A NEW FAMILY OF BALANCED INCOMPLETE BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

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Abstract

In this paper we present a method of constructing balanced incomplete block designs with nested rows and columns (BIBRC) in which each block has two rows and the number of treatments $v \equiv 5 \pmod 8$ is a prime or a prime power, $v > 5$. Our construction requires the existence of a special type of primitive element in $GF(v)$. We have verified the existence of these for all such primes $v, v \leq 1000$. We conjecture that this primitive element must exist for all such primes. Each such design, in conjunction with a result of Uddin [11] gives rise to an infinite family of BIBRC's where the number of treatments is v^a, for any positive integer a.

1 INTRODUCTION

A balanced incomplete block design with nested rows and columns is an arrangement of v treatments in b blocks if the following conditions are satisfied:

(i) each block is a $p \times q$ array of pq plots,

(ii) every treatment occurs at most once in each block,

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(iii) every treatment occurs in exactly r blocks,

(iv) for every pair of treatments $i \neq i'$,

$$p\lambda^{R}{}_{i,i'} + q\lambda^{C}{}_{i,i'} - \lambda_{i,i'} = \lambda = \frac{r(p-1)(q-1)}{v-1}.$$

Here $\lambda^{R}{}_{i,i'}$ and $\lambda^{C}{}_{i,i'}$ denote, respectively, the number of rows and columns of the blocks in which treatment pair (i, i') occurs, and $\lambda_{i,i'}$ denotes the number of blocks in which (i, i') occurs. We shall let BIBRC (v, b, r, p, q, λ), or BIBRC for short, denote a design which satisfies (i) through (iv).

Block designs with nested rows and columns were introducted by Srivastava [8]. Methods of construction are given in [1, 2, 4, 5, 6, 7, 9, 10, 11]. In this paper we give a method of constructing BIBRC's in which each block has two rows and the number of treatments $v \equiv 5 \pmod 8$ is a prime power, $v > 5$. Our construction requires the existence of a special type of primitive element in $GF(v)$. We have verified the existence of these for all such primes $v, v \leq 1000$. We conjecture that this primitive element must exist for all such primes. Each such design, in conjunction with a result of Uddin [11] gives rise to an infinite family of BIBRC's where the number of treatments is v^a for any positive integer a.

2 CONSTRUCTION

Our construction is based on the method of differences introduced by Bose [3] in his fundamental theorem. In this paper we are primarily concerned with BIBRC's with one initial block in which each block has exactly two rows. Construction of these designs when the number of treatments is $v \equiv 3 \pmod 4$, (v is a prime power) is given by Sreenath [7]. The general case $v \equiv 1 \pmod 4$ is still open. Sreenath [7] obtains trial and error solution when $v = 13$ and $v = 17$.

The following theorem provides such designs when $v \equiv 5 \pmod 8$, v is a prime or a prime power. As will be seen, our construction requires special types of primitive elements in $GF(v)$.

We now introduce some notation:

In what follows $v \equiv 1 \pmod 4$ will denote a prime or a prime power. Let x be a primitive element of $GF(v)$.

Write $v = 4t + 1$ and for $i = 0, 1$, let C_i be the set of quadratic and nonquadratic residues in $GF(v)$ and for $m = 0, 1, 2, 3$, let D_m be the set of quartic residues and its multiplicative cosets. Note that $C_0 = D_0 \cup D_2$ and $C_1 = D_1 \cup D_3$.

Theorem 1 *Let $v = 4t + 1$, where t is odd (thus $v \equiv 5 \pmod 8$), v is a prime or a prime power. Let x be a primitive element in $GF(v)$ such that $x^2 - 1 \in C_0$. Then there exists a BIBRC with parameters*

$$\left(v; b = v; r = v-1; p = 2, q = \frac{(v-1)}{2}, \lambda = \frac{(v-3)}{2}\right).$$

Proof. It suffices to produce one initial block of the required design.

We assert that

$$A = \begin{pmatrix} x^0 & x^4 & \dots & x^{4t-4} & x^2 & x^6 & \dots & x^{4t-2} \\ x & x^5 & \dots & x^{4t-3} & -x & -x^5 & \dots & -x^{4t-3} \end{pmatrix}$$

would serve as the desired initial block.

Write $A = (a_{ij})$, $i = 1, 2$; $j = 1, ..., 2t$. For each $y \neq 0$ in G define:

(i) $\lambda_y{}^R$ to be the number of times the list of differences $\pm(a_{ij} - a_{ij'})$, for $j \neq j', i = 1, 2$, contains y;

(ii) $\lambda_y{}^C$ to be the number of times the list of differences $\pm(a_{ij} - a_{i'j}), i \neq i', j = 1, 2, \dots, 2t$, contains y; and

(iii) λ_y to be the number of times the list of differences $\pm(a_{ij} - a_{i'j'}), (i, j) \neq (i', j')$, contains y.

