Isomorphisms and Classification of Cayley Graphs of Small Valencies on Finite Abelian Groups

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Abstract

In this paper, we completely classify all connected Cayley graphs of valency at most 5 on abelian groups and show that, with a few simple families of exceptions, the graph is sufficient to determine up to isomorphism the group and the set giving the edges. It follows that the Ádám's conjecture about isomorphisms of circulant graphs with valency at most 5 is true. This is the best possible for this conjecture in the sense that for every $k \geq 6$ there exist counterexamples of valency k.

1 Introduction

Let G be a finite group and S a *Cayley subset* (that is, $1 \notin S$) of G. The Cayley (di)-graph $X(G,S)$ of G with respect to S has the elements of G as vertices and the pairs $\{g, sg\}$, $g \in G$, $s \in S$, as edges. By the definition, $X(G,S)$ is connected if and only if $\langle S \rangle = G$, and $X(G,S)$ is undirected if and only if $S = S^{-1}$. The group G_R, the right regular representation of G (that is, $g_R : x \to xg$), is a subgroup of automorphisms of $X(G,S)$ and acts transitively on vertices. For two groups G and H, if $G \cong^{\sigma} H$ then σ induces an isomorphism from $X(G,S)$ to $X(H,S^{\sigma})$ in the obvious way. For any fixed element $g \in G$, $g_R\sigma$ is an isomorphism from $X(G,S)$ to $X(H,S^{\sigma})$, called an *Ádám isomorphism*. It is of course possible for two Cayley graphs $X(G,S)$ and $X(H,T)$ to be isomorphic without any Ádám isomorphism from $X(G,S)$ to $X(H,T)$, see Proposition 2.2 for some examples. A lot of people have been interested in the following question: does there exist some class of groups, and some families of elements, such that for any two groups G, H in the same class, $X(G,S) \cong X(H,T)$ implies that there is an Ádám isomorphism from $X(G,S)$ to $X(H,T)$,

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see [1, 2, 6, 4] for references. In 1967, A. Ádám [1] conjectured that the cyclic groups are examples of this class. Elspas and Turner [5] disproved this conjecture by constructing counterexamples for digraphs of valency 3 and for (undirected) graphs of valency 6. But Toida [7] proved that the conjecture is true for cubic graphs; Delorme, Favaron and Mahéo [4] proved that it is true for the graphs of valency 4, which had been open in [3]. In fact, [4] completely determined isomorphisms of Cayley graphs of valency 4 on abelian groups, although in which some simple cases were missed, see Corollary 1.1.1 of this paper. In this paper we determine the isomorphism classes of connected Cayley graphs of valency at most 5 on abelian groups and as a corollary we prove that Ádám's conjecture is true for undirected graphs of valency 5. Since the Elspas and Turner's counterexample for undirected graphs is of valency 6, our this result is the best possible for Ádám's conjecture for undirected graphs.

In the following, all graphs $X(G,S)$ we mention are connected and undirected, so $\langle S\rangle = G$ and $S = S^{-1}$. Let $\Gamma = (V,E)$ be a graph with vertex-set V and edge-set E. Let $\Gamma_i(x) = \{y \in V \mid d(x,y) = i\}$, where $d(x,y)$ denotes the distance between x and y, and $\mathrm{Aut}\Gamma$ be the automorphism group of Γ. A graph Γ is called *symmetric* if $\mathrm{Aut}\Gamma$ acts transitively on the set of arcs of Γ. For two graphs Γ_1 and Γ_2, $\Gamma_1 \times \Gamma_2$ and $\Gamma_1[\Gamma_2]$ denote the direct product and lexicographic product of Γ_1 and Γ_2, respectively. Let Q_n be the cube graph of dimension n and $Q_n^+ = Q_n + E'$, where $E' = \{\{x,y\} \mid x,y \in V(Q_n), d(x,y) = n\}$. For $n > 2$, let C_n be a cycle of length n. A circulant graph $C(n;\ n_1,\cdots,n_d)$ is a graph with vertex set $VC = \{0,1,\cdots,n-1\}$ and edge set $EC = \{(i,j) \bigm| |j-i| = n_1,\cdots,n_{d-1} \ or\ n_d \ (mod\ n)\}$, which has order n and valency $2d$ or $2d-1$. Thus $C_n = C(n;1)$. If n is even then the graph $C(n;1,\frac{n}{2})$ is of valency 3, denoted by M_n. In section 2, we will define a type of product $\Gamma_1 \times_{(q,\tau)} \Gamma_2$ of two circulant graphs Γ_1 and Γ_2. The main results of this paper are as follows.

Theorem 1.1 *Let Γ be a connected Cayley graph of valency 4 on an abelian group. Then*

(1) *either $\Gamma \cong M_r \times K_2$ for some $r > 3$, or $\Gamma \cong C_m \times_{(q,\tau)} C_n$ for some integers m,n,q and τ, where $C_{2q} \times_{(q,1)} C_{2q} \cong C_{4q} \times_{(4q,2q-1)} C_{4q} \cong C_{2q}[\overline{K_2}]$;*

(2) *if Γ is symmetric then $\Gamma \cong C_{rq} \times_{(q,r)} C_{rq}$, where $\tau^2 = \pm 1\ (mod\ q)$, or $\tau = 0$.*

Moreover, these graphs are nonisomorphic except for the stated cases.

Corollary 1.1.1 *Any two finite isomorphic connected Cayley graphs of valency 4 coming from abelian groups are Ádám isomorphic, unless one of the following cases occurs:*

