On the volume of trades in triple systems

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Abstract. This paper gives a complete answer to the question: For which values of v and s does there exist a graph G, having v non-isolated vertices, and a pair of disjoint sets of s triangles each forming a partition of the edge set of G?

1 Introduction

A (v, k, t) *trade of volume* s is a pair (T_1, T_2) where T_1 and T_2 are collections of k-sets chosen from a v-set V, such that $T_1 \cap T_2 = \emptyset$, $|T_1| = |T_2| = s$ and T_1 and T_2 cover the same t-subsets of V. The *foundation* of a trade is the set of elements covered by T_1 and T_2.

Many results on trades can be found in the literature. For example, see the survey [5] by Street and the references therein. Hwang [2] constructed (v, k, t) trades of volume 2^t and proved the non-existence of (v, k, t) trades having volume less than 2^t or volume $2^t + 1$. Mahmoodian and Soltankhah [3] have given further existence results and have shown that, for given integers $v > k > t \geq 1$, there does not exist a (v, k, t) trade of volume s for $2^t < s < 2^t + 2^{t-1}$.

In this paper we are only concerned with $(v, 3, 2)$ trades in which T_1 (and hence also T_2) cover no pair of V more than once. We determine, for any given v and s, whether or not there exists such a $(v, 3, 2)$ trade of volume s and foundation size v.

A *partial triple system* is any collection of triples no two of which intersect in more than one element. The number of distinct elements occurring in the triples of a partial triple system is called its *order* and a partial triple system of order v is denoted PTS(v). A partial triple system which covers every possible pair of elements chosen from a v-set is a Steiner triple system of order v.

Example 1.1 The partial triple systems

$$T_1 = \{135, 146, 236, 245\} \text{ and } T_2 = \{136, 145, 235, 246\}$$

form a (6,3,2) trade of volume 4 with foundation size 6.

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We make use of terminology from Graph Theory to simplify the presentation of the arguments we use. Clearly, a partial triple system is a pair (G, T) where T is a set of triangles which form a partition of the the edge set of a simple graph G. We do not allow isolated vertices in G so that the order of the partial triple system is the number of vertices in G. We use the following definition throughout the remainder of the paper.

Definition 1.2 A *trade of volume s and foundation size v*, a $T(s, v)$, is a triple (G, T_1, T_2) where:

(1) $|T_1| = |T_2| = s$;
(2) $T_1 \cap T_2 = \emptyset$;
(3) (G, T_1) and (G, T_2) are PTS(v)s.

We say that G *admits* a $T(s, v)$.

In this paper, we give necessary and sufficient conditions on s and v for the existence of a $T(s, v)$ (see Table 2.1). In our terminology, a *maximum packing (of triples) of order v* is a PTS(v) which has the maximum possible number of triples (for given v). A *group divisible triple system* with g groups each of size m is a PTS(gm) (G, T) where G is the complete multi-partite graph with m vertices in each of its g parts. The *intersection problem for maximum packings of triples* involves finding two maximum packings of order v having precisely I common triples for all possible integers v and I. This problem was solved by Quattrocchi [4]. The *intersection problem for group divisible triple systems* involves finding two group divisible triple systems with g groups of size m which have precisely I common triples for all possible integers g, m and I. This problem was solved by Butler and Hoffman [**1**]. The results of this paper rely on the above two results..

2 Necessary Conditions

For given v, an upper bound on s (necessary for the existence of a $T(s, v)$) is easily obtained. If there exists a $T(s, v)$, s can be no larger than the number of triples in a maximum packing of order v. To find a lower bound for s we note that if G admits a $T(s, v)$ then every vertex of G must have degree at least 4. Hence G contains at least $2v$ edges and so s is at least $\frac{2v}{3}$. We summarise these results in Proposition 2.1.

