Minimal Completely Separating Systems of Sets

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Abstract

Let $[n]$ denote an n-set. A subset S of $[n]$ separates i from j if $i \in S$ and $j \notin S$. A collection of k-sets $\mathcal{R}$ is called a (n, k) completely separating system if, for each ordered pair $(i, j) \in [n] \times [n]$ with $i \neq j$, there is a set $S \in \mathcal{R}$ which separates i from j.

Let $R(n, k)$ denote the size of a smallest (n, k) completely separating system. Amongst other things, it will be shown that $R(n,k) = \lceil 2n/k \rceil$ for $n > k^2/2$, except when $n = \binom{k+1}{2} - 1$, and $R(n,k) = k+1$ for $\binom{k}{2} \leq n \leq k^2/2$. These results build on and extend those in Ramsay et al [8].

1 Introduction

In 1961 Rényi [9] raised the problem of finding minimum separating systems in the context of solving certain problems in information theory. Subsequently, several variants have been treated in the literature. (See, for example, [1, 2, 3, 4, 5, 10, 11, 12].) Completely separating systems were introduced by Dickson [4]. It is the purpose of this paper to extend the results of Ramsay et al in [7] and [8]. Basic notation and definitions are included in this section.

Definition 1 Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A set $A \subseteq [n]$ **separates** i from j if $i \in A$ and $j \notin A$. A collection $\mathcal{S}$ of subsets of $[n]$ is a **separating system** if, for each $i, j \in [n]$ with $i \neq j$, there is a set A in $\mathcal{S}$ that separates i from j *or* a set B that separates j from i. If, for each $i, j \in [n]$ with $i \neq j$, there is a set A in $\mathcal{S}$ that separates i from j *and* a set B that separates j from i, then $\mathcal{S}$ is called a **completely separating system**.

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Observe that any complete separator is a separator, but not vice-versa. For example, in $\{\{1,2\},\{1,3\}\}$, 1 is separated from 2 by $\{1,3\}$, but 2 is not separated from 1.

The generic problem is to find separators of smallest size; that is, containing the least number of sets. This paper will be concerned exclusively with completely separating systems, abbreviated by CSSs. Let $R(n)$ denote the size of a smallest CSS on $[n]$. Dickson [4] showed that $R(n) \sim \log_2 n$. Spencer [10] obtained the sharper result that

$$R(n) = \min\{r : \binom{r}{\lfloor r/2 \rfloor} \geq n\}$$

by exploiting the connection between CSSs and Sperner families (antichains).

Notation (1) (n,k)CSS denotes a CSS of k-sets on an n-set.
(2) (n,a,k)CSS denotes a CSS of sets on an n-set where each set A in the system has $a \leq |A| \leq k$.

Definition 2 Let S be an n-set and let $\mathcal{C}$ be the collection of all completely separating systems on S in which no set occurs more than once. Then:
(1) $R(n,k) = \min_{\mathcal{R}\in\mathcal{C}}\{|\mathcal{R}| : |A| = k, \forall A \in \mathcal{R}\}$. That is, $R(n,k)$ is the minimum number of k-sets that completely separate n elements.
(2) $R(n,a,k) = \min_{\mathcal{R}\in\mathcal{C}}\{|\mathcal{R}| : a \leq |A| \leq k, \forall A \in \mathcal{R}\}$. That is, $R(n,a,k)$ is the minimum number of sets of cardinality between a and k (inclusive) which completely separate n elements.

Example $R(6,3) = 4$, since $\mathcal{R} = \{\{1,2,3\}, \{1,5,6\}, \{2,4,6\}, \{3,4,5\}\}$ is a $(6,3)$CSS and no fewer number of 3-sets will completely separate 6 elements. For $k \neq 3$, $R(6,k) = 6$. Note that any collection of k-subsets of $[n]$ that is a superset of a (n,k)CSS is also a (n,k)CSS.

Note 1 (1) The definitions above consider only the number of sets in the CSS. Another notion of minimality which is relevant to a variant of the $R(n,a,k)$ problem is the notion of *strong* minimality. Here, the number of sets is required to be minimised and then the sum of the cardinality of the sets is also required to be minimised. In general it is possible to have one but not the other of these numbers achieving a minimum amongst their possible values.
(2) The definition of $\mathcal{C}$ is for formal convenience, to ensure that only finite collections are considered. If a set A occurs more than once in a CSS $\mathcal{R}$ then $\mathcal{R}$ cannot be a minimal CSS as the removal of one occurrence of A would produce a smaller CSS.
(3) Definition 2.2 is of particular interest in this paper in the cases when $a = 1$ or 2. Only when $a = 1$ can an element of $[n]$ occur once only in a CSS, and then as a singleton set.
(4) Cai [2] shows that

$$R(n,1,k) = \lceil 2n/k \rceil, \quad \text{if } n > k^2/2 \geq 2.$$

(5) The main focus of this paper is on $R(n,k)$, and, unless otherwise stated, all material refers to the consideration of the $R(n,k)$ problem.

(6) It is easily seen that $R(n,1) = R(n,2) = n$ for all n. Henceforth, unless otherwise stated, it will be assumed that $k > 2$.

It is obvious that $R(n) \leq R(n,1,k) \leq R(n,2,k) \leq \cdots \leq R(n,k-1,k) \leq R(n,k)$. Combining this observation with the result of Spencer, the following lower bound on $R(n,k)$ is obtained.

Lemma 1 $R(n,k) \geq \min\{r : \binom{r}{\lfloor r/2 \rfloor} \geq n\}$.

Definition 3 (1) Given a CSS $\mathcal{R}$ of R k-sets on an n-set S with $2n \leq Rk < 3n$ define $E = Rk - 2n$. E is called the **excess**. E is the maximum number of elements of S which can occur more than twice in $\mathcal{R}$ and $n - E$ is the minimum number of elements which must occur at most twice in $\mathcal{R}$. For $k > 1$, "at most twice" may be read as "exactly twice".
(2) A CSS $\mathcal{R}$ on an n-set S is said to be **fair** if there exists an integer p, such that every element of S occurs in either p or $p+1$ sets of $\mathcal{R}$. That is, a CSS $\mathcal{R}$ is fair if each element of S is used, as far as is possible, the same number of times in $\mathcal{R}$.

Where no confusion arises, collections of sets are denoted in an abbreviated form, omitting the braces and the commas of the contained sets. Thus, the example given earlier can be written $\mathcal{R} = \{123, 156, 246, 345\}$. Elements greater than 9 are sometimes represented using the letters $A, B, \ldots$
In [8] it was shown that $R(n,k) \geq \lceil 2n/k \rceil$, $k > 1$, and that this bound can be achieved for $n \geq k(k-1)$. It was also shown that, in general, $R(n,1,k) \neq R(n,k)$. Here, these results are extended to show that $R(n,k) = \lceil 2n/k \rceil$ for $n > k^2/2$, except when $n = \binom{k+1}{2} - 1$. The solution to the $R(n,k)$ problem is also extended to include all $n \geq \binom{k}{2}$. The proofs of these results are constructive, thus example minimal CSSs are included for each value of n and k. Constructive examples and values for $R(n,2,k)$ are also included in the proofs.

2 Review

In this section, required results from [7, 8] are presented, with a brief discussion.

Lemma 2 *The symmetry $R(n,k) = R(n,n-k)$ holds for all $1 \leq k < n$.*

Proof By taking complements, a (n,k)CSS becomes a $(n,n-k)$CSS. □
The importance of this result lies in the fact that one normally need only consider values of $k \leq n/2$.

Lemma 3 *For all $2 \leq k < n$,*

$$R(n,k) \geq \left\lceil \frac{2n}{k} \right\rceil. \tag{1}$$

Proof For $k \geq 2$ every element of $[n]$ must appear in at least two k-sets of any (n,k)CSS. Thus $k \cdot R(n,k) \geq 2n$, from which (1) follows. □

This important lower bound is a consequence of the fact that every element of $[n]$ must occur at least twice in a (n,k)CSS, when $k \neq 1$. The main result in [8] was

Theorem 1 *If $n \geq k(k-1)$, $1 < k < n$, then*

$$R(n,k) = \left\lceil \frac{2n}{k} \right\rceil.$$

The proof of this is based on the construction of a matrix, the rows of which form a CSS. See [8] for the details.