G	S	$X(G,S)$
$Z_2^3 = \langle a,b,c\rangle$	$\{a,b,ab,c\}$	$M_4 \times K_2$
$Z_4 \times Z_2 = \langle a,b\rangle$	$\{a,a^{-1},a^2,b\}$	
$Z_2^4 = \langle a,b,c,d\rangle$	$\{a,b,c,d\}$	$C_4 \times C_4 \cong Q_4$
$Z_4 \times Z_4 = \langle a,b\rangle$	$\langle a,a^{-1},b,b^{-1}\rangle$	
$Z_4 \times Z_2^2 = \langle a,b,c\rangle$	$\{a,a^{-1},b,c\}$	
$Z_n \times Z_4 = \langle a,b\rangle$	$\{a,a^{-1},b,b^{-1}\}$	$C_n \times C_4,\ n \neq 4$
$Z_n \times Z_2^2 = \langle a,b,c\rangle$	$\{a,a^{-1},b,c\}$	
$Z_2^3 = \langle a,b,c\rangle$	$\{a,b,c,abc\}$	$C_4 \times_{(2,1)} C_4 \cong Q_3^+$
$Z_4 \times Z_2 = \langle a,b\rangle$	$\{a,a^{-1},b,a^2b\}$	
$Z_n \times Z_4 = \langle a,b\rangle$	$\{ab^2,a^{-1}b^2,b,b^{-1}\}$	$C_{2n} \times_{(2,1)} C_4,\ n$ *odd*
$Z_{2n} \times Z_2 = \langle a,b\rangle$	$\{a,a^{-1},b,a^nb\}$	
$Z_{4n} = \langle a\rangle$	$\{a,a^{-1},a^{2n-1},a^{2n+1}\}$	$C_{2n}[\overline{K}_2]$
$Z_{2n} \times Z_2 = \langle a,b\rangle$	$\{a,a^{-1},ab,a^{-1}b\}$	

This result was got in [4, Theorem 1], but some simple cases were missed.

Corollary 1.1.2 (1) *All circulant graphs of valency* 4 *are* $C_{rq} \times_{(q,\tau)} C_{sq}$ *with* $(r,s) = 1$.

(2) *All symmetric circulant graphs of valency* 4 *are exactly* $C_q \times_{(q,\tau)} C_q \cong C(q;1,\tau)$, *where* $\tau^2 = 1 \pmod q$.

Theorem 1.2 (1) *Let* Γ *be a symmetric Cayley graph of valency* 5 *on an abelian group. Then* Γ *is isomorphic to one of* K_6, $K_{5,5}$, $K_{6,6} - 6K_2$, Q_4^+ *and* Q_5.

(2) *Let* Γ *be a Cayley graph of valency* 5 *on an abelian group. Then* Γ *is isomorphic to one of the following:* $(C_m \times_{(q,\tau)} C_n) \times K_2$, $M_m \times_{(q,\tau)} C_n$, $C_m \times_{(q,\tau)} M_n$ *and* $\Gamma(m,n,q,\tau)$, *where* $(C_{4q} \times_{(4q,2q-1)} C_{4q}) \times K_2 \cong (C_{2q} \times_{(q,1)} C_{2q}) \times K_2$, $C_{4q} \times_{(4q,2q-1)} M_{4q} \cong C_{2q} \times_{(q,1)} M_{2q}$ *and* $M_{4q} \times_{(4q,2q-1)} C_{4q} \cong M_{2q} \times_{(q,1)} C_{2q}$. ($\Gamma(m,n,q,\tau)$ is defined in Definition 2.3).

Moreover, all graphs listed here are nonisomorphic except for the stated cases.

Corollary 1.2.1 *Any two finite isomorphic connected Cayley graphs of valency* 5 *coming from abelian groups are Ádám isomorphic, unless one of the following cases occurs:*

G	S	$X(G,S)$
$H_1 \times Z_2 = \langle T_1\rangle \times \langle e\rangle$	$T_1 \cup \{e\}$	$X(H_1,T_1) \times K_2$ ($X(H_1,T_1) \cong X(H_2,T_2)$ as in Corollary1.1.1)
$H_2 \times Z_2 = \langle T_2\rangle \times \langle e'\rangle$	$T_2 \cup \{e'\}$	
$Z_n \times Z_4 = \langle a,b\rangle$	$\{a,a^{-1},b,b^{-1},b^2\}$	$C_n \times K_4$
$Z_n \times Z_2^2 = \langle a,b,c\rangle$	$\{a,a^{-1},b,c,bc\}$	
$Z_2^4 = \langle a,b,c,d\rangle$	$\{a,b,c,d,abcd\}$	Q_4^+
$Z_4 \times Z_2^2 = \langle a,b,c\rangle$	$\{a,a^{-1},b,c,a^2bc\}$	
$Z_4^2 = \langle a,b\rangle$	$\{a,a^{-1},b,b^{-1},a^2b^2\}$	
$Z_2^3 = \langle a,b,c\rangle$	$\{a,b,ab,c,abc\}$	$K_4 \times_{(2,1)} C_4 \cong C_4[K_2]$
$Z_4 \times Z_2 = \langle a,b\rangle$	$\{a,a^{-1},a^2,b,a^2b\}$	
$Z_{4n} = \langle a\rangle$	$\{a,a^{-1},a^{2n},a^{2n-1},a^{2n+1}\}$	$C_{2n}[K_2]$
$Z_{2n} \times Z_2 = \langle a,b\rangle$	$\{a,a^{-1},b,ab,a^{-1}b\}$	

Corollary 1.2.2 *If Γ and Γ' are two Cayley graphs of valency 5 on a cyclic group and $\Gamma \cong \Gamma'$, then there is an Ádám isomorphism between Γ and Γ', that is, the Ádám's conjecture is true for undirected graphs of valency 5.*

In section 2, we define a type of product, called (q,τ)-product, of two circulant graphs and give some properties. In section 3, we prove Theorem 1.1 and its corollaries. In section 4, we prove Theorem 1.2 and its corollaries.

2 Preliminary

In this section, we quote some results and give some definitions which will be used in the following sections.

Proposition 2.1 *All Cayley graphs of valency 3 for abelian groups are exactly $C_n \times K_2$ and M_n, where $C_4 \times K_2 \cong Q_3$, $M_4 \cong K_4$ and $M_6 \cong K_{3,3}$, which are symmetric.*

PROOF: It is clear. ∎

There are many pairs of non-isomorphic groups which give isomorphic graphs.