Proposition 2.1 If there is a $T(s, v)$ then:

(1) $\frac{2v}{3} \leq s \leq \frac{v(v-2)}{6}$ when $v \equiv 0 \pmod 6$;
(2) $\frac{2v+1}{3} \leq s \leq \frac{v(v-1)}{6}$ when $v \equiv 1 \pmod 6$;
(3) $\frac{2v+2}{3} \leq s \leq \frac{v(v-2)}{6}$ when $v \equiv 2 \pmod 6$;
(4) $\frac{2v}{3} \leq s \leq \frac{v(v-1)}{6}$ when $v \equiv 3 \pmod 6$;
(5) $\frac{2v+1}{3} \leq s \leq \frac{v^2-2v-2}{6}$ when $v \equiv 4 \pmod 6$;
(6) $\frac{2v+2}{3} \leq s \leq \frac{v^2-v-8}{6}$ when $v \equiv 5 \pmod 6$.

We now proceed to establish some further necessary conditions by proving the non-existence of some trades of small volume. First we need some more notation. The unique (up to isomorphism) $T(4,6)$ given in Example 1.1 is commonly called a *Pasch configuration* and we also use this term to describe the graph (isomorphic to $K_{2,2,2}$) which admits this trade.

Lemma 2.2 If G admits a $T(s,v)$ and G has at least $v-1$ vertices of degree 4, then G is the union of edge disjoint Pasch configurations. Moreover, these Pasch configurations are also vertex disjoint except possibly for one vertex (of degree > 4) which is in more than one of the Pasch configurations.

Proof Let (G, T_1, T_2) be a $T(s,v)$ and let G have at least $v-1$ vertices of degree 4. Let a_0 be a vertex of degree 4 and let a_1, a_2, a_3 and a_4 be its neighbours. Without loss of generality we can assume that a_2, a_3 and a_4 each have degree 4 and that

$$a_0a_1a_2, a_0a_3a_4 \in T_1 \quad \text{and} \quad a_0a_1a_3, a_0a_2a_4 \in T_2.$$

The edge a_1a_3 must occur in some triangle of T_1 and clearly the third vertex of this triangle cannot be a_0, a_2 or a_4. So let $a_5 \in V(G)$ and $a_1a_3a_5 \in T_1$. Now, since a_3 has degree 4, the triple $a_3a_4a_5 \in T_2$ and so $a_4a_5 \in E(G)$. Similarly, since a_4 has degree 4, $a_2a_4a_5 \in T_1$ and so $a_2a_5 \in E(G)$. But since a_2 has degree 4, $a_1a_2a_5 \in T_2$. The union of the triangles thus far described is clearly a Pasch configuration. Moreover, since only one vertex can have degree more than 4, there are no further edges a_ia_j for $0 \leq i, j \leq 5$. □

Corollary 2.3 If G admits a trade and G has exactly one vertex of degree greater than 4, then this vertex has degree $4x$ for some positive integer $x \geq 2$.

Corollary 2.4 If $v \equiv 3 \pmod 6$, then there is no $T(\frac{2v}{3}, v)$.

Proof Let $v = 6x+3$ and suppose G admits a $T(\frac{2v}{3}, v)$. Then $|E(G)| = 12x+6$ and each vertex of G must have degree 4. Hence by Lemma 2.2, G is the union of edge and vertex disjoint Pasch configurations. But since 6 does not divide v this is impossible. □

Corollary 2.5 If $v \equiv 1 \pmod 3$, then there is no $T(\frac{2v+1}{3}, v)$.

Proof Let $v = 3x+1$ and suppose G admits a $T(\frac{2v+1}{3}, v)$. Then $|E(G)| = 6x+3$ and G must have $3x$ vertices of degree 4 and one of degree 6. This is impossible by Corollary 2.3. □

Lemma 2.6 If G admits a $T(s,v)$, then G does not have degree sequence $4, 4, \ldots, 4, 6, 8$.