Theorem 1 says that the lower bound on $R(n,k)$ can be achieved for sufficiently large n compared to k. The following lemma says that it is not possible to achieve the bound on $R(n,k)$ in Lemma 3 for sufficiently small n compared to k, in the case where $k \mid 2n$. That is, at least one element of $[n]$ must be used more than twice in a CSS.

Lemma 4 *If $n < \binom{k+1}{2}$, $1 < k < n$, and $k \mid 2n$, then $R(n,k) > \lceil 2n/k \rceil$.*

Proof As $k \mid 2n$, $\lceil 2n/k \rceil = 2n/k$. If $R(n,k) \leq 2n/k$ then, by Lemma 3, $R(n,k) = 2n/k$. Thus each element of $[n]$ occurs exactly twice in a separator. Without loss of generality it can be assumed that $\{1,\ldots,k\}$ is a member of a (n,k)CSS. To separate these elements, using each only once more, each element of $[k]$ must appear in a set by itself and so there are at least $k+1$ sets in the (n,k)CSS. As each element of $[n]$ occurs twice, $2n \geq k(k+1)$, as required. □

The next two lemmas show how one can use a (n,k)CSS to construct separators for larger values of n and k.

Lemma 5 *If $R(n,k) \leq k+1$, $1 < k < n$, then $R(n+k+1, k+1) \leq k+2$.*

Proof Let $\mathcal{R} = \{R_1, R_2, \ldots, R_{k+1}\}$ be a (n,k)CSS in k+1 sets. Note that a (n,k)CSS can be extended as necessary, whilst maintaining the complete separation property, by adding arbitrary k-subsets of $[n]$. Consider the set system $\mathcal{C} = \{C_0, C_1, \ldots, C_{k+1}\}$ where $C_0 = \{n+1, n+2, \ldots, n+k+1\}$ and $C_i = R_i \cup \{n+i\}$ for $i = 1, 2, \ldots, k+1$. It is easy to verify that $\mathcal{C}$ is a $(n+k+1, k+1)$CSS, in k+2 sets. □

Lemma 6 *If $R(n,k) \geq k+1$, $1 < k < n$, then $R(n+k+1, k+1) \leq 1 + R(n,k)$.*

The proof of this involves a similar construction to that of Lemma 5, but here it has to be shown that one can extend all the k-sets to $(k+1)$-sets, using elements from $\{n+1, \ldots, n+k+1\}$, whilst maintaining complete separation. See [8] for the details.

These results are examples of a number of similar results that allow one to sharpen the upper bounds on $R(n,k)$ from the trivial bound of n. To see that n is an upper bound, consider the collection $\mathcal{R} = \{\{1, \ldots, k\}, \{2, \ldots, k+1\}, \ldots, \{n, 1, \ldots, k-1\}\}$. Lemmas 5 and 6 will be important in the proofs in Sections 3.3 and 3.6.

A complete solution of the $R(n,k)$ problem is given for the cases $k = 1, 2, 3, 4$ & 5 in [8]. If $k = 1$ or 2 then $R(n,k) = n$. See Table 1 for the solutions for $k = 3$, 4 and 5. By Lemma 2, this also gives a complete solution for the $k = n - i$, $1 \leq i \leq 5$, cases. The results in this paper allow this complete solution to be extended to the $k = 6$ and $k = n - 6$ cases.

3 Results

The main result of this paper is the following theorem.

Theorem 2 *For $k < n$:*
(1) If $n \geq \binom{k+1}{2}$, $k \geq 2$, then $R(n,k) = \lceil 2n/k \rceil$;
(2) If $n = \binom{k+1}{2} - 1$, $k \geq 3$, then $R(n,k) = k + 2 = \lceil 2n/k \rceil + 1$;
(3) If $k^2/2 \leq n \leq \binom{k+1}{2} - 2$, $k \geq 5$, then $R(n,k) = k+1$, with $R(n,k) = \lceil 2n/k \rceil$ except for $n = k^2/2$;
(4) If $\binom{k}{2} \leq n < k^2/2$, $k \geq 5$, then $R(n,k) = k + 1 > \lceil 2n/k \rceil$.

Theorem 2 is a compilation of several lemmas and theorems. These results, and their proofs, are presented in Sections 3.1–3.5. Section 3.6 presents a recursive construction for parts (3) & (4) of Theorem 2, using Lemma 5.

3.1 Bounds on n for $R(n,k) \leq k$

In this section it is shown that, if $n > \binom{k}{2} - k/3$, then $R(n,k) > k$. This lower bound on $R(n,k)$ is used in the proof of Theorem 2.4 and is also an improvement on $\lceil 2n/k \rceil$ for $n < k^2/2$.

Assume that $\mathcal{R}$ is a minimal (n,k)CSS and that $|\mathcal{R}| = R \leq k$. Consider a matrix M with R rows, with the rows forming the sets in $\mathcal{R}$. Assume, without loss of generality, that row 1 is $[k]$.

Note 2 (1) All elements in a minimal (n,k)CSS $\mathcal{R}$ on $[n]$ occur at least twice in $\mathcal{R}$, as $k \neq 1$.
(2) To completely separate the elements of $[k]$, not all elements of $[k]$ can occur exactly twice in M else they could not be completely separated in $R - 1 < k$ sets.

Lemma 7 *Let M be a matrix whose rows form a minimal (n,k)CSS, with $R(n,k) = R \leq k$, $k \geq 6$. Then at most $R(k-5)/2$ elements of $[n]$ occur exactly twice in M.*

Proof Assume the R rows of a matrix M form a minimal (n,k)CSS. It will be shown that, if $R \le k$, then every row of M contains at most $k-5$ elements which occur exactly twice in M. Once this is shown, then the number of positions in M filled by elements which occur exactly twice in M is at most $R(k-5)$. Hence, at most $R(k-5)/2$ distinct elements of $[n]$ occur exactly twice in M.

Consider any row of M with $p \ge k-4$ of its elements occuring exactly twice in M. Then, without loss of generality, it can be assumed that this row is the ordered set $[k]$ and $[k-p]$ is the set of elements of $[k]$ which occur more than twice in M. To completely separate the elements of $[k]$ which occur twice only in M, p elements of $[k]$ must occur in rows of M with no other element of $[k]$. Then there are at most $R-p-1$ rows of M which can be used to completely separate the elements of $[k-p]$. By assumption, in these rows each element of $[k-p]$ must occur at least twice.

As $R \le k$ and $p \ge k-4$, it follows that $k-p \le 4$ and $R-p-1 < k-p$. It is not difficult to check, by exhaustion if necessary, that if each element must occur at least twice, it is impossible to completely separate i elements in less than i sets, for $i \le 4$. Hence, the elements of $[k-p]$ cannot be completely separated in the number of sets available. This analysis is valid for each choice of row in M and the result follows. □

Lemma 8 *If $R(n,k) \le k$, $k \ge 6$, then $n \le R(3k-5)/6 \le \binom{k}{2} - k/3$.*

Proof Given R and k, the theoretical maximum value of n occurs when as many as possible elements are used twice in the Rk total elements of a CSS, with the other elements being used three times. If $R \le k$, the previous lemma shows that there are at most $R(k-5)/2$ elements of $[n]$ which occur exactly twice in M. If this bound is attained, and the remaining elements are used exactly three times, then $5R/3$ elements of $[n]$ occur three times in M. Combining these two bounds gives the first inequality. The second inequality follows upon substitution of k for R and rearrangement of the expression. □

Incorporating the results for the $k \le 5$ cases and applying Lemma 8 yields,

Lemma 9 *If $n > \binom{k}{2} - k/3$, then $R(n,k) > k$.*

3.2 Proof of Theorem 2.1

The first step is to show that the lower bound of $\lceil 2n/k \rceil$ can be achieved for $n \ge \binom{k+1}{2}$. Example constructions of minimum CSSs for $n \ge \binom{k+1}{2}$ appear at the end of this section. Once the construction is known it is straightforward to obtain a CSS for each value of n and k, $n \ge \binom{k+1}{2}$.