Proposition 2.2 *Suppose $G = K \times Z_2^2$ and $H = K \times Z_4$, where K is abelian, $Z_2^2 = \langle a, b\rangle$ and $Z_4 = \langle c\rangle$. Let $S = T_1 \cup T_2 ab \cup \{a,b\}$ and $S' = T_1 \cup T_2 c^2 \cup \{c, c^{-1}\}$, where $T_1 \subseteq K - \{1\}$ and $T_2 \subseteq K$. Then $X(G,S) \cong X(H,S')$, but there is no Ádám isomorphism between $X(G,S)$ and $X(H,S')$.*

PROOF: Let ρ be a map from G to H such that $1_G^\rho = 1_H$ and inductively, if $x^\rho = x'$ then

$$\rho: \quad \begin{array}{l} xt \to x't, \text{ for } t \in T_1, \\ xa \to x'c, \\ xb \to x'c^{-1}, \\ xab \to x'c^2, \\ xtab \to x'tc^2, \text{ for } t \in T_2. \end{array}$$

It is easy to see that ρ is an isomorphism from $X(G,S)$ to $X(H,S')$. ∎

Similarly, we can easily get S and S' from $G = K \times Z_2^{2n}$ and $H = K \times Z_4^n$, respectively, such that $X(G,S) \cong X(H,S')$. Now we define *(q,τ)-product of two circulant graphs.*

Definition 2.3 *For an integer q, let τ satisfy either $0 \le \tau < q$ and $(\tau, q) = 1$, or $q = 1$ and $\tau = 0$. A (q,τ)-product, denoted by $C_{rq} \times_{(q,\tau)} C_{sq}$, of C_{rq} by C_{sq} is defined as a graph Γ with vertex set V and edge set E, where*

$$V = \{(i,j,k) \mid 0 \le i \le r-1,\ 0 \le j \le s-1,\ 0 \le k \le q-1\},$$

$\big((i_1,j_1,k_1),\ (i_2,j_2,k_2)\big) \in E$ *if and only if either*

$i_2 = i_1 \pm 1\ (mod\ r),\ j_2 = j_1,\ k_2 = k_1 + [\frac{i_2-i_1\mp 1}{r}]\ (mod\ q)$; *or*

$i_2 = i_1,\ j_2 = j_1 \pm 1\ (mod\ s),\ k_2 = k_1 + [\frac{j_2-j_1\mp 1}{s}]\tau\ (mod\ q)$.

If rq is even, then let E_1 be the set of $\big((i_1,j_1,k_1),\ (i_2,j_2,k_2)\big)$ such that

$$\begin{cases} i_2 = i_1,\ j_2 = j_1,\ k_2 = k_1 + \frac{q}{2}\ (mod\ q), & q \text{ even;} \\ i_2 = i_1 + \frac{r}{2}\ (mod\ r),\ j_2 = j_1,\ k_2 = k_1 + \frac{q-1}{2}\ (mod\ q), & q \text{ odd.} \end{cases}$$

If sq is even, then let E_2 be the set of $\big((i_1,j_1,k_1),\ (i_2,j_2,k_2)\big)$ such that

$$\begin{cases} i_2 = i_1,\ j_2 = j_1,\ k_2 = k_1 + \frac{q}{2}\tau\ (mod\ q), & q \text{ even;} \\ i_2 = i_1,\ j_2 = j_1 + \frac{s}{2}\ (mod\ s), k_2 = k_1 + \frac{q-1}{2}\tau\ (mod\ q), & q \text{ odd.} \end{cases}$$

For odd q and even r,s, let E_3 be the set of $\big((i_1,j_1,k_1),(i_2,j_2,k_2)\big)$ such that

$$i_2 = i_1 + \tfrac{r}{2}\ (mod\ r),\ j_2 = j_1 + \tfrac{s}{2}\ (mod\ s), \text{and } k_2 = k_1 + \tfrac{q-1}{2}\ (mod\ q).$$

Define $M_{rq} \times_{(q,\tau)} C_{sq}$, $C_{rq} \times_{(q,\tau)} M_{sq}$ and $\Gamma(rq,sq,q,\tau)$ as the graph with vertex-set V, with edge-set $E \cup E_1$, $E \cup E_2$ and $E \cup E_3$, respectively.

Note that in the definition of $\Gamma(rq,sq,q,\tau)$, E_3 is the set of edges which join the pairs of vertices with the largest distance in $C_{rq} \times_{(q,\tau)} C_{sq}$. It is easy to show that $C_r \times_{(1,0)} C_s \cong C_r \times C_s$; $C_q \times_{(q,\tau)} C_{sq} \cong C(sq;1,s)$; and $C_{rq} \times_{(q,\tau)} C_q \cong C(rq;1,r)$.

3 The graphs of valency 4

In this section, we prove Theorem 1.1 and its corollaries.

Proof of Theorem 1.1: We first list all Cayley graphs of valency 4 on abelian groups and then determine isomorphism relations among them. Suppose that $\Gamma = X(G,S)$ for some abelian group G and some $S \in G - \{1\}$ such that $|S| = 4$.

First, assume $S = \{a, a^{-1}, b, b^{-1}\}$, where $o(a), o(b) > 2$. Let $q = |\langle a\rangle \cap \langle b\rangle|$. If $q = 1$, then $X(G,S) = X(\langle a\rangle, \{a,a^{-1}\}) \times_{(1,0)} X(\langle b\rangle, \{b,b^{-1}\})$. Now assume $q > 1$. Let

$$r = min\{i \geq 1 \mid a^i \in \langle b\rangle\},\ s = min\{i \geq 1 \mid b^i \in \langle a\rangle\},$$
$$r^* = min\{i \geq 1 \mid a^i = b^s\},\ s^* = min\{i \geq 1 \mid b^i = a^r\}.$$

Then clearly $o(a) = rq$ and $o(b) = sq$. Without loss of generality, assume $r \geq s$. Since $a^{r^*} = b^s$ and $b^{s^*} = a^r$, we have $a^{r^*} \in \langle b\rangle$ and $b^{s^*} \in \langle a\rangle$. So $r \mid r^*$ and $s \mid s^*$, that is, $r^* = \tau r$ and $s^* = \tau^* s$ for some $1 \leq \tau, \tau^* < q$. Since $a^r = b^{s^*} = b^{\tau^* s} = a^{\tau^* r^*} = a^{\tau^*\tau r}$, it follows that $a^{r(\tau\tau^*-1)} = 1$ and so $\tau\tau^* = 1\ (mod\ q)$. Clearly the vertex set of $X(G,S)$ can be written

$$VX = G = \{a^{rk+i}b^j \mid 0 \leq i \leq r-1,\ 0 \leq j \leq s-1,\ 0 \leq k \leq q-1\}.$$

In the following, we use (i,j,k) to denote the vertices $a^{rk+i}b^j$ for all admissible i,j,k. Then two vertices (i_1,j_1,k_1) and (i_2,j_2,k_2) are adjacent in $X(G,S)$ if and only if

$$a^{r(k_2-k_1)+i_2-i_1}b^{j_2-j_1} = (a^{rk_1+i_1}b^{j_1})^{-1}(a^{rk_2+i_2}b^{j_2}) \in \{a, a^{-1}, b, b^{-1}\}.$$

