B_8	B_9	
$6_0 4_1 8_1$	$8_0 1_1 2_1$	$1_0 11_0 12_0$	$5_1 7_1 10_1$	
$a_6 = 10$	$a_7 = 7$	$a_8 = 8$	$a_9 = 12$	

In [23], [26], a search for a restricted class of starter-adder KS(v) was conducted where the underlying STS(v) contains a maximum order subsystem, i.e. a subsystem of order $\frac{v-1}{2}$. In this way, several KS(39) and KS(51) were found.

Baker [2] gave a construction for an infinite class of starter-adder KSs.

Theorem 3.1 *For all $m \equiv 1 \pmod 3$ there exists a KS($2^{2m+2} - 1$).*

In particular, this gives a KS(63).

Fuji-Hara and Vanstone [7] have shown that every affine space AG($2^{i+1}, 3$), $i \geq 1$, admits a skew resolution, and hence is doubly resolvable (as the affine resolution and a skew resolution are orthogonal). In particular, this gives a KS(81).

Another construction giving infinite classes of starter-adder KSs is obtained from the following theorem.

Theorem 3.2 *If there exists a starter-adder KS($2q + 1$) on the set of elements $GF(q) \times \{0,1\} \cup \{\infty\}$ then for each $n \geq 1$ there exists a KS($2q^n + 1$) on the set of elements $GF(q^n) \times \{0,1\} \cup \{\infty\}$.*

We also have the following PBD-closure theorem for Kirkman squares.

Theorem 3.3 *If there exists a PBD$(v, K, 1)$ and if for each $k \in K$ there exists a KS$(2k+1)$ then there exists a KS$(2v+1)$.*

Proof. Let $(V, \mathcal{D})$ be a PBD$(v, K, 1)$. Put $W = V \times \{0,1\} \cup \{\infty\}$ where ∞ is a new element. Label a square S of side v with the elements of V. For a block $B \in \mathcal{D}$, $|B| = k$, let S_B be the subsquare of S whose rows and columns are labelled by the elements of B. Place a copy of a KS$(2k+1)$ (which exists by the assumption) whose elements are $B \times \{0,1\} \cup \{\infty\}$ on S_B in such a way that the diagonal cell (i,i) contains the triple $\{\infty, i_1, i_2\}$ for each $i \in B$.

Performing the above for each block $B \in D$ but placing for each $i \in V$ only one copy of the triple $\{\infty, i_1, i_2\}$ in the diagonal cell (i,i) yields a KS$(2v+1)$ whose set of elements is W. □

Theorem 3.3 together with some small known KSs suffices to prove the following asymptotic existence result.

Theorem 3.4 *For all $v \equiv 3 \pmod 6$, v sufficiently large, there exists a KS(v).*

Proof. KS(v) for v=39 and 81 are known to exist, so choose $K = \{19, 40\}$. By Wilson's theorem (see, for example, [3]), there exists v_0 such that for all $v \equiv 1 \pmod 3$ and $v > v_0$, there exists a PBD$(v, K, 1)$. □

Two further recursive constructions for Kirkman squares are due to Vanstone [28] and to Colbourn, Curran and Vanstone [4]. We state these as the next two theorems.

Theorem 3.5 *(The singular direct product for Kirkman squares). If there exists a KS$(2v_1+1)$, and a KS$(2v_2+1)$ with a sub-KS$(2v_3+1)$, and there exist three MOLS$(v_2 - v_3)$ then there exists a KS$(2[(2v_1+1)(v_2-v_3)+v_3]+1)$.*

Theorem 3.6 *(The Kirkman frame construction for Kirkman squares). If there exists a (t, u) Kirkman frame, a KS$(2(us+l)+1)$ with a sub-KS$(2l+1)$, and there exist 3 MOLS(s) then there exists a KS$(2t(us+l)+1)$.*

It has been conjectured that KS(v) exists for all $v \equiv 3 \pmod 6$, $v \geq 27$. However, at present there are only 5 orders $v \leq 100$ for which KS(v) is known to exist: v=27, 39, 51, 63, 81.

4 Kirkman cubes

A regular MKD$(3, 1, v)$ is a *Kirkman cube*. It is *strong* (sKC) if it is of strength 2, and *weak* (wKC) if it is of strength 3.