Proof Suppose that (G, T_1, T_2) is a $T(s, v)$ and that G has degree sequence $4, 4, \ldots, 4, 6, 8$. Let a_0 be a vertex of degree 6 and let $a_1, a_2, \ldots, a_6$ be its neighbours. Without loss of generality we can assume that a_2, a_3, a_4, a_5 and a_6 each have degree 4 and that

$$a_0a_1a_2, a_0a_3a_4, a_oa_5a_6 \in T_1 \quad \text{and} \quad a_0a_2a_3, a_0a_4a_5, a_0a_6a_1 \in T_2.$$

The third vertex of the triangle in T_1 which contains the edge a_2a_3 cannot be a_0, a_1 or a_4 since they have already occurred in triangles of T_1 with either a_2 or a_3. Neither can it be a_5 or a_6 since then this vertex would have degree greater than 4. Hence the third vertex must be a new vertex a_7; with $a_2a_3a_7 \in T_1$.

Now, since a_2 has degree 4, $a_1a_2a_7 \in T_2$ and since a_3 has degree 4, $a_3a_4a_7 \in T_2$. If a_7 has degree 4 then $a_1a_4a_7 \in T_1$ and this forces a_4 to have degree greater than 4. Hence a_7 has degree 8 and a_1 has degree 4. Since a_1 has degree 4, $a_1a_6a_7 \in T_1$ and since a_4 has degree 4, $a_4a_5a_7 \in T_1$. This forces $a_5a_6a_7 \in T_2$.

Now, thus far a_7 is in three triangles of T_1 and three triangles of T_2 and since it has degree 8 it must occur in only one more triangle of each of T_1 and T_2. But then $|T_1 \cap T_2| > 0$ which is impossible. □

Lemma 2.7 If G admits a $T(s, v)$ and G has degree sequence $4, 4, \ldots, 4, 6, 6, 6$ then G has one component with degree sequence $4, 4, 4, 4, 4, 4, 6, 6, 6$ and any further components are Pasch configurations.

Proof Suppose that (G, T_1, T_2) is a $T(s, v)$ and that G has degree sequence $4, 4, \ldots, 4, 6, 6, 6$. Since any component of G is a trade in its own right, there can be no component of G with a exactly one vertex of degree 6 (by Corollary 2.3). Hence the three vertices of degree 6 must all be in one component. We now show that this component must also contain exactly 6 vertices of degree 4.

Let a_0 be a vertex of degree 6 and let its neighbours be $a_1, a_2, a_3, a_4, a_5, a_6$. At least 4 of these vertices have degree 4 and without loss of generality we can assume that

$$a_0a_1a_2, a_0a_3a_4, a_oa_5a_6 \in T_1 \quad \text{and} \quad a_0a_2a_3, a_0a_4a_5, a_0a_6a_1 \in T_2.$$

Also, without loss of generality we can assume that exactly one of the following is true:

(1) a_1, a_2, a_3, a_4 have degree 4;
(2) a_1, a_2, a_3, a_5 have degree 4;
(3) a_1, a_2, a_4, a_5 have degree 4;

We consider these three cases one at a time.
(1) The third vertex of the triangle in T_1 which contains the edge a_2a_3 cannot be a_0, a_1 or a_4 since they have already occurred in triangles of T_1 with either a_2 or a_3. The third vertex cannot be a_5 since a_3 has degree 4 and we cannot have $a_3a_4a_5 \in T_2$. Similarly, the third vertex cannot be a_6 since a_2 has degree 4 and we cannot have $a_0a_1a_6 \in T_2$. Hence the third vertex must be a new vertex a_7; with $a_2a_3a_7 \in T_1$.

Now, since a_2 has degree 4, $a_1a_2a_7 \in T_2$ and since a_3 has degree 4, $a_3a_4a_7 \in T_2$. Also, since a_4 has degree 4, $a_4a_5a_7 \in T_1$ and since a_1 has degree 4, $a_1a_6a_7 \in T_1$. Hence, a_7 has degree 6 and so at least one of a_5 and a_6 has degree 4.