Unfortunately, the proof that the construction works is not simple. This section provides the rules for the construction and, through a series of technical lemmas and notes, eventually proves the validity of the construction for all $n \ge \binom{k+1}{2}$.

In the process, constructions for minimal $(n,2,k)$CSSs are provided for each $n \geq \binom{k+1}{2}$. In this case the constructions also provide strong minimal $(n,2,k)$CSSs. This result appears as Theorem 4.

The proof requires a matrix construction, several lemmas and Theorem 4, which is extended to a (n,k)CSS. The initial step in the proof of Theorem 5 is given below as Construction M.

Construction M Assume $n \geq \binom{k+1}{2}$ and let $R = \lceil 2n/k \rceil$. An $R \times k$ matrix M will be constructed where the R rows of M form a (n,k)CSS. Let m_{ij} denote the element of M in row i column j. Initialise all elements of M to zero. Note that with the given assumptions, $R = k+1$ only when $n = \binom{k+1}{2}$, and $R \geq k+2$ otherwise.

For each $m \in [n]$ initially include m in M in exactly 2 positions as follows. For each m, in lexicographic order, include m in turn in the two positions of M defined by:

$$\min_j \min_i \{m_{ij} : m_{ij} = 0\},$$
$$\min_i \min_j \{m_{ij} : m_{ij} = 0\}.$$

That is, m is placed in the first row of M containing 0, in the first 0-valued place in that row. m is then also placed in the first column of M containing 0, in the first 0-valued place in that column. Clearly M is sufficiently large to do this. This concludes Construction M.

Now consider the special case when $n = \binom{k+1}{2}$, $k > 1$. This is the only case when $R = k+1$.

Theorem 3 *If* $n = \binom{k+1}{2}$, $k > 1$, *then* $R(n,k) = \lceil 2n/k \rceil = k+1$.

Proof Using Construction M note that $m_{ij} = m_{j+1,i}$ for all $1 \leq i,j \leq k$, and that there are no 0-valued elements left in M. Hence each element of $[n]$ appears in exactly two positions in M and, given any pair of elements (m_{ij}, m_{il}) occurring in the same row i of M, m_{ij} and m_{il} occur once more in two different rows. Hence the rows of M form a minimal (n,k)CSS. □

Corollary 1 *If* $n = \binom{k+1}{2}$, $k > 1$, *then* $R(n,2,k) = k+1$.

Henceforth assume $n > \binom{k+1}{2}$. The following notes and technical lemmas are needed and concern M after the above replacement of 0-valued elements has occurred.

Note 3 The elements of M in a given row may be partitioned into 3 parts, allowing for some of these parts to be empty. For a given row r let H or H_r denote the set of consecutive integers in row r which are the first occurrence of those integers in M using Construction M. Let D or D_r denote the set of integers in row r which are the second occurrence of those integers in M using Construction M. Let B or B_r denote the 0-valued elements of row r. Let h_r denote the least element of H_r.

Note 4 $|B| = 0$ for any row above a row which contains a non-empty set H. This follows immediately from the construction. Hence there is in M at most one row with both $|H|$, $|B| > 0$.

Note 5 $|H_r| \leq |H_{r-1}|$ for all $r > 1$. This follows immediately from the construction.

Lemma 10 *For $n > \binom{k+1}{2}$:*
(1) in row r, and in terms of column order, all elements of D_r occur before all elements of H_r which occur before all elements of B_r;
(2) $|H_R| = 0$.

Proof (1) For any row r and any $m \in D_r$ with $m \in H_s$ for some $s < r$, it must be that m occurs in row r before any element of H_r.

Assume $r < R$. In Construction M, if an element h of H_r is included in M at m_{rj}, then $m_{r+1,j}$ is 0 at this stage. Therefore, the latest occurrence of the second occurrence of h in M is in column j. Hence no second occurrence of an element of H_r can occur in row r after H_r. Hence all elements of D_r occur before all elements of H_r. It is clear that no 0-valued element can occur in row r before a non-zero element. The case $r = R$ is dealt with in (2).

(2) Assume $|H_R| > 0$. By Note 4, $|B_{R-1}| = 0$. By Note 5, $|H_{R-1}| \geq |H_R|$. By Construction M, if h_{R-1} is $m_{R-1,j}$ then $m_{R-1,j-1}$ is non-zero and equal to an element of D_{R-1}. Hence the first column in row R where an element of H_{R-1} can be placed the second time is at least column $j-1$. Therefore the elements of H_{R-1}, when they have been placed for the second time, leave at most one 0-valued element in row R of M. This contradicts the choice of the size for M as it then leaves insufficient places in M for the insertion of elements of H_R in at least 2 positions in M. It follows that $|H_R| = 0$ and that part (1) of the lemma is true for $r = R$. □

Note 6 Hence, in construction M, the rows of M in numeric order can be partitioned into non-empty collections with parts $\mathcal{H}$, $\mathcal{DH}$, $\mathcal{DHB}$, $\mathcal{DB}$ or $\mathcal{H}$, $\mathcal{DH}$, $\mathcal{DB}$ where, for example, the collection $\mathcal{H}$ represents the set of rows in M with $|H| > 0$ and $|D| = |B| = 0$. $\mathcal{DH}$ represents the set of rows of M with $|D|, |H| > 0$ and $|B| = 0$.

The next lemma is pivotal to the proof. It shows that in each row i of M there are at least $|H_i|$ rows below row i.

Lemma 11 *For $n > \binom{k+1}{2}$, in each row i of M, $|H_i| \leq R - i$.*

Note 7 This lemma implies that if h_i is in column j then h_{i+1} is in column j or $j+1$.

Proof of Lemma 11 The proof involves four claims.

Claim 1 *If such a row exists, the first row of M with $|H_r| > R - r$ is not the first row i of M for which h_i is m_{ij} with $j < i$.*

Proof of claim 1 Note that M has a sequence of rows for which $h_i = m_{ij}$ with $i = j$. Let r be the first row for which $j < i$. By Note 5 $|H_r| \leq |H_{r-1}|$ for all r. As $R > k + 1$, row $r - 1$ has at least $|H_{r-1}| + 1$ rows below it in M. Therefore row r has at least $|H_r|$ rows below it. □

Claim 2 *If such a row exists, the first row of M with $|H_r| > R - r$ is not a row t in M with D_t, H_t, B_t each non-empty.*

Proof of claim 2 Assume row t has each of D, H and B non-empty and suppose that row t is the first for which $|H_t| > R - t$. By Note 5 $H_t \leq |H_{t-1}|$. By Note 4 $|B_{t-1}| = 0$. By assumption $|B_t| > 0$. Therefore $|H_t| < |H_{t-1}|$. Hence, as row $t - 1$ has at least $|H_{t-1}|$ rows below it, row t has at least $|H_t|$ rows below it. □

Claim 3 *Each row r with $|D_r|, |H_r| > 0$ and $|B_r| = 0$ has at least $|H_r|$ rows below it in M.*

Proof of claim 3 Assume row r is the first row of M with $|H_r| > R - r$, $|D_r| > 0$, $|B_r| = 0$. Assume $h_r = m_{rj}$.

The following assumptions may be made:

1. $|H_r| > 1$. This follows from the assumption that $|H_r| > 0$ and $|H_r| \neq 1$ as $r \neq R$ by Lemma 10.2. Hence there is at least one row below row r in M.

2. $m_{r,j-1}$ is equal to an element of H_i for some $i \leq r - 1$. This follows from the construction and the assumed position of h_r in M.