(1). Assume $a^{r(k_2-k_1)+i_2-i_1}b^{j_2-j_1} = a^\delta$, where $\delta = \pm 1$. Since $j_1, j_2 < s$, $j_2 = j_1$ and so

$$\begin{cases} r(k_2-k_1)+i_2-i_1 = \delta \ (mod\ rq), \\ j_2 - j_1 = 0. \end{cases}$$

It follows that $i_2 - i_1 - \delta = 0 \ (mod\ r)$. Since $0 \le i_2, i_1 < r$, we have $-r \le i_2 - i_1 - \delta \le r$. Thus $i_2 - i_1 - \delta = \varepsilon r$, where $\varepsilon = 0, 1$ or -1. Thus,

$$r(k_2-k_1) + \varepsilon r = 0 \ (mod\ rq), \text{ that is, } k_2 - k_1 = -\varepsilon \ (mod\ q).$$

So $i_2 - i_1 = \delta \ (mod\ r)$ and $k_2 - k_1 = -\varepsilon = -\frac{i_2-i_1-\delta}{r} \ (mod\ q)$.

(2). Assume $a^{r(k_2-k_1)+i_2-i_1}b^{j_2-j_1} = b^\delta$, where $\delta = \pm 1$. Since $a^r = b^{s\tau^*}$,

$$a^{i_2-i_1}b^{s\tau^*(k_2-k_1)+j_2-j_1} = a^{r(k_2-k_1)+i_2-i_1}b^{j_2-j_1} = b^\delta.$$

Since $i_1, i_2 < r$, if $i_2 \ne i_1$ then $a^{i_2-i_1} \notin \langle b \rangle$, a contradiction. Thus

$$\begin{cases} i_2 - i_1 = 0, \\ s\tau^*(k_2-k_1)+j_2-j_1 = \delta \ (mod\ qs). \end{cases}$$

It follows that $j_2 - j_1 - \delta = 0 \ (mod\ s)$. Since $0 \le j_2, j_1 < s$, we have $-s \le j_2 - j_1 - \delta \le s$. Thus $j_2 - j_1 - \delta = \varepsilon s$, where $\varepsilon = 0, 1$ or -1. Thus

$$s\tau^*(k_2-k_1) + \varepsilon s = 0 \ (mod\ sq), \text{ that is, } k_2 - k_1 = -\varepsilon\tau \ (mod\ q).$$

Hence $j_2 - j_1 = \delta \ (mod\ q)$ and $k_2 - k_1 = -\varepsilon\tau = -\frac{j_2-j_1-\delta}{s}\tau \ (mod\ q)$. So $X(G,S) \cong C_{rq} \times_{(q,\tau)} C_{sq}$.

Now assume $S = \{a, a^{-1}, d, e\}$, where $o(a) > o(d) = o(e) = 2$. If $|\langle a \rangle \cap \langle d, e \rangle| = 1$, then

$$X(G,S) \cong X(\langle a \rangle, \{a, a^{-1}\}) \times X(\langle d, e \rangle, \{d, e\}) \cong C_{o(a)} \times C_4.$$

If $|\langle a \rangle \cap \langle d, e \rangle| > 1$, then $o(a)$ is even and let $o(a) = 2r$. It is easy to see that at most one of d, e and de belongs to $\langle a \rangle$. If one of d, e belongs to $\langle a \rangle$, for example, $d \in \langle a \rangle$, then $d = a^r$ and $X(\langle a, d \rangle, \{a, a^{-1}, d\}) \cong C(2r; 1, r) = M_{2r}$. If $de \in \langle a \rangle$ then $de = a^r$. So

$$X(G,S) = \begin{cases} X(\langle a, d \rangle, \{a, a^{-1}, d\}) \times X(\langle e \rangle, \{e\}) \cong M_{2r} \times K_2, & \text{if } d = a^r, \\ X(\langle a \rangle, \{a, a^{-1}\}) \times_{(2,1)} X(\langle d, e \rangle, \{d, e\}) \cong C_{2r} \times_{(2,1)} C_4, & \text{if } de = a^r. \end{cases}$$

Finally, assume that $S = \{a, b, c, d\}$, where a, b, c, d are all of order 2. It is easy to show that $X(G,S)$ is either $C_4 \times_{(1,0)} C_4 \cong Q_4$, or $M_4 \times K_2$, or $C_4 \times_{(2,1)} C_4$.

Therefore, there are three cases:

(i) $S = \{a, a^{-1}, b, b^{-1}\}$, where $a^2 \ne 1$ and $b^2 \ne 1$;

(ii) $S = \{a, a^{-1}, d, e\}$, where $a^2 \ne 1$, $d^2 = e^2 = 1$;

(iii) $S = \{a, b, c, d\}$, where all elements have order 2.

In case (i), $X(G,S)$ is $C_{rq} \times_{(q,\tau)} C_{sq}$, where $q = |\langle a\rangle \cap \langle b\rangle|$, $r = |\langle a\rangle/(\langle a\rangle \cap \langle b\rangle)|$, $s = |\langle b\rangle/(\langle a\rangle \cap \langle b\rangle)|$, and τ is the least non-negative integer such that $a^{r\tau} = b^{s}$. In case (ii), $X(G,S)$ is $M_{2r} \times K_2$ or $C_{2r} \times_{(2,1)} C_4$, where $r = o(a)/2$. If $X(G,S) = M_{2r} \times K_2$, then $d \in \langle a\rangle$ and $e \notin \langle a\rangle$; if $X(G,S) = C_{2r} \times_{(2,1)} C_4$ then $d, e \notin \langle a\rangle$ but $de \in \langle a\rangle$. In case (iii), $X(G,S) = Q_4$, $M_4 \times K_2$ or $C_4 \times_{(2,1)} C_4$. If $X(G,S) = Q_4$ then $G = Z_2^4$ and S is a minimal generating set of G; if $X(G,S) = M_4 \times K_2$ then $G = Z_2^3$ and $\langle a,b,c\rangle = Z_2^2$; if $X(G,S) = C_4 \times_{(2,1)} C_4$ then $G = Z_2^3$ and $ab = cd$.