If both a_5 and a_6 have degree 4 then the triples given above define a complete component (a $T(6,8)$) but this is impossible because all the vertices of degree 6 are in the one component. On the other hand, if only one of them has degree 6 then this vertex must occur in exactly one more triangle in each of T_1 and T_2. But then $|T_1 \cap T_2| > 0$ which is impossible.

(2) The third vertex of the triangle in T_1 which contains the edge a_2a_3 cannot be a_0, a_1 or a_4 since they have already occurred in triangles of T_1 with either a_2 or a_3. The third vertex cannot be a_5 since then a_5 would have degree greater than 4. The third vertex cannot be a_6 since a_2 has degree 4 and we cannot have $a_1a_2a_6 \in T_2$. Hence the third vertex must be a new vertex a_7; with $a_2a_3a_7 \in T_1$.

Now, since a_2 has degree 4, $a_1a_2a_7 \in T_2$ and since a_3 has degree 4, $a_3a_4a_7 \in T_2$. Then, since a_1 has degree 4, $a_1a_6a_7 \in T_1$. This forces a_7 to be a vertex of degree 6 since it already has five neighbours (a_1, a_2, a_3, a_4 and a_6) in G. If its sixth neighbour is a_5 then this forces $a_4a_5a_7 \in T_1$ and $a_5a_6a_7 \in T_2$.

If both a_4 and a_6 have degree 4 then the triangles given thus far define a complete component (a $T(6,8)$) but this is impossible because all the vertices of degree 6 are in the one component. On the other hand, if only one of them has degree 6 then this vertex must occur in exactly one more triangle in each of T_1 and T_2. But then $|T_1 \cap T_2| > 0$ which is impossible.

Hence the sixth neighbour of a_7 must be a new vertex a_8 and we must have $a_4a_7a_8 \in T_1$ and $a_6a_7a_8 \in T_2$. But this forces both a_4 and a_6 to have degree greater than 4 which is impossible (since we already have a_0 and a_7 with degree 6).

(3) The third vertex of the triangle in T_1 which contains the edge a_2a_3 cannot be a_0, a_1 or a_4 since they have already occurred in triangles of T_1 with either a_2 or a_3. The third vertex cannot be a_5 since then a_5 would have degree greater than 4. The third vertex cannot be a_6 since a_2 has degree 4 and we cannot have $a_1a_2a_6 \in T_2$. Hence the third vertex must be a new vertex a_7; with $a_2a_3a_7 \in T_1$.

Now, since a_2 has degree 4, $a_1a_2a_7 \in T_2$ and since a_1 has degree 4, $a_1a_6a_7 \in T_1$. The third vertex of the triangle in T_1 which contains the edge a_4a_5 must be either a_7 or a new vertex a_8. If $a_4a_5a_7 \in T_1$ then we forced to have $a_3a_4a_7 \in T_2$ and $a_5a_6a_7 \in T_2$, but this gives us a complete component which we have seen in (i) and (ii) is impossible.

Hence $a_4a_5a_8 \in T_1$ and (since a_4 and a_5 have degree 4) this forces $a_3a_4a_8 \in T_2$ and $a_5a_6a_8 \in T_2$. Now, both a_3 and a_6 have degree greater than 4 and so a_0, a_3 and a_6 must be the vertices of degree 6. Hence, a_7 and a_8 have degree 4 and so $a_3a_6a_7 \in T_2$ and $a_3a_6a_8 \in T_1$. This defines a complete component: a $T(7,9)$ having degree sequence 4,4,4,4,4,4,6,6,6. □

Corollary 2.8 If G admits a trade and G has degree sequence $4, 4, \ldots, 4, 6, 6, 6$ then G must have $6x + 9$ vertices for some non-negative integer x.

Proof By Lemma 2.7, G must have one component with 9 vertices and all other components with 6 vertices. □

Corollary 2.9 If $v \equiv 0 \pmod 6$, there is no $T(\frac{2v}{3}+1, v)$.