3. $|H_r| = |H_{r-1}|$. This follows from Notes 4 & 5, Lemma 10 and our choice of r.

This means that row $r - 1$ has exactly $|H_{r-1}|$ rows below it. Thus $m_{r,j-1}$ must be equal to an element of H_{r-1} and $m_{r-1,j-1}$ must be equal to an element of H_{r-2}. Therefore the first occurrence of an element of H_{r-1} below row $r - 1$ is at $m_{r,j-1}$. That is, h_{r-1} occurs at $m_{r,j-1}$ for the second time.

4. h_{r-2} occurs at $m_{r-2,j-1}$. To see this, suppose h_{r-2} is at $m_{r-2,l}$. If $l > j$, this contradicts Note 5. If $l = j$, then by the above discussion, the element at $m_{r-1,j-1}$ is an element of H_{r-2} while the element at $m_{r-2,j-1}$ is an element of H_{r-3}. This implies that $|H_{r-2}| = 1$, contradicting Assumption 1 and Note 3. If it is assumed that $l < j - 1$, then $|H_{r-2}| > R - (r - 2)$, contradicting the choice of r. Thus, h_{r-2} is at $m_{r-2,j-1}$.

The proof of Claim 3 needs Claim 4.

Claim 4 *With the assumptions stated immediately above for each $i < r$, row i has exactly $|H_i|$ row below it in M.*

Claim 4 immediately leads to a contradiction as $|H_1| = k$ and $R > k+1$. That is, row 1 has more than k rows below it. Thus Claim 3 is proved, once Claim 4 is proved. □

Proof of claim 4 Induct on i. The claim is true for $i = r-1$ by Assumption 3. Assume that the claim is true for all i, $p \leq i \leq r-1$. Assume h_{p+1} is at $m_{p+1,q}$.

It is clear that $m_{p+1,q-1}$ is not equal to an element of H_{p+1}, and as H_p has exactly $|H_p|$ rows below it, $m_{p+1,q-1}$ must be equal to an element of H_p. Further, $m_{p,q-2}$ is not an element of H_p. Assume $|H_{p-1}| = |H_p|$. Then $m_{p-1,q-2}$ is not an element of H_{p-1}.

Therefore the first ocurrence of an element of H_{p-1} below row $p-1$ is in column $q-2$ at or below row p. As $|H_{p-1}| = |H_p|$ and $m_{p,q-1}$ is an element of H_p, the first occurrence of an element of H_p below row p is at $m_{R,q-2}$ or at $m_{p+1,q-1}$.

If the first occurrence of an element of H_p below row p is at $m_{R,q-2}$, then the first occurrence of an element of H_{p+1} is at $m_{R,q-1}$. As each row i, $p \leq i \leq r-1$, has exactly $|H_i|$ rows below it, h_{r-1} is at $m_{R,j-2}$. This contradicts Assumption 3.

If the first occurrence of an element of H_p below row p is at $m_{p+1,q-1}$, then the first occurrence of h_{r-1} is at m_{rj}. Again this contradicts Assumption 3.

Thus $|H_{p-1}| > |H_p|$. If $|H_{p-1}| > |H_p| + 1$ then H_{p-1} has less than $|H_{p-1}|$ rows below it, contradicting our choice of r. Hence $|H_{p-1}| = |H_p| + 1$ and Claim 4 is proved. □

This completes the proof of Lemma 11. □

Theorem 4 *For $n > \binom{k+1}{2}$ the rows of M in the construction M, ignoring the 0-valued elements, form a minimal $(n, 2, k)CSS$.*

Proof Consider the rows of M as being sets consisting of the non-zero valued elements in the rows. Each element of $[n]$ occurs exactly twice in M, and $R = \lceil \frac{2n}{k} \rceil$.

By taking $n = pk + r$, $0 \leq r < k$, it is easily seen that the number of 0-valued elements of M are at most $Rk - 2n \leq k - 2$. Therefore M has no singleton set, and it can be concluded that $R(n, 2, k) \geq \lceil \frac{2n}{k} \rceil$. It needs to be shown that the rows of M, ignoring the 0-valued elements, form a completely separating system. For any D_i, each element of D_i is separated from each element of H_i as each element of D_i occurs above row i and no element of H_i occurs above row i.

By Lemma 11 the elements of D_i will each occur in different rows above row i as members of different H sets. By Lemma 11 the elements of H_i appear in different rows of M below row i. Therefore each of these are separated from one another. The elements of H_i are also separated from all elements of D_i as no element of D_i appears below row i in M. Thus the rows of M from a completely separating system. □

Lemma 12 *Assume $n > \binom{k+1}{2}$.*
(1) Each row of M containing a non-empty H has the least element of H in position m_{ij} where $j \leq i$.
(2) Let $t \leq k$ be the last row in M with $|H| \neq 0$. Then the least element of H_t is in position m_{tj} where $j < t$.

Proof (1) Note that by Lemma 11, if h_{r-1} is $m_{r-1,j}$ then h_r is m_{rj} or $m_{r,j+1}$. Hence as h_1 is in column 1 the result is true for all subsequent rows.

(2) Assume the conditions of Lemma 12.2. By part (1), $j \leq t$. To show that $j < t$, assume $j = t$. Then by Note 7 and Lemma 12.1 $|H_r| = k - r + 1$ for all $r \leq t$. Therefore $n = \sum_{i=1}^{t} |H_i| \leq \sum_{i=1}^{k} i = \binom{k+1}{2}$. This contradicts $n > \binom{k+1}{2}$. □

Lemma 13 *Let* $n > \binom{k+1}{2}$. *Let row* t *be the last row* t *of* M *with* $|H_t| > 0$. *Then* $|D_t| \leq t - 2$.

Proof Let row t be as in the statement of the lemma. Lemma 11 ensures that the second occurence of elements in a given H do not occur in adjacent positions in the same row. Hence if h_i is at m_{ij} then h_{i+1} is at $m_{i+1,j}$ or at $m_{i+1,j+1}$. This, together with Lemma 12.2, implies that $|D_t| \leq t-2$ when $t \leq k$. If $t > k$ then as $|D_t| \leq k-1$, $|D_t| \leq t-2$. □

Lemma 14 *If* $n > \binom{k+1}{2}$ *and* $k|2n$ *then the rows of* M *are a minimal* (n,k)*CSS.*

Proof If $k|2n$ then M has no 0-valued elements at this stage. Hence, as shown in Theorem 4, the rows of M form an appropriate completely separating system. □

Lemma 15 *If* $n > \binom{k+1}{2}$ *and* $k \not| 2n$ *then the 0-valued elements in* M *can be replaced by elements of* $[n]$ *to form a minimal* (n,k)*CSS.*

Proof The 0-valued elements of M need to be replaced whilst ensuring that the complete separation property is maintained. This is done in numeric order of the rows. Consider two cases, with t defined as in Lemma 13 to be the last row of M with $|H_t| > 0$.

(i) Assume that row t of M has $|H_t| \neq 0$ and $|B_t| \neq 0$. By Note 4 there is at most one of these rows. Each element of D_t occurs in exactly one row above row t. No element of H_t occurs in a row above t. By Lemma 13 there are at least $|D_t| + 1$ rows above row t. Hence there is a row r above row t which contains no element of row t. The elements of row r will be used to fill B_t.

It must be ensured that the elements of row r used in B_t are already separated from the elements of row t. Note that at this stage any two elements of row r are already completely separated in M.

The elements of D_t appear in exactly $|D_t|$ rows of M above row t. Hence they occur with at most $|D_t|$ different elements of row r. These elements of row r cannot be used to place in B_t as this would destroy the complete separation property.

It is necessary to ensure that the elements of H_t are separated in a row below row t from the set of elements of row r used to replace B_t. To do this note that the elements of H_t occur in exactly $|H_t|$ different rows below row t. Hence at most $|H_t|$ elements of row r occur with elements of H_t in these lower rows.