So every connected Cayley graph of valency 4 on abelian groups is one of the graphs listed in the theorem. Now we determine the isomorphisms between these graphs.

Clearly, if $n_1 \neq n_2$ then $M_{n_1} \times K_2 \not\cong M_{n_2} \times K_2$. For any $n \geq 4$, let $A = \mathrm{Aut}(M_n \times K_2)$ act on the edge-set of $M_n \times K_2$. Then A_1 has at least one orbit of length 1 on the neighborhood of 1. For graph $C_{rq} \times_{(q,\tau)} C_{sq}$, let $G = \langle a, b \mid ab = ba,\ a^{rq} = b^{sq} = 1,\ b^{s} = a^{r\tau}\rangle$ and $S = \{a, b, a^{-1}, b^{-1}\}$. Then we have shown that $C_{rq} \times_{(q,\tau)} C_{sq} \cong X(G,S)$ as above. Let $B = \mathrm{Aut}(C_{rq} \times_{(q,\tau)} C_{sq})$. Since there is an $\alpha \in \mathrm{Aut}(G)$ such that $a^{\alpha} = a^{-1}$ and $b^{\alpha} = b^{-1}$, each orbit of B_1 on S has length 2 or 4. So $M_n \times K_2 \not\cong C_{rq} \times_{(q,\tau)} C_{sq}$ for any r, s, q, τ, n.

Now suppose $C_{rq} \times_{(q,\tau)} C_{sq} \cong C_{r'q'} \times_{(q',\tau')} C_{s'q'}$ for some r', s', q', τ'. Let

$$H = \langle a', b' \mid a'b' = b'a',\ a'^{r'q'} = b'^{s'q'} = 1,\ b'^{s'} = a'^{r'\tau'}\rangle \text{ and } T = \{a', b', a'^{-1}, b'^{-1}\}.$$

Then $C_{r'q'} \times_{(q',\tau')} C_{s'q'} \cong X(H,T)$.

First suppose that there are $x, y \in S$ such that $|\Gamma(x) \cap \Gamma(y) - \{1\}| \geq 2$ but $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z) - \{1\}| = 0$ for any $z \in S$. It follows that $a^2 = b^{2\varepsilon}$, where $\varepsilon = \pm 1$. Without loss of generality, we may assume that $a^2 = b^2$. Then $S = a\langle e\rangle \cup a^{-1}\langle e\rangle$ where $e = a^{-1}b$ is an involution, and $\Gamma(a) \bigcap \Gamma(b) - \{1\} = \{a^2 = b^2, ab = ba\}$. Let $V_i = \{a^i, a^{i-1}b\}$ for $i = 1, 2, \cdots, n$. Then $G = \bigcup_i V_i$ is a partition of G and every vertex in V_i is adjacent to every vertex in V_{i+1}. So $X(G,S) \cong C_n[\overline{K}_2]$. If G is cyclic then G is of even order. Let $|G| = 2n$. Since $a^2 = b^2$, without loss of generality we may assume $G = \langle a\rangle$ and $b = a^{l}$ for some $1 < l < 2n-1$. Now $a^{2l} = b^2 = a^2$. Thus $a^{2l-2} = 1$ and so $2n = 2l-2$, that is, $l = n+1$. Clearly, $(l, o(a)) = 1$ if and only if n is even. If $(l, o(a)) = 1$, then $a \in \langle b\rangle$ and $n = 2q$ for some positive integer q. We have shown before that $X(G,S) \cong C_{4q} \times_{(4q,2q+1)} C_{4q}$. If $(l, o(a)) \neq 1$, then $o(b) = n$. Thus $|\langle a\rangle \cap \langle b\rangle| = n$ and $\tau = \frac{l}{2} = \frac{n+1}{2}$. So $X(G,S) \cong C_{2n} \times_{(n,\frac{n+1}{2})} C_n$. If G is noncyclic, then let $q = |\langle a\rangle \cap \langle b\rangle|$. It follows that $X(G,S) \cong C_{2q} \times_{(q,1)} C_{2q}$. Thus $C_{4q} \times_{(4q,2q-1)} C_{4q} \cong C_{2q}[\overline{K}_2] \cong C_{2q} \times_{(q,1)} C_{2q}$.

Next suppose that there are $x, y, z \in S$ such that $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z) - \{1\}| \geq 1$. It follows that $b = a^{3\varepsilon}$ or $a = b^{3\varepsilon'}$ where $\varepsilon, \varepsilon' = \pm 1$. Without loss of generality, assume that $b = a^3$. Since $X(G,S) \cong X(H,T)$, there are $x', y', z' \in T$ such that $|\Gamma(x') \cap \Gamma(y') \cap \Gamma(z') - \{1\}| \geq 1$. It follows that $b' = a'^{\pm 3}$ or $a' = b'^{\pm 3}$. Hence G and H are cyclic of the same order. It is easy to show that there is an isomorphism α from G to H such that $S^{\alpha} = T$.

Finally suppose that, for any $x, y, z \in S$, $|\Gamma(x) \cap \Gamma(y) - \{1\}| \leq 1$ and $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z) - \{1\}| = 0$. If $X(G,S) \cong X(H,T) \cong C_4 \times C_4$, then it is easy to show that $G \cong H \cong Z_4 \times Z_4$ and $S^\sigma = T$ for some isomorphism σ from G to H. Thus we may assume that $X(G,S) \ncong C_4 \times C_4$. Since $X(G,S)$ is vertex-transitive, there is an isomorphism ρ from $X(G,S)$ to $X(H,T)$ such that $1^\rho = 1$. If $\{a, a^{-1}\}^\rho = \{a'^{\varepsilon_1}, b'^{\varepsilon_2}\}$, then $\{b, b^{-1}\}^\rho = \{a'^{-\varepsilon_1}, b'^{-\varepsilon_2}\}$ where $\varepsilon_1, \varepsilon_2 = 1$ or -1, and so $\Gamma(a) \cap \Gamma(a^{-1}) - \{1\} = \{a^2 = a^{-2}\}$ and $\Gamma(b) \cap \Gamma(b^{-1}) - \{1\} = \{b^2 = b^{-2}\}$. Thus $X(G,S) \cong C_4 \times C_4$, a contradiction. Therefore, we may suppose that $\{a, a^{-1}\}^\rho = \{a', a'^{-1}\}$, $\{b, b^{-1}\}^\rho = \{b', b'^{-1}\}$ and

$$\rho: \ 1 \to 1,\ a \to a',\ b \to b',\ a^{-1} \to a'^{-1},\ b^{-1} \to b'^{-1}.$$