Proof Suppose G admits such a trade and let $v = 6x$; so $\frac{2v}{3}+1 = 4x+1$. Then $|E(G)| = 12x+3$ and G must have degree sequence which sums to $24x+6$. But this is impossible since:

(1) $4, 4, \ldots, 4, 10$ is excluded by Corollary 2.3;
(2) $4, 4, \ldots, 4, 6, 8$ is excluded by Lemma 2.6;
(3) $4, 4, \ldots, 4, 6, 6, 6$ is excluded by Corollary 2.8. □

Lemma 2.10 There is no $T(7, 8)$.

Proof Suppose G admits a $T(7, 8)$ and consider the complement G' of G. If G' contains a triangle, then the trade with this triangle added to T_1 and T_2 would give a maximum packing of order 8 that has intersection 1. No such maximum packing exists, see [4], and so G' is triangle free. Now, it is easy to see that G must have degree sequence 4,4,4,6,6,6,6,6 and so G' has degree sequence 1,1,1,1,1,3,3,3. It is straightforward to check that the only triangle free graph with this degree sequence is that shown in Figure 1.

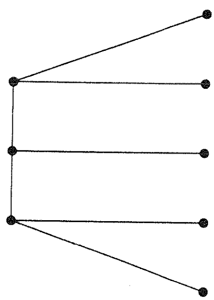

Figure 1

Hence G must be as shown in Figure 2.

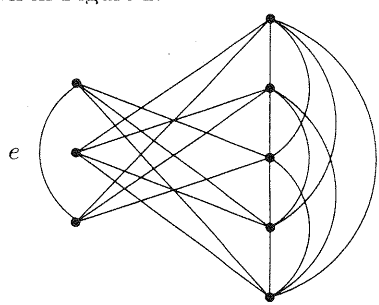

Figure 2

But then there is only one triangle containing the edge e which is impossible. $\square$

At this stage we introduce some convenient notation. For each $v \geq 6$, the results of Section 2 leave us with a set $P(v)$ of possible values of s for which there may exist a $T(s,v)$. We summarise these results in Table 2.1.

v	$P(v)$
$v \leq 5$	$\emptyset$
$v = 8$	$\{6, 8\}$
$v \geq 6$ and $v \equiv 0 \pmod 6$	$\{\frac{2v}{3}, \frac{2v}{3}+2, \frac{2v}{3}+3, \ldots, \frac{v(v-2)}{6}\}$
$v \geq 7$ and $v \equiv 1 \pmod 6$	$\{\frac{2v+4}{3}, \frac{2v+4}{3}+1, \ldots, \frac{v(v-1)}{6}\}$
$v \geq 14$ and $v \equiv 2 \pmod 6$	$\{\frac{2v+2}{3}, \frac{2v+2}{3}+1, \ldots, \frac{v(v-2)}{6}\}$
$v \geq 9$ and $v \equiv 3 \pmod 6$	$\{\frac{2v+3}{3}, \frac{2v+3}{3}+1, \ldots, \frac{v(v-1)}{6}\}$
$v \geq 10$ and $v \equiv 4 \pmod 6$	$\{\frac{2v+4}{3}, \frac{2v+4}{3}+1, \ldots, \frac{v^2-2v-2}{6}\}$
$v \geq 11$ and $v \equiv 5 \pmod 6$	$\{\frac{2v+2}{3}, \frac{2v+2}{3}+1, \ldots, \frac{v^2-v-8}{6}\}$

Table 2.1

Let $E(v)$ be the set of values of s for which there exists a $T(s,v)$. The remainder of this paper is devoted to proving that for each $v \geq 6$, $E(v) = P(v)$.

3 Existence Results

Lemma 3.1 Let (G, S_1) and (G, S_2) be a pair of PTS(v)s where:

(1) $|S_1| = |S_2| = b$;
(2) $|S_1 \cap S_2| = I$;
(3) the maximum degree of the complement G' of G is d.

Then, if $v - d - 1 > 2I$, there is a $T(b - I, v)$.