Thus there are at least $k - |D_t| - |H_t| = |B_t|$ elements of row r that can be used to replace the 0-valued elements in row t, whilst maintaining the complete separation property.

(ii) Consider any row s of M with $|H_s| = 0$ and $|B_s| > 0$. Then $s > t$ by Note 4. By Lemmas 11 and 13, $|D_s| \le |D_t| + 1 \le t - 1$. Then if $|B_t| = 0$ there is at least one row r at or above row t which contains no element of row s. Each element of row s occurs in at most one row at or above row t and hence with at most $|D_s|$ elements of row r. Hence there are at least $k - |D_s| = |B_s|$ elements of row r which can be used to replace the elements of $|B_s|$ whilst maintaining the complete separation property. Note that $|D_s| = |D_t| + 1$ if and only if D_s contains an element of H_t. Therefore, if $|B_t| > 0$ and D_s contains no element h of H_t, then $|D_s| \le |D_t|$. If $|B_t| > 0$ and D_s contains an element h of H_t then $|D_s - \{h\}| \le |D_t|$. Note that in this case h does not occur above row t in M. Therefore, in either case, by applying Lemma 13, there is at least one row r above row t which contains no elements of D_s. Each element of row D_s occurs in at most one row above row t and hence with at most $|D_s|$ elements of row r. Hence there are at least $k - |D_s| = |B_s|$ elements of row r which can be used to replace the elements of $|B_s|$ whilst maintaining the complete separation property. This completes the proof of the lemma. □

Theorem 5 *If* $n \ge \binom{k+1}{2}$, $k > 1$, *then* $R(n,k) = \lceil 2n/k \rceil$.

Proof Combine Theorem 3 and Lemmas 14 & 15. □

As an example of this construction, consider the following three matrices, the rows of which are minimal separators for the (10,4), (13,4) and (16,5)CSS cases. The elements used to fill the 0-valued positions of M have been offset for clarity.

1	2	3	4
1	5	6	7
2	5	8	9
3	6	8	10
4	7	9	10

1	2	3	4
1	5	6	7
2	7	8	9
3	8	10	11
4	9	12	13
5	10	12	2
6	11	13	2

1	2	3	4	5
1	6	7	8	9
2	7	10	11	12
3	8	12	13	14
4	9	13	15	16
5	10	14	16	6
6	11	15	1	3

Note that this construction is not fair in general. In the second example above, 2 occurs four times and there is no other possible choice with the given construction.

3.3 Proof of Theorem 2.2

One of the open questions posed in [7] was whether or not the values of $R(n,k)$ are monotonic with n, for fixed $k \ne 4,5$ and $n \ge 2k$. This question is answered in the negative, by showing that the lower bound of $\lceil 2n/k \rceil = k+1$ cannot be achieved for $n = \binom{k+1}{2} - 1$, whilst it is achieved for $n = \binom{k+1}{2} - 2$ and, as seen above, for $n = \binom{k+1}{2}$. Note that the $R(\binom{k+1}{2} - 1, k) = R(\frac{k(k+1)}{2} - 1, k)$ values are related by the $(n', k') = (n+k+1, k+1)$ construction method of Lemma 6.

Lemma 16 *If $k \geq 3$ then $R(\binom{k+1}{2} - 1, k) = k + 2 = \lceil 2n/k \rceil + 1$.*

Proof The proof is by induction, with the base case being the result that $R(5,3) = R(5,2) = 5$. Note that, if $k \geq 3$, then the lower bound $\lceil 2n/k \rceil$ is $k+1$.

Assume that the result holds for all values of $k \leq k'$, for some $k' \geq 3$. Let $k = k' + 1$ and assume that $R(n,k) = k+1$, with $n = (k(k+1)/2) - 1$. If this is the case, then one element of $[n]$ occurs 4 times or two elements of $[n]$ occur 3 times in the separator, with all other elements occuring exactly twice. That is, the excess is 2. It can be assumed, without loss of generality, that $\{1, \ldots, k\}$ occurs in the separator. There are four cases for the other k sets in the separator.

(a) If each of $1, \ldots, k$ occurs only once more then, to be separated, they must occur singly in the remaining sets. Thus, the k $(k-1)$-sets formed by removing each of $1, \ldots, k$ from these sets must form a $(n-k, k-1)$CSS in k sets. That is, a $((k-1)k/2 - 1, k-1)$CSS. This is not possible, by the inductive hypothesis.

(b) Assume that 1 occurs twice and each of $2, \ldots, k$ occur once in the remaining k sets. To separate $2, \ldots, k$, with only one occurence of each, they must appear in $k-1$ separate sets of the k available. Since the 1's cannot be in the same set, at least one of $2, \ldots, k$ must occur with 1 and thus cannot be separated from it.

(c) Assume that 1 occurs three times and each of $2, \ldots, k$ occur once in the remaining k sets. To separate $2, \ldots, k$, with only one occurence of each, they must appear in $k-1$ separate sets of the k available. Since the 1's cannot all be in the same set, at least two of $2, \ldots, k$ must occur with 1 and thus cannot be separated from it.

(d) Assume that 1 and 2 each occur twice and each of $3, \ldots, k$ occur once in the remaining k sets. If 1 or 2 occur singly in the remaining two sets then at least one of $3, \ldots, k$ must occur with 1 or 2 and thus cannot be separated from it. If 1 and 2 occur together in the remaining two sets then they are not separated from each other. If 1 and 2 occur in the $k-2$ sets with $3, \ldots, k$ then at least one of $3, \ldots, k$ is not separated from at least one of 1 and 2.

Thus, in all cases there is a contradiction, so $R(n,k) > k+1$. That $R(n,k) = k+2$ follows from the inductive hypothesis and the base case of $n = 5, k = 3$, via the construction of Lemma 6. □

As an example of the construction provided by Lemma 6, consider the following four matrices, the rows of which are minimal separators for the $k = 3, 4, 5, 6$ cases. Note that these separators are fair.

1	2	3
2	3	4
3	4	5
4	5	1
5	1	2

1	2	3	6
2	3	4	6
3	4	5	7
4	5	1	8
5	1	2	9
6	7	8	9

1	2	3	6	10
2	3	4	6	10
3	4	5	7	11
4	5	1	8	12
5	1	2	9	13
6	7	8	9	14
10	11	12	13	14

1	2	3	6	10	15
2	3	4	6	10	15
3	4	5	7	11	16
4	5	1	8	12	17
5	1	2	9	13	18
6	7	8	9	14	19
10	11	12	13	14	20
15	16	17	18	19	20

3.4 Proof of Theorem 2.3

Lemma 17 *If $k^2/2 \le n \le \binom{k+1}{2} - 2$, $k > 1$, then:*
(1) $R(n,2,k) \ge k+1$;
(2) $R(n,1,k) \ge k+1$.

Proof (1) Assume the conditions of the lemma. Assume $R(n,2,k) \le k$. In any $(n,2,k)$CSS each element must occur in at least two different sets. Therefore, the number of sets in a minimum $(n,2,k)$CSS is at least $2n/k$. For $n > k^2/2$ this means that $n > k$.

If $n = k^2/2$ then $2n/k = k$ so at least k k-sets are required in a CSS. If $R(n,2,k) = k$, each element must occur in exactly two sets in a $R(n,2,k)$CSS. Assume $[k]$ is one of the sets in the minimal CSS. Then, as each element of $[k]$ must occur exactly once more without the other elements of $[k]$, at least k more sets are required in the CSS. Hence the minimum CSS requires at least $k+1$ sets.