Then

$$\{a^i b^j\} = \Gamma(a^i) \cap \Gamma(a^j) - \{1\} \to \Gamma(a'^i) \cap \Gamma(a'^j) - \{1\} = \{a'^i b'^j\},$$

where $i, j = \pm 1$. Thus

$$\{a^2\} = \Gamma(a) - \{1, ab, ab^{-1}\} \to \Gamma(a') - \{1, a'b', a'b'^{-1}\} = \{a'^2\};$$

similarly, $a^{-2} \to a'^{-2}$, $b^2 \to b'^2$ and $b^{-2} \to b'^{-2}$. That is,

$$(a^i b^j)^\rho = a'^i b'^j, \quad \text{for } |i| + |j| \leq 2.$$

Now we inductively suppose that $(a^i b^j)^\rho = a'^i b'^j$ for all integers i, j such that $|i| + |j| \leq l$ where $l \geq 2$. Let $x = a^{i'} b^{j'}$ be a vertex of $X(G,S)$ such that $|i'| + |j'| = l - 1$. Then by the induction assumption, $x \to x' = a'^{i'} b'^{j'}$ and $(xa, xb, xa^{-1}, xb^{-1}) \to (x'a', x'b', x'a'^{-1}, x'b'^{-1})$. Thus

$$\{xab\} = \Gamma(xa) \cap \Gamma(xb) - \{x\} \to \Gamma(x'a') \cap \Gamma(x'b') - \{x'\} = \{x'a'b'\},$$

similarly, $xa^h b^k \to x'a'^h b'^k$ for $|h| + |k| = 2$. It follows that $(a^i b^j)^\rho = a'^i b'^j$ for all integers i, j such that $|i| + |j| = l + 1$. By induction, $(a^i b^j)^\rho = a'^i b'^j$ for all integers i, j. Therefore, ρ induces an isomorphism α from G to H such that $S^\alpha = T$. It follows that $(r, s, q, \tau) = (r', s', q', \tau')$. ∎

Proof of Corollary 1.1.1: By the proof of Theorem 1.1, if $G \cong H$ then there is an Ádám isomorphism between $X(G,S)$ and $X(H,T)$. Since $G = \langle S \rangle$ and $|S| = 4$, all 2-Sylow subgroups of G are of rank at most 4. By Proposition 2.2 and Theorem 1.1, it is easy to get our conclusion. ∎

Proof of Corollary 1.1.2: (1). Let Γ be a circulant graph of valency 4. Then there is a cyclic group G and $S \subseteq G - \{1\}$ with $|S| = 4$ such that $\Gamma \cong X(G,S)$. Since there is at most one involution in G, $S = \{a, a^{-1}, b, b^{-1}\}$ for some $o(a), o(b) > 2$. By the proof of Theorem

1.1, $X(G,S) \cong C_{rq} \times_{(q,\tau)} C_{sq}$, where $o(a) = rq$, $o(b) = sq$, $|G| = rsq$, $q = |\langle a\rangle \cap \langle b\rangle|$, $a^r = b^{s\tau^*}$, $b^s = a^{r\tau}$ and $\tau\tau^* = 1$. Since G is cyclic, there exists an element $c \in G$ such that $c = a^ib^j$ for some i, j and $G = \langle c\rangle$, that is, $o(a^ib^j) = rsq$. Clearly, $o(a^ib^j) = rsq$ if and only if $(r,s) = 1$.

(2). By (1) and the proof of Theorem 1.1, this is clear. ∎

4 The graphs of valency 5

In this section, we first determine symmetric Cayley graphs of valency 5 on abelian groups and then classify general graphs.

Proof of Theorem 1.2 (1): Let Γ be a symmetric Cayley graph of valency 5 for some abelian group G. Then there is a $S = \{a_1, a_2, a_3, a_4, a_5\}$ such that $\Gamma \cong X(G,S)$. Set $X = X(G,S)$ and $A = \mathrm{Aut}X$. Then A_1, the stabilizer of 1 in A, is transitive on S. Without loss of generality, suppose $\rho \in A_1$ such that $\rho : a_i \to a_{i+1}$, $i = 1,2,3,4$ and $a_5 \to a_1$. For $R \subseteq S$, let $\Gamma(R) = \bigcap_{x\in R}\Gamma(x)$ and $\Gamma^*(R) = \Gamma(R) - \bigcup_{x\in S-R}\Gamma(x)$, that is, $\Gamma^*(R)$ is the set of all elements which are joined to every elements of R and to no element of $S - R$. If $R = \{a_{i_1}, \cdots, a_{i_h}\}$ then set $\Gamma^*(R) = \Gamma^*(a_{i_1}, \cdots, a_{i_h})$. Clearly, for every $x \in \Gamma_2(1)$, there is a unique $R \subseteq S$ such that $x \in \Gamma^*(R)$. If $|\Gamma^*(S)| \leq 1$, then $|\Gamma^*(R)| \geq 1$ for some $R \subset S$. Noting that A_1 is transitive on S, R can be taken as follows.

Case 1. $|\Gamma^*(S)| \geq 2$. Let $w, y \in \Gamma^*(S)$. Then there exist $u_1, \cdots, u_5$; $v_1, \cdots, v_5 \in S$ such that

$$\begin{cases} a_1u_1 = a_2u_2 = a_3u_3 = a_4u_4 = a_5u_5 = w, \\ a_1v_1 = a_2v_2 = a_3v_3 = a_4v_4 = a_5v_5 = y, \end{cases}$$

where $u_i \neq v_i$ for $1 \leq i \leq 5$ and $\{u_1, \cdots, u_5\} = \{v_1, \cdots, v_5\} = \{a_1, \cdots, a_5\}$. Thus $a_1u_1.a_iv_i = a_1v_1.a_iu_i$ and so $u_1v_i = v_1u_i$ for $i = 2,3,4,5$. Since $u_1v_1 = v_1u_1$, we have that $u_1S = v_1S$. Since A_1 is transitive on S, it follows that $a_1S = a_2S = a_3S = a_4S = a_5S$, that is, $\Gamma(a_1) = \Gamma(a_2) = \Gamma(a_3) = \Gamma(a_4) = \Gamma(a_5)$. So $X(G,S) \cong K_{5,5}$.