Proof Since $v - d - 1 > 2I$, every vertex occurs in $S_1 \setminus (S_1 \cap S_2)$ (and in $S_2 \setminus (S_1 \cap S_2)$). Hence $(G \setminus (S_1 \cap S_2), S_1 \setminus (S_1 \cap S_2), S_2 \setminus (S_1 \cap S_2))$ is a $T(b - I, v)$. $\square$

Notice that if $I > 0$ in Lemma 3.1 then G has a set of 3 independent vertices (a set of vertices is independent if it contains no pair of adjacent vertices); consider the vertices of any triangle in $S_1 \cap S_2$.

By using the solution of the intersection problem for maximum packings (see [4]) and the above result we obtain $T(s,v)$s for s and v as given in Table 3.2. For convenience, the solution of the intersection problem for maximum packings of triples is given in Table 3.1 (note that for $v \equiv 1, 3 \pmod 6$, $d = 0$; for $v \equiv 0, 2 \pmod 6$, $d = 1$; for $v \equiv 4 \pmod 6$, $d = 3$; and for $v \equiv 5 \pmod 6$, $d = 2$).

Order of maximum packing	*Possible intersection sizes*
$v = 6$	$I \in \{0, 4\}$
$v = 7$	$I \in \{0, 1, 3, 7\}$
$v = 8$	$I \in \{0, 2, 8\}$
$v = 9$	$I \in \{0, 1, 2, 3, 4, 6, 12\}$
$v \equiv 0, 2 \pmod 6$, $v \geq 12$	$I \in \{0, 1, 2, \dots, v(v-2)/6 = q\}$ $\setminus\{q-1, q-2, q-3, q-5\}$
$v \equiv 1, 3 \pmod 6$, $v \geq 13$	$I \in \{0, 1, 2, \dots, v(v-1)/6 = q\}$ $\setminus\{q-1, q-2, q-3, q-5\}$
$v \equiv 4 \pmod 6$, $v \geq 10$	$I \in \{0, 1, 2, \dots, (v^2-2v-2)/6 = q\}$ $\setminus\{q-1, q-2, q-3, q-5\}$
$v \equiv 5 \pmod 6$, $v \geq 11$	$I \in \{0, 1, 2, \dots, (v^2-v-8)/6 = q\}$ $\setminus\{q-1, q-2, q-3, q-5\}$

Table 3.1

Corollary 3.2 For each v and each corresponding value of s in Table 3.2 there exists a $T(s, v)$.

v	s
$v = 6$	$s = 4$
$v = 7$	$s = 6, 7$
$v = 8$	$s = 6, 8$
$v = 6x \geq 12$	$6x^2 - 5x + 2 \leq s \leq 6x^2 - 2x$
$v = 6x + 1 \geq 13$	$6x^2 - 2x + 1 \leq s \leq 6x^2 + x$
$v = 6x + 2 \geq 14$	$6x^2 - x + 1 \leq s \leq 6x^2 + 2x$
$v = 6x + 3 \geq 9$	$6x^2 + 2x + 1 \leq s \leq 6x^2 + 5x + 1$
$v = 6x + 4 \geq 10$	$6x^2 + 3x + 2 \leq s \leq 6x^2 + 6x + 1$
$v = 6x + 5 \geq 11$	$6x^2 + 6x + 2 \leq s \leq 6x^2 + 9x + 2$

Table 3.2

Lemma 3.3 Suppose G_1 has a set of m independent vertices and G_1 admits a $T(s_1, v_1)$. If there exists a $T(s_2, v_2)$, then there exists a $T(s_1 + s_2, v_1 + v_2 - n)$ for $n = 0, 1, \dots, \min(m, v_2)$.