The starter-adder method of direct construction is applicable to Kirkman cubes as well. The first example of a strong Kirkman cube, an sKC(255), was constructed by Vanstone [29]. The only other sKC obtained by a direct construction (of starter-adder type) is an sKC(39) (see Rosa and Vanstone [24]). Stinson and Vanstone [26] have constructed a set of 6 pairwise orthogonal packings in PG(5,2) which implies the existence of a sKC(63) (actually, this yields a 6-dimensional Kirkman cube of order 63 and strength 2). Since the PBD-closure theorem of §3 holds for Kirkman cubes as well, we obtain the following asymptotic result (see [26]).

Theorem 4.1 *For every $v \equiv 3 \pmod{12}$, v sufficiently large, there exists a sKC(v).*

Lamken [17] has shown that Theorem 4.1 holds for all sufficiently large v satisfying $v \equiv 3 \pmod 6$.

Weak Kirkman cubes are somewhat easier to come by. While there is no wKC(9), a wKC(15) does exist (see [23]). In fact, wKC(v) is known to exist for all $v \equiv 3$ (mod 6), $15 \le v \le 63$ except for v=33 and 57. Also, Theorem 3.4 holds for wKC as well.

Theorem 4.2 *For every $v \equiv 3 \pmod 6$, $v \ge v_0$, there exists a wKC(v).*

It has been conjectured (see [23]) that v_0=15. Proving this, or at least obtaining a good bound on v_0 is contingent on the existence of a wKC(33) which is unknown at present.

5 Doubly resolvable twofold triple systems

All MKD(d, λ, v) considered so far had λ=1. Next we consider the case where the underlying triple system is a twofold triple system TTS(v). The corresponding MKD$(2, 2, v)$ is called a *twofold Kirkman square*; its side is $v - 1$. Colbourn and Vanstone [5] used the starter-adder method to construct twofold Kirkman squares of orders v=15, 18, 21, 24, 27, and 30. They chose the underlying TTS(v) to be 1-rotational; this is a convenient choice since the 1-rotational automorphism partitions the triples of a TTS(v) into $\frac{v}{3}$ orbits, and $\frac{v}{3}$ is also the number of triples in a parallel class. One of the resolutions is relatively easy to obtain: one just has to choose a starter containing one block from each orbit.

An analogue of Theorem 3.5 and 4.2 (the asymptotic existence of twofold Kirkman squares for all sufficiently large orders $v \equiv 0 \pmod 3$) is easily obtained. In fact, Lamken [16] proves the following result:

Theorem 5.1 *A twofold Kirkman square of order v exists if and only if $v \equiv 0$* (mod 3) *except when $v \in \{6, 9\}$ and possibly when $v \in \{72, 78, 90, 114, 117, 126\}$.*

6 A further generalization: Room rectangles and parallelepipeds

In the definition of a d-orthogonal set of resolutions, the resolutions consist of parallel classes (which may be viewed as 1-designs on the underlying set V). Relaxing this condition by allowing the resolutions to consist of other designs led Kramer and Mesner [14] to the following definition (actually, their definition is more general; we confine ourselves here to the case when the underlying design is still a triple system).

A *Room rectangle* RR$(m, n; (v, v_1, v_2), (\lambda, \lambda_1, \lambda_2))$ is an $m \times n$ array R such that

(i) each cell is either empty or contains a triple of an underlying TS(v, λ) $(V, \mathcal{B})$,

(ii) each triple of $\mathcal{B}$ is contained in exactly one cell of the array, and

(iii) the triples in the nonempty cells of any row [column] are the triples of a $TS(v_1, \lambda_1)$ [$TS(v_2, \lambda_2)$].

Thus in a Room rectangle, a resolution does not necessarily consist of parallel classes but is rather a partition of the underlying design into subdesigns each of which is a $TS(v_i, \lambda_i)$. Here we permit $\lambda=0$, in which case the $TS(v_i, 0)$ must be a 1-design.