Case 2. $|\Gamma^*(S - \{a_1\})| \geq 1$. Since A_1 is transitive on S, $|\Gamma^*(S-\{a_i\})| \geq 1$ for each $1 \leq i \leq 5$. If $a_1 \in \Gamma^*(S-\{a_1\})$, then clearly $X(G,S) \cong K_6$. If $a_1 \notin \Gamma^*(S-\{a_1\})$, then $a_i \notin \Gamma^*(S-\{a_i\})$ for every $1 \leq i \leq 5$ and so there exists $b_i \neq a_i$ and $b_i \in \Gamma^*(S - \{a_i\})$ for every i. It follows that for any i, j with $i \neq j$, (a_i, b_j) is an edge of $X(G,S)$. Since $(a_i, 1) \in E(X)$ and $|\Gamma(a_i)| = 5$, $\Gamma_2(1) = \{b_1, \cdots, b_5\}$. Let W be a subset of G and $X[W]$ be the subgraph of X spanned by W, that is, $X[W]$ has vertex set W and edge set $E(X) \cap (W \times W)$. Then in the subgraph $X[\{1\} \cup \Gamma(1) \cup \Gamma_2(1)]$, every vertex in $\{1\} \cup \Gamma(1)$ is of valency 5 and every vertex in $\Gamma_2(1)$ is of valency 4. Thus $\Gamma_3(1) \neq \emptyset$. If $E(X[\Gamma_2(1)]) \neq \emptyset$ then at most 3 vertices in $\{1\} \cup \Gamma(1) \cup \Gamma_2(1)$ are adjacent to the vertices in $\Gamma_3(1)$. So X is at most 3-connected.

But the vertex 1 is not in any 3-cut set of X, contrary to the vertex transitivity of X. Hence, $E(X[\Gamma_2(1)]) = \emptyset$. Clearly $\Gamma(a_1) = \{1, b_2, b_3, b_4, b_5\}$ and $a_2, a_3, a_4, a_5 \in \Gamma_2(a_1)$. Since $X(G,S)$ is vertex transitive, $|\Gamma_2(a_1)| = |\Gamma_2(1)| = 5$ and so there is $z \in \Gamma_3(1)$ such that $\Gamma_2(a_1) = \{a_2, a_3, a_4, a_5, z\}$. It follows that z is adjacent to every b_i for $i \neq 1$. Consider a_2. We have that z is adjacent to every b_i for $i \neq 2$. Thus z is adjacent to every b_i. It follows that $X(G,S) \cong K_{6,6} - 6K_2$.

Case 3. $|\Gamma^*(a_1, a_2, a_i)| \geq 1$ for $i = 3$ or 4. Since A_1 is transitive on S, there are at least five $\{i,j,k\} \subset \{1,2,3,4,5\}$ such that $|\Gamma^*(a_i, a_j, a_k)| \geq 1$. Clearly, $\Gamma^*(a_i, a_j, a_k) \cap \{a_i^2, a_j^2, a_k^2\} \neq \emptyset$. Thus there are at least five distinct square elements $a_h^2 \in \Gamma_2(1)$, contrary to that S contains an element of order 2. So this case is impossible.

Case 4. $|\Gamma^*(a_1, a_i)| \geq 1$ for some $1 < i \leq 5$. If $|\Gamma^*(a_1, a_i)| > 1$, then $\Gamma^*(a_1, a_i) = \{a_1^2 = a_i^2,\ a_1 a_i = a_i a_1\}$. Thus $\Gamma^*(a_1, a_i)^{\rho^{i-1}} = \Gamma^*(a_i, a_{2i-1}) = \{a_i^2 = a_{2i-1}^2,\ a_i a_{2i-1} = a_{2i-1} a_i\}$. Thus $a_1^2 = a_i^2 = a_{2i-1}^2$, a contradiction. So $|\Gamma^*(a_1, a_i)| = 1$ and similarly $|\Gamma^*(x,y)| = 1$ for any $x, y \in S$. If a_5 is the unique involution of S, then $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, a_5\}$. Thus $\{a_1^{-2} = a_1^2\} = \Gamma^*(a_1, a_1^{-1})$ and $\{a_2^{-2} = a_2^2\} = \Gamma^*(a_2, a_2^{-1})$, that is, $a_1^4 = 1$ and $a_2^4 = 1$. Since $|\Gamma^*(a_1, a_2)| = 1$, $a_1^2 \neq a_2^2$. So $X(\langle a_1, a_2 \rangle, \{a_1, a_1^{-1}, a_2, a_2^{-1}\}) \cong C_4 \times C_4 = Q_4$. If $a_5 \notin \langle a_1, a_2 \rangle$ then $X(G,S) \cong Q_5$. If $a_5 \in \langle a_1, a_2 \rangle$ then clearly $a_5 = a_1^2 a_2^2$ and $X(G,S) \cong Q_4^+$. If $S = \{a_1, a_1^{-1}, a_3, a_4, a_5\}$, then $o(a_1) > o(a_3) = o(a_4) = o(a_5) = 2$. Thus $\Gamma(a_1, a_1^{-1}) = \{a_1^2 = a_1^{-2}\}$ and so $o(a_1) = 4$. Arguing as above, $X(G,S) \cong Q_4^+$ or Q_5. If all elements in S are of order 2, then similarly we have $X(G,S) \cong Q_4^+$ or Q_5. ∎

Proof of Theorem 1.2 (2): Let $\Gamma = X(G,S)$, where G is an abelian group and $S = \{a, b, c, d, e\}$ such that if $o(a) > 2$ then $b = a^{-1}$ and if $o(c) > 2$ then $d = c^{-1}$. So S contains at least one involution. Let e be an involution of S and $R = S - \{e\}$. Then $X(\langle R \rangle, R)$ is a Cayley graph of valency 4 on abelian group. So it is one of the graphs listed in Theorem 1.1. If $e \notin \langle R \rangle$ then clearly $X(G,S) = X(\langle R \rangle, R) \times X(\langle e \rangle, \{e\})$. So

$$X(G,S) \cong \begin{cases} (C_m \times_{(q,r)} C_n) \times K_2, & \text{if } X(\langle R \rangle, R) \cong C_m \times_{(q,r)} C_n, \\ M_m \times C_4, & \text{if } X(\langle R \rangle, R) \cong M_m \times K_2. \end{cases}$$