(2) Assume the conditions of the lemma and that, for some n, k, there exists a $(n,1,k)$CSS $\mathcal{R}$ with $|\mathcal{R}| \le k$. By part (1), $\mathcal{R}$ must contain a singleton set and therefore $\sum_{R\in\mathcal{R}} |R| \le k(k-1)+1 = k^2-(k-1)$. As $2n \ge k^2$, there must be at least $k-1$ elements of $[n]$ which occur in only one set in $\mathcal{R}$ and hence must occur in singletons sets in $\mathcal{R}$. As there are at least $k-1$ such singleton sets and $|\mathcal{R}| \le k$, it is clearly impossible for $\mathcal{R}$ to completely separate the remaining elements of $[n]$. □

Theorem 6 *If $k^2/2 \le n \le \binom{k+1}{2} - 2$, $k > 1$, then $R(n,k) = k+1$. In each case, a fair minimal CSS exists.*

Proof Assume the conditions of the theorem. The theorem is vacuously true for $k \le 3$ so assume $k \ge 4$. By Lemma 9 $R(n,k) > k$ if $n \ge k^2/2$. An $R \times k = (k+1) \times k$ matrix M will be constructed such that its row vectors form a (n,k)CSS. Note that the excess E has $4 \le E \le k$ and E is always even.

The initial step is to use construction M as given in Section 3.2. This provides a fair $(n,2,k)$CSS and a fair $(n,1,k)$CSS. The fairness and the complete separation property of the system is clear.

Once construction M has been completed consider the position of the 0-valued elements remaining in matrix M. Consideration of construction M, with the given size of matrix M, easily leads to the truth of the following statement for all values of E. With the possible exception of one column, each column in M which now contains a 0-valued element contains at least three 0-valued elements. The notation h_t, H_t, B_t as defined in the proof of Theorem 2.1 is used where appropriate in the remainder of this proof. Assume h_t occurs at m_{tt}. Define the submatrix A of M by $A = \{m_{ij} \in M : i,j \ge t\}$ if $|B_t| > 0$ and $A = \{m_{ij} \in M : i,j > t\}$ if $|B_t| = 0$. Assume A is a $r \times (r-1)$ matrix with column vectors A_i, $i = 1,\ldots,r-1$. There are four cases to consider.

1) If row t contains no 0-valued elements each A_j contains at least three 0-valued elements. Then, in row order, replace the first 0-valued element in A_j by $m_{k+1,j}$ and

for each remaining 0-valued element in row i and in A_j replace it by $m_{i-1,j}$. Given that construction M forms a fair $(n,2,k)$CSS it is easy to see that in this case the rows of M form a fair (n,k)CSS.

For the remaining cases assume row t contains some 0-valued elements.

2) If A is a 3×2 matrix set $m_{k+1,t} = m_{tt}$. The remaining columns of A can be dealt with as in case 1.

3) If A is a 4×3 matrix and $|H_t| = 2$ set $m_{k+1,t} = m_{t,t}$. The remaining columns of A can be dealt with as in case 1.

4) If A is an $r \times (r-1)$ matrix with $r > 3$, set $m_{ii} = m_{it}$ for $t < i \leq k$, set $m_{k+1,k+1} = m_{tt}$ and then set $m_{it} = 0$ for $i > t$. Now each column of M which contains a 0-valued element contains at least three such elements. The remaining 0-valued elements can be replaced in a similar way to that outlined in case 1. Let a_{ij} denote the element in row i column j of A with $i = t, \ldots, k+1$ and $j = 1, \ldots, r-1$. For each column vector $A_j = \{a_{ij}\}$ form the column vector $C_j = \{a_{ij}\}$ of elements in column j of M. In row order, replace the first 0-valued element in A_j by the last element of C_j in the same row as a 0-valued element of A_j. For each a_{ij}, $i > 1$, simultaneously set $a_{ij} = c_{mj}$ where $m < i$ is the row index of the first 0-valued element immediately above a_{ij} in A_j.

By the nature of construction M, to show that the rows of M now form a fair (n,k)CSS, it is only necessary to consider the elements of the matrix A. All elements of $[n]$ not in A are completely separated from elements of A in some row above row t. The elements of different A_j, other than elements of H_t, are completely separated from one another in rows above row t. Elements of the same A_j are completely separated from one another in the corresponding sets A_j and C_j.

The elements of H_t are completely separated from one another in A. They are completely separated from elements of the A_j by occurring in row t or, in the case of h_t when $|B_t| = 0$, by occurring three times in A.

The fairness of the system is clear as every element occurs either two or three times in M. This completes the proof. $\square$

As an example of this construction, consider the following three matrices, the rows of which are minimal separators for the $k = 10$, $n = 50, 51, 52$ & 53 cases. These illustrate, respectively, cases 4, 3, 1 & 2 of the proof.

1	2	3	4	5	6	7	8	9	10
1	11	12	13	14	15	16	17	18	19
2	11	20	21	22	23	24	25	26	27
3	12	20	28	29	30	31	32	33	34
4	13	21	28	35	36	37	38	39	40
5	14	22	29	35	41	42	43	44	45
6	15	23	30	36	41	46	47	48	49
7	16	24	31	37	42	46	50	19	26
8	17	25	32	38	43	47	10	50	24
9	18	26	33	39	44	48	8	16	25
10	19	27	34	40	45	49	9	18	50

1	2	3	4	5	6	7	8	9	10
1	11	12	13	14	15	16	17	18	19
2	11	20	21	22	23	24	25	26	27
3	12	20	28	29	30	31	32	33	34
4	13	21	28	35	36	37	38	39	40
5	14	22	29	35	41	42	43	44	45
6	15	23	30	36	41	46	47	48	49
7	16	24	31	37	42	46	50	51	19
8	17	25	32	38	43	47	50	10	16
9	18	26	33	39	44	48	51	8	17
10	19	27	34	40	45	49	50	9	18

1	2	3	4	5	6	7	8	9	10
1	11	12	13	14	15	16	17	18	19
2	11	20	21	22	23	24	25	26	27
3	12	20	28	29	30	31	32	33	34
4	13	21	28	35	36	37	38	39	40
5	14	22	29	35	41	42	43	44	45
6	15	23	30	36	41	46	47	48	49
7	16	24	31	37	42	46	50	51	52
8	17	25	32	38	43	47	50	10	19
9	18	26	33	39	44	48	51	8	17
10	19	27	34	40	45	49	52	9	18

1	2	3	4	5	6	7	8	9	10
1	11	12	13	14	15	16	17	18	19
2	11	20	21	22	23	24	25	26	27
3	12	20	28	29	30	31	32	33	34
4	13	21	28	35	36	37	38	39	40
5	14	22	29	35	41	42	43	44	45
6	15	23	30	36	41	46	47	48	49
7	16	24	31	37	42	46	50	51	52
8	17	25	32	38	43	47	50	53	10
9	18	26	33	39	44	48	51	53	8
10	19	27	34	40	45	49	52	53	9

Corollary 2 *If $k^2/2 \leq n \leq \binom{k+1}{2} - 2$ then minimal $(n, 2, k)$CSSs and minimal $(n, 1, k)$CSSs contain $k + 1$ sets. In each case fair minimal CSSs exist.*

Proof Lemma 17 ensures that $R(n, 1, k) > k$ and $R(n, 2, k) > k$. Construction M, in Section 3.2, provides a fair $(n, 1, k)$CSS and a fair $(n, 2, k)$CSS using $k+1$ sets. □

3.5 Proof of Theorem 2.4

Lemma 18 *If $\binom{k}{2} \leq n < k^2/2$, $k \geq 5$, then $R(n, k) = k + 1 \neq \lceil 2n/k \rceil$ and a fair minimal separator exists in each case.*

Proof Assume the conditions of the theorem. Then $R(n, k) > k$ by Lemma 9.

Let $R = k + 1$. An $R \times k$ matrix M will be constructed with the R row vectors of M forming a (n, k)CSS. Initialise all elements of M to zero. Let m_{ij} denote the element of M in row i, column j.

Partition M into four parts defined by:

$$
\begin{aligned}
A &= \{m_{1j} : 1 \leq j \leq k\}; \\
B &= \{m_{ij} : 2 \leq j \leq k+1, 1 \leq j \leq 2\}; \\
C &= \{m_{ij} : 3 \leq i \leq k, 3 \leq j \leq k\}; \\
D &= \{m_{k+1,j} : 3 \leq j \leq k\}.
\end{aligned}
$$

The elements of these parts are now defined for various cases.