The first row of A consists of elements of C_0 and the second one elements of C_1. It is well known that C_0 and C_1 are the initial blocks of B.I.B. design whose 'λ' value is $(v-3)/2$.

Thus $\lambda_y{}^R = \frac{(v-3)}{2}$ for each $y \neq 0$ in $GF(v)$.

Likewise since the entries of A provide an initial block of the trivial $(v, v-1, v-2)$ design we conclude that $\lambda_y = v - 2$ for all $y \neq 0$ in $GF(v)$.

It now remains to calculate the exact value of $\lambda_y{}^C$. To do this we consider all the 'column' differences, which look like

$$\pm x^0 (1-x), \pm x^4 (1-x), \dots, \pm x^{4t-4} (1-x) \tag{1}$$

$$\pm x (1+x), \pm x^5 (1+x), \dots, \pm x^{4t-3} (1+x) \tag{2}$$

Using the fact that $x^{2t} = -1$ and t is odd, we see that the differences in (1) account for all the elements in

$$(1-x)(D_0 \cup D_2) = (1-x) C_0 \tag{3}$$

and those in (2) account for elements in

$$(1+x)(D_1 \cup D_3) = (1+x) C_1. \tag{4}$$

As $v \equiv 1 \pmod 4$; $-1 \in C_0$ and hence $1 - x^2 = -(x^2 - 1) \in C_0$

Thus *either* both $1 + x$ and $1 - x \in C_0$ *or* both $1 + x$ and $1 - x \in C_1$.

This observation together with (3) and (4) yields that $\lambda_y{}^C = 1$ for all $y \neq 0$ in $GF(v)$.

Hence $\lambda = 2\lambda_y{}^R + \frac{(v-1)}{2}\lambda_y{}^C - \lambda_y = 2(v-3)/2 + 1(v-1)/2 - (v-2) = (v-3)/2$, completing the proof of Theorem 1.

Example 1 *If $v = 29$ then $x = 8$ is a primitive element of $GF(29)$ such that $x^2 - 1 = 5$ is a quadratic residue of (mod 29). Hence by Theorem 1,*

$$A = \begin{pmatrix} 1 & 7 & 20 & 24 & 23 & 16 & 25 & 6 & 13 & 4 & 28 & 22 & 9 & 5 \\ 8 & 27 & 15 & 18 & 10 & 12 & 26 & 21 & 2 & 14 & 11 & 19 & 17 & 3 \end{pmatrix}$$

is the initial block of a BIBRC with parameters $(v = 29, b = 29, r = 28, p = 2, q = 14, \lambda = 13)$.

Remark 1 *The design in Example 1 is new. We make the following conjecture:*

For each prime power v, $v \equiv 5 \pmod 8$, $v > 5$, $GF(v)$ contains a primitive element x such that $x^2 - 1 \in C_0$ (i.e. $x^2 - 1$ is a square in $GF(v)$).

Remark 2 *Obviously our conjecture does not hold for $v = 5$. We have verified this conjecture for all primes $v, v \equiv 5 \pmod 8$, $5 < v < 1000$ and tabulate our results below:*

(In Table 1, x denotes the primitive element that satisifies $x^2 - 1 \in C_0$).

Table 1

v	x	v	x	v	x
5	–	269	10	653	8
13	2	277	48	661	2
29	8	293	32	677	128
37	2	317	8	701	8
53	8	349	2	709	2
61	2	373	2	733	6
101	27	389	10	757	2
109	6	397	346	773	32
149	8	421	2	797	32
157	142	461	128	821	8
173	128	509	10	829	2
181	2	541	2	853	2
197	8	557	8	877	2
229	10	613	2	941	59
				997	7

We now invoke a recent theorem of Uddin [11] in conjunction with Theorem 1 to obtain the following result

Theorem 2 *Let $v \equiv 5 \pmod 8$, where v is a prime or a prime power. Let x be a primitive element in $GF(v)$ satisfying $x^2 - 1 \in C_0$. Then for each positive integer a, there exists a BIBRC with parameters*

$$\left(v^a; b = v^a\left(v^a - 1\right)/\left(v - 1\right); r = v^a - 1; p = 2, q = \left(v - 1\right)/2, \lambda = \left(v - 3\right)/2\right).$$

Proof Apply our Theorem 1, and Theorem 2.2 of Uddin [11].

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