Now assume $e \in \langle R \rangle$. Then $\langle R \rangle = G$. If $X(G,R) \cong M_r \times K_2$, then clearly $X(G,S) \cong M_r \times_{(2,1)} C_4$. If $X(G,T) \cong C_m \times_{(q,r)} C_n$ for some m, n, q, r, then it follows from the proof of Theorem 1.1 that $X(G,T) = X(\langle a,b \rangle, \{a,b\}) \times_{(q,r)} X(\langle c,d \rangle, \{c,d\})$. Thus

$$X(G,S) = \begin{cases} X(\langle a,b \rangle, \{a,b,e\}) \times_{(q,r)} X(\langle c,d \rangle, \{c,d\}) \cong M_m \times_{(q,r)} C_n, & \text{if } e \in \langle a,b \rangle, \\ X(\langle a,b \rangle, \{a,b\}) \times_{(q,r)} X(\langle c,d \rangle, \{c,d,e\}) \cong C_m \times_{(q,r)} M_n, & \text{if } e \in \langle c,d \rangle. \end{cases}$$

Assume, $e \notin \langle a,b \rangle \cup \langle c,d \rangle$. If $o(a), o(c) > 2$, then $e = a^{\frac{m}{2}} c^{\frac{n}{2}}$. Clearly, any two vertices which are joined by an e-edge have the largest distance in $C_m \times_{(q,r)} C_n$. Thus $X(G,S) \cong \Gamma(m,n,q,r)$. Assume $o(a) > 2$ and $o(c) = o(d) = 2$. If $e = a^{\frac{m}{2}} cd$, then $X(G,S) \cong$

$\Gamma(m,4,1,0)$. Otherwise, without loss of generality, suppose $e = a^{\frac{m}{2}}c$. Then $\langle a,c,e\rangle$ contains just three involutions and so $X(\langle a,c,e\rangle,\{a,b,c,e\}) \cong C_m \times_{(2,1)} C_4$. Thus $d \notin \langle a,c,e\rangle$. So $X(G,S) \cong (C_m \times_{(2,1)} C_4) \times K_2$. So every Cayley graph of valency 5 on abelian groups is one of the graphs listed in this theorem. Now we determine isomorphism relations among these graphs.

Let $X(G,S)$ be a graph listed in the theorem and $S = \{a,b,c,d,e\}$, where if $o(a) > 2$ then $b = a^{-1}$; if $o(c) > 2$ then $d = c^{-1}$; and $o(e) = 2$. Let $A = \mathrm{Aut}(X(G,S))$.

(i). If A_1 is transitive on S, then by (1) of the theorem, $X(G,S)$ is isomorphic to $C(6;1,2,3) \cong K_6$, $C(10;1,3,5) \cong K_{5,5}$, $\Gamma(4,4,1,0) \cong Q_4^+$, $(C_4 \times C_4) \times K_2 \cong Q_5$ or $\Gamma(6,6,3,1) \cong K_{6,6} - 6K_2$. Clearly no two of these graphs are isomorphic.

(ii). Assume that A_1^S has two orbits $S_1 = \{a,b\}$, $S_2 = \{c,d,e\}$. Then $X(\langle S_2\rangle,S_2)$ is $A^{\langle S_2\rangle}$-symmetric. By Proposition 2.1, $X(\langle S_2\rangle,S_2) \cong Q_3, K_4$ or $K_{3,3}$. If $|\langle S_1\rangle \cap \langle S_2\rangle| = 1$, then $X(G,S) \cong C_m \times K_4$, $C_m \times K_{3,3}$ or $C_m \times Q_3$, where $m = 4$ or $o(a)$. Note that $K_4 \cong M_4$ and $K_{3,3} \cong M_6$. Assume that $|\langle S_1\rangle \cap \langle S_2\rangle| > 1$. Then $X(\langle S_2\rangle,S_2) \cong K_{3,3}$ or Q_3. If $X(\langle S_2\rangle,S_2) \cong K_{3,3}$, then $|\langle S_1\rangle \cap \langle S_2\rangle| = 3$ and

$$X(G,S) \cong \begin{cases} K_{3,3} \times_{(3,1)} C_m, & \text{if } m \le 6,\\ C_m \times_{(3,1)} K_{3,3}, & \text{if } m > 6.\end{cases}$$

If $X(\langle S_2\rangle,S_2) \cong Q_3$, then $|\langle S_1\rangle \cap \langle S_2\rangle| = 2$ and $e_1 = e_2$, where e_1 is the unique involution in $\langle a,b\rangle - \{a,b\}$ and $e_2 = cde$ or c^2e, depending on $o(c) = 2$ or $o(c) = 4$, respectively. Since $X(\langle S_i\rangle,S_i)$ is symmetric for $i = 1,2$, $|\langle S_1\rangle \cap \langle c,d\rangle| = 1$. Thus $X(G,\{a,b,c,d\}) \cong C_{o(a)} \times C_4$. Since $e = cde_1$ or c^2e, $X(G,S) \cong \Gamma(o(a),4,1,0)$. Clearly, no two of these graphs as above are isomorphic.

(iii). Assume that A_1^S has an orbit $\{e\}$ of length 1. Then e is an involution and $X(G,S) = X(G,R) \cup X(G,\{e\})$, and $X(G,R)$ and $X(G,\{e\})$ is invariant under A. By Theorem 1.1, $X(\langle R\rangle,R) \cong M_r \times K_2$ or $C_m \times_{(q,\tau)} C_n$. If $X(\langle R\rangle,R) \cong M_r \times K_2$, then $X(G,S) \cong M_r \times C_4$ or $M_r \times_{(2,1)} C_4$ and e belongs to an orbit of length 2 of A_1^S, a contradiction. Thus $X(\langle R\rangle,R) \cong C_m \times_{(q,\tau)} C_n$. By Theorem 1.1, up to isomorphism, $X(\langle R\rangle,R)$ is uniquely determined by m,n,q,τ unless $C_{2q} \times_{(q,1)} C_{2q} \cong C_{4q} \times_{(4q,2q-1)} C_{4q}$. So it is easy to get isomorphism relations in the theorem. This completes the proof of the theorem. ∎

Proof of Corollary 1.2.1 and 1.2.2: By Proposition 2.2 and the proof of Theorem 1.2, it is easy to get Corollary 1.2.1 and it follows Corollary 1.2.2. ∎

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