Case 1: Assume $n = \binom{k}{2}$.

For part A, set $m_{1j} = j$ for $1 \leq j \leq k$.

For part B, set $m_{k+1,2} = 1$ and for $i \neq k+1$, $j \neq 2$ set $m_{ij} = i + j - 2$ for $1 \leq i \leq 2$ and $2 \leq j \leq k+1$.

For part C, use the construction M on the set $\{k+1, \ldots, n\}$. At this stage C has exactly two 0-valued elements at $m_{k-1,k}$ and $m_{k,k}$. Set $m_{k-1,k} = m_{k-2,3}$ and $m_{kk} = m_{k-1,3}$.

Part D is filled by setting $m_{k+1,j} = m_{jj}$, $3 \leq j \leq k-1$ and $m_{k+1,k} = m_{k,3}$.

The elements of A are completely separated in B and are completely separated in A from all other elements of $[n]$. It can be noted that the element n first occurs exactly at $m_{k-3,k}$ and hence all elements of $[n]$ occur in a row above row $k-1$. The fact that the elements of C are completely separated from one another is then easily seen as a feature of the construction with the only special cases being the separation of $m_{k,3}$ from $m_{k-1,3}$ at $m_{k+1,k}$ and the separation of $n-1$ from $m_{k-3,3}$ at $m_{k+1,k-1}$. The choice of elements of D other than $m_{k+1,k}$ ensures the separation of the elements of C from the elements of B. Hence the row vectors of M form a fair CSS on $[n]$.

Case 2: Assume $n = \binom{k}{2} + 1$.

The construction in this case is the same as for case 1, with one modification. For this case set $m_{i,k} = n$ for $k-1 \leq i \leq k+1$. It is easier than in case 1 to see that M is a fair CSS in this case.

Case 3: Assume $\binom{k}{2} + 1 < n \leq k^2/2 - 1$.

For this case note that, with the assumed bounds on n, there are less than $k/2$ elements of $[n]$ greater than $\binom{k}{2}$. For ease of notation, assume there are r elements greater than $\binom{k}{2}$, denoted by $d_1, \ldots, d_r$.

To construct a fair CSS on $[n]$ now proceed as for case 2. Then the remaining elements not yet included in M are used to replace elements of M as follows: For $2 \leq i \leq r$, set $m_{i,2} = d_i$ and $m_{k+1,i+1} = d_i$. Set $m_{k+1,2} = m_{r+1,1}$.

Given that $k \geq 6$ and $r < k/2$ it is a small matter to check that the rows of M form a fair CSS on $[n]$. To check that complete separation is maintained note that the elements replaced in D are the ones which now occur in the same row as only one element of $[k]$. Hence they no longer need to be repeated in D to separate them from the elements of $[k]$. The change in value of $m_{k+1,2}$ is important to ensure that r is separated from $r+1$. It is clear that the elements $d_1, \ldots, d_r$ are completely separated from one another with the construction.

Hence the theorem is proven. □

As an example of this construction, consider the following four matrices, the rows of which are minimal separators for the $k = 7$, $n = 21, 22, 23$ & 24 cases.

1	2	3	4	5	6	7
1	2	8	9	10	11	12
2	3	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	10
6	7	12	16	19	21	11
7	1	8	13	17	20	12

1	2	3	4	5	6	7
1	2	8	9	10	11	12
2	3	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	22
6	7	12	16	19	21	22
7	1	8	13	17	20	22

1	2	3	4	5	6	7
1	23	8	9	10	11	12
2	3	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	22
6	7	12	16	19	21	22
7	2	23	13	17	20	22

1	2	3	4	5	6	7
1	23	8	9	10	11	12
2	24	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	22
6	7	12	16	19	21	22
7	3	23	24	17	20	22

3.6 An Alternative Construction

Corollary 3 proves an alternative method of constructing a fair (n,k)CSS in $k+1$ sets, for $\binom{k}{2} \leq n \leq \binom{k+1}{2} - 2$, based on Lemma 5. Note that $\left(\binom{k+1}{2} - 2\right) - \binom{k}{2} = k-2$. The construction is by induction, with the base cases being the construction of Lemma 18 (for the $n = \binom{k}{2}$ case only) and some known $(n,5)$CSSs.

The inductive step is based on the observations that, if $\binom{k}{2} < n$ then $\binom{k-1}{2} \leq n-k$, and if $n \leq \binom{k+1}{2} - 2$ then $n - k \leq \binom{k}{2} - 2$. Thus, the construction of Lemma 5 can be used for the inductive step.

Corollary 3 *If $\binom{k}{2} \leq n \leq \binom{k+1}{2} - 2$, $k \geq 5$, then $R(n,k) \leq k+1$ and a fair (n,k)CSS in $k+1$ sets exists.*

Proof Lemma 18 establishes the result for the case $n = \binom{k}{2}$. The collections {129AB, 13678, 24578, 345AB, 5689B, 4679A}, {12345, 16ABC, 26789, 389BC, 479AC, 578AB}, and {12345, 16789, 26BCD, 37ACD, 48ABD, 59ABC} establish the result for the remaining cases when $k = 5$. Together, these form the basis for the induction, which is on k.

Assume that the corollary is true for some $k' \geq 5$ and consider the case $k = k' + 1$. If $n = \binom{k}{2}$, the result follows from the base case. If $\binom{k}{2} < n \leq \binom{k+1}{2} - 2$ then $\binom{k-1}{2} \leq n - k \leq \binom{k}{2} - 2$. By the inductive hypothesis the result is true for $k - 1 = k'$ and Lemma 5 can be used to construct a CSS of k-sets in $k+1$ sets, from the CSS of k'-sets in $k' + 1 = k$ sets. Thus $R(n,k) \leq k+1$.

Note that the construction of Lemma 5 adds the new elements exactly twice. Thus, if the original separator is fair, with all elements occuring two or three times, then the new separator is fair. So the inductively constructed CSSs are fair and the result follows. □

As an example of the construction of Lemma 5, consider the following six matrices, the rows of which form CSSs. The examples in the top row are, from left to right, the base cases for $R(13,5)$, $R(15,6)$ and $R(21,7)$. The examples in the second row, for $R(19,6)$, $R(22,7)$ and $R(29,8)$, are built from the examples above via the construction in Lemma 5.

1	2	3	4	5
1	6	7	8	9
2	6	11	12	13
3	7	10	12	13
4	8	10	11	13
5	9	10	11	12

1	2	3	4	5	6
1	2	7	8	9	10
2	3	7	11	12	13
3	4	8	11	14	15
4	5	9	12	14	8
5	6	10	13	15	9
6	1	7	11	14	10

1	2	3	4	5	6	7
1	2	8	9	10	11	12
2	3	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	10
6	7	12	16	19	21	11
7	1	8	13	17	20	12

1	2	3	4	5	14
1	6	7	8	9	15
2	6	11	12	13	16
3	7	10	12	13	17
4	8	10	11	13	18
5	9	10	11	12	19
14	15	16	17	18	19

1	2	3	4	5	6	16
1	2	7	8	9	10	17
2	3	7	11	12	13	18
3	4	8	11	14	15	19
4	5	9	12	14	8	20
5	6	10	13	15	9	21
6	1	7	11	14	10	22
16	17	18	19	20	21	22

1	2	3	4	5	6	7	22
1	2	8	9	10	11	12	23
2	3	8	13	14	15	16	24
3	4	9	13	17	18	19	25
4	5	10	14	17	20	21	26
5	6	11	15	18	20	10	27
6	7	12	16	19	21	11	28
7	1	8	13	17	20	12	29
22	23	24	25	26	27	28	29

3.7 $k = 6$

Theorem 2, together with the result that $R(n,1) = R(n,2) = n$ and Lemma 2, provides a complete solution to the $R(n,k)$ problem for all $k \leq 5$. Using also Lemma 1 and the results in [8], the only remaining unknown case for $k = 6$ is $R(13,6)$. The final result provides a solution for this case.