A *Room parallelepiped* $RP((n_1, \ldots, n_d); (v, v_1, \ldots, v_d); (\lambda, \lambda_1, \ldots, \lambda_d))$ of dimension d, is an $n_1 \times \ldots \times n_d$ array R satisfying (i) and (ii) above, and

(iv) the triples in the nonempty cells with fixed coordinate i_c in the ith coordinate direction are the triples of a $TS(v_i, \lambda_i)$.

We can define a d-orthogonal set of resolutions of this more general kind in a direct analogy to the definition given in §1. The following extension of Theorem 1.1 is then immediate.

Theorem 6.1 *A Room parallelepiped of dimension d exists if and only if there exists a d-orthogonal set of d resolutions of the underlying $TS(v, \lambda)$.*

The definition of a Room rectangle allows one, for instance, to extend considerations of §4 to twofold triple systems $TS(v,2)$ when $v \equiv 1 \pmod 3$. The underlying design in this case is an *near resolvable* $TS(v,2)$, and the orthogonal resolutions consist of near parallel classes. The Room rectangle corresponding to such a doubly near resolvable $TTS(v)$ is an $RR(v, v; (v, v-1, v-1); (2, 0, 0))$. The example below for $v=10$ is taken from Colbourn and Vanstone [5].

ABF				CHI			DEG		
DIJ	BCG							AEH	
	EFJ	CDH							ABI
		AFG	DEI		BCJ				
			BGH	AEJ		CDF			
			ACJ			BEH		FGI	
				BDF			ACI		GHJ
CEG					FHI			BDJ	
	ADH					GIJ			CEF
		BEI			ADG		FHJ		

Colbourn and Vanstone use the starter-adder method to find doubly near resolvable $TTS(v)$ for v=13, 16, 19, 25, and 28. Lamken [15] has the strongest result in this area:

Theorem 6.2 *A doubly near resolvable $TTS(v)$ exists for $v \equiv 1 \pmod 3$, $v \geq 10$, except possibly when $v \in \{34, 70, 85, 88, 115, 124, 133, 142\}$.*

One of the earliest (and nicest) examples of Room parallelepipeds are the Steiner tableaus of Kramer and Mesner [13]. Each of the 3 nonisomorphic Steiner tableaus T(3,7) constructed by Kramer and Mesner is an RP((7,7,7); (9,9,9,9); (7,1,1,1)) which is regular of strength 3 (where the meaning of "regular of strength t" is the same as that given at the beginning of §1). In other words, it is a 3-dimensional cube whose underlying design consists of all triples of a 9-set, and the collection of triples in the nonempty cells of any plane in any of the three directions is an STS(9). It follows that the 7 planes in any of the 3 directions constitute a large set of STS(9). We refer the reader for further details to [13].

Another interesting example is that of a (square) RR((8,8);(8,7,7);(5,1,1)) constructed originally by Mullin and Rosa [20] as a labelled orthogonal resolutions square from a pair of orthogonal S(3,4,8). The triples in the nonempty cells of any row (column) are the triples of an STS(7), and the collection of the 8 rows (8 columns) constitutes an overlarge set of 8 STS(7). This Room rectangle is displayed below.

	235	346	457	156	267	137	124
467		037	056	236	024	257	345
157	456		014	067	347	035	136
126	247	567		025	017	145	046
237	057	135	167		036	012	256
134	367	016	246	127		047	023
245	034	147	027	357	123		015
356	026	045	125	013	146	234	

For many other interesting examples of Room rectangles and Room parallelepipeds (many of them "proper", i.e. non-square and non-cube), see [12], [13], [10] and [11].

7 Open problems

We end with several open problems.

1. What is the spectrum for GRS(v)? In particular, does there exist a GRS(27)?

2. Does there exist a Kirkman square KS(v) for every $v \equiv 3 \pmod 6$, $v \geq 27$?

3. Does there exist a starter-adder KS(v) for every $v \equiv 3 \pmod 6$, $v \geq 27$?

4. Determine the precise spectrum for existence of sKC(v).

5. Does there exist wKC(v) for every $v \equiv 3 \pmod 6$, $v \geq 15$?

6. Settle the remaining possible exceptions in the spectrum for doubly resolvable TTS(v).

7. Does there exist a starter-adder generated twofold Kirkman square for every $v \equiv 0 \pmod 3$, $v \geq 15$?

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