Lemma 19 $R(13,6) = 7$.

Proof By Lemma 1, $R(13,6) \geq 6$. That $R(13,6) \leq 7$ follows from consideration of the collection {12345D, 12345C, 16789A, 2678BD, 369ABC, 479BCD, 58ABCD}. To prove that $R(13,6) \neq 6$, assume $\mathcal{C}$ is a $(13,6)$CSS with $|\mathcal{C}| = 6$.

Here the excess, $E = 10$ so there are at least three elements, say 1, 2 and 3, which occur exactly twice in sets in $\mathcal{C}$. If i of these occur in one set A, then to obtain complete separation there are i other sets which contain exactly one of these elements of A. This leaves $6 - i$ elements in A to be separated in less than $6 - i$ sets. It is not difficult to check that this is impossible for $i > 1$.

Assume all sets of $\mathcal{C}$ contain exactly one element which occurs exactly twice in sets in $\mathcal{C}$. Assume 1 occurs in sets A and B. Let the other sets in $\mathcal{C}$ be C, D, E and F. It can be assumed that $2 \in C, D$ and $3 \in E, F$.

A contains exactly five elements which occur more than twice in C. None of these can occur in B to ensure that 1 is separated from each of the other elements of A. It can be checked that there is exactly one way, up to labelling of elements, to separate five elements in no more than four sets; namely, $\{456, 478, 57, 68\}$. These four sets must be subsets of C, D, E and F. For all possible arrangements of these sets it is now easy to check that either 2 or 3 cannot now be separated from at least one element of A other than 1. Hence $R(13,6) \neq 6$, and the result follows. □

4 Final Remarks

Combining the results in this paper with those in [8], the bounds on the values of $R(n,k)$ shown in Table 1 are obtained. The points where $n = 2k$ are bracketed to highlight the symmetry due to Lemma 2.

It appears to be increasingly difficult to calculate $R(n,k)$ as n approaches $2k$ from above. It is hoped that various methods currently under consideration, including the close connection between CSSs and antichains as shown by Cai in [2], can be used to increase the range of known values of $R(n,k)$. A useful result in regard to this would be a proof of the following conjecture due to Lieby [6].

Definition 4 A family of sets $\mathcal{A}$ is an **antichain** if, for all distinct $A, B \in \mathcal{A}$, $A \not\subseteq B$ and $B \not\subseteq A$.

Definition 5 An antichain $\mathcal{A}$ is said to be **flat** if there exists an x such that, $\forall A \in \mathcal{A}$, $|A| = x$ or $|A| = x + 1$.

Conjecture [Flat antichain (FAC) conjecture]
Let $\mathcal{A}$ be an antichain on an n-set S, with $\sum_{A\in\mathcal{A}} |A| = t$. Then there exists a flat antichain $\mathcal{A}'$ on S with $|\mathcal{A}'| = |\mathcal{A}|$ and $\sum_{A'\in\mathcal{A}'} |A'| = t$.

Determining exact values and constructions for minimal (n,a,k)CSSs remains an interesting open problem. There are many variations of problems on CSSs which can be formulated by the imposition of additional constraints on the nature of the systems. There appears potential for applications of CSSs in various disciplines.

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n	k = 1	2	3	4	5	6	7	8	9	10	11	12	13
2	[2]												
3	3	3											
4	4	[4]	4										
5	5	5	5	5									
6	6	6	[4]	6	6								
7	7	7	5	5	7	7							
8	8	8	6	[5]	6	8	8						
9	9	9	6	6	6	6	9	9					
10	10	10	7	5	[6]	5	7	10	10				
11	11	11	8	6	6	6	6	8	11	11			
12	12	12	8	6	6	[6]	6	6	8	12	12		
13	13	13	9	7	6	7	7	6	7	9	13	13	
14	14	14	10	7	7	7	[6–8]	7	7	7	10	14	14
15	15	15	10	8	6	7	6–9	6–9	7	6	8	10	15
16	16	16	11	8	7	7	6–8	[6–7]	6–8	7	7	8	11
17	17	17	12	9	7	7	7–8	6–9	6–9	7–8	7	7	9
18	18	18	12	9	8	7	7–8	6–9	[6–10]	6–9	7–8	7	8
19	19	19	13	10	8	7	8	6–9	6–11	6–11	6–9	8	7
20	20	20	14	10	8	8	8	7–9	6–10	[6–8]	6–10	7–9	8
21	21	21	14	11	9	7	8	7–9	7–10	7–10	7–10	7–10	7–9
22	22	22	15	11	9	8	8	7–9	7–10	7–11	[7–12]	7–11	7–10
23	23	23	16	12	10	8	8	8–9	7–10	7–11	7–13	7–13	7–11
24	24	24	16	12	10	8	8	8–9	7–10	7–11	7–12	[7–8]	7–12
25	25	25	17	13	10	9	8	8–9	7–10	7–11	7–12	7–10	7–10
26	26	26	18	13	11	9	8	9	8–10	7–11	7–12	7–11	[7–12]
27	27	27	18	14	11	9	9	9	8–10	7–11	7–12	7–13	7–13
28	28	28	19	14	12	10	8	9	8–10	7–11	7–12	7–13	7–14
29	29	29	20	15	12	10	9	9	8–10	7–13	7–12	7–13	7–14
30	30	30	20	15	12	10	9	9	9–10	8–11	7–12	7–13	7–14
31	31	31	21	16	13	11	9	9	9–10	8–11	7–12	7–13	7–14
32	32	32	22	16	13	11	10	9	9–10	8–11	7–12	7–13	7–14
33	33	33	22	17	14	11	10	9	9–10	8–11	8–12	7–13	7–14
34	34	34	23	17	14	12	10	9	10	9–11	8–12	7–13	7–14
35	35	35	24	18	14	12	10	10	10	9–11	8–12	7–14	7–14
36	36	36	24	18	15	12	11	9	10	9–11	8–12	8–13	8–14
37	37	37	25	19	15	13	11	10	10	9–11	8–12	8–13	8–14
38	38	38	26	19	16	13	11	10	10	10–11	9–12	8–13	8–14
39	39	39	26	20	16	13	12	10	10	10–11	9–12	8–13	8–14
40	40	40	27	20	16	14	12	10	10	10–11	9–14	8–13	8–14
41	41	41	28	21	17	14	12	11	10	10–11	9–12	8–13	8–14
42	42	42	28	21	17	14	12	11	10	11	9–12	9–13	8–14
43	43	43	29	22	18	15	13	11	10	11	10–12	9–13	8–14
44	44	44	30	22	18	15	13	11	11	11	10–12	9–13	8–14
45	45	45	30	23	18	15	13	12	10	11	10–12	9–13	8–14
46	46	46	31	23	19	16	14	12	11	11	10–12	9–13	9–14
47	47	47	32	24	19	16	14	12	11	11	11–12	10–13	9–14
48	48	48	32	24	20	16	14	12	11	11	11–12	10–13	9–16
49	49	49	33	25	20	17	14	13	11	11	11–12	10–13	9–14
50	50	50	34	25	20	17	15	13	12	11	11–12	10–13	9–14
51	51	51	34	26	21	17	15	13	12	11	11–12	10–13	9–14
52	52	52	35	26	21	18	15	13	12	11	12	11–15	10–14
53	53	53	36	27	22	18	16	14	12	11	12	11–13	10–14
54	54	54	36	27	22	18	16	14	12	12	12	11–13	10–14
55	55	55	37	28	22	19	16	14	13	11	12	11–13	10–14
56	56	56	38	28	23	19	16	14	13	12	12	11–13	10–14

Table 1: Known bounds on $R(n,k)$ for $2 \le n \le 56$ and $1 \le k \le \min\{13, n-1\}$.

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