Proof Let (G_1, T_1, T_2) be a $T(s_1, v_1)$ (G_1 having a set of m independent vertices) and let (G_2, T_1', T_2') be a $T(s_2, v_2)$. Identify n vertices of G_2 with n independent vertices of G_1. Then $(G_1 \cup G_2, T_1 \cup T_1', T_2 \cup T_2')$ is the required trade. □

Lemma 3.3 is particularly useful for constructing new trades when we have existing trades with a large number of independent vertices, such as the trades obtained from the solution of the intersection problem for group divisible triple systems [1]. We only need to use group divisible triple systems with three groups. By using the solution of the intersection problem for group divisible triple systems with three groups and Lemma 3.1 we obtain the following Corollary.

Corollary 3.4 There exists a subgraph G of the complete tripartite graph $K_{x,x,x}$ which admits a $T(s, 3x)$ for the values of s and x given in Table 3.3.

x	s
$x = 2$	4
$x = 3$	9
$x \geq 4$	$x^2 - x + 1, x^2 - x + 2, \ldots, x^2$

Table 3.3

Lemma 3.5 Let (G, S_1) and (G, S_2) be a pair of group divisible triple systems with three groups and where $|S_1| = |S_2| = x^2$ and $|S_1 \cap S_2| = I$.

(1) If there exists $T(s_1, x), T(s_2, x)$ and $T(s_3, x)$ there exists $T(s_1 + s_2 + s_3 + x^2 - I, 3x)$.

(2) If there exists $T(s_1, x + 1), T(s_2, y)$ and $T(s_3, z)$, where y and z are either x or $x + 1$, there exists $T(s_1 + s_2 + s_3 + x^2 - I, 3x + 1)$.

(3) If there exists $T(s_1, x + 2), T(s_2, y)$ and $T(s_3, z)$, where y and z are either $x, x + 1$ or $x + 2$ and where at most one of the trades of order $x + 2$ has no pair of independent vertices, there exists $T(s_1 + s_2 + s_3 + x^2 - I, 3x + 2)$.

Proof We can assume (by Lemma 3.1) that there is a $T(x^2 - I, 3x)$ which contains three disjoint sets of x independent vertices. The condition that $v - d - 1 > 2I$ in Lemma 3.1 is only necessary to ensure that there are no isolated vertices. Here, we include extra triples from the trades of volume s_1, s_2 and s_3 so that in the end, there will be no isolated vertices and we need not be concerned with this condition. Hence, the result follows by Lemma 3.3 (applied three times). □

Lemma 3.6 There exists a $T(x, x + 2)$ for all even integers $x \geq 4$.

Proof Let G be the graph with vertex set

$$V(G) = \{\infty_1, \infty_2\} \cup \{1, 2, \ldots, x\}$$

and edge set

$$E(G) = \{\infty_1 1, \infty_1 2, \ldots, \infty_1 x, \infty_2 1, \infty_2 2, \ldots, \infty_2 x, 12, 23, 34, \ldots, (x-1)x, x1\}.$$

Also, let

$$T_1 = \{\infty_1 12, \infty_1 34, \ldots, \infty_1 (x-1)x, \infty_2 23, \infty_2 45, \ldots, \infty_2 x1\}$$

and

$$T_2 = \{\infty_1 23, \infty_1 45, \ldots, \infty_1 x1, \infty_2 12, \infty_2 34, \ldots, \infty_2 (x-1)x\}.$$

Then (G, T_1, T_2) is a $T(x, x + 2)$. □

Lemma 3.7 For $v \leq 15$, $E(v) = P(v)$.

Proof For $v \leq 5$ there is nothing to prove and for $v = 6, 7$ and 8, the result follows from Corollary 3.2. To show $E(9) = P(9)$, again by Corollary 3.2, we only need a $T(7,9)$ and a $T(8,9)$; both of which are given in the Appendix. We consider the remaining values of v, $10 \leq v \leq 15$, one at a time.
$\underline{v = 10.}$ By Corollary 3.2, we only need to show that 8,9,10 $\in E(10)$. A $T(8,10)$ exists by Lemma 3.6 and $T(9,10)$ and $T(10,10)$ are given in the Appendix.

$\underline{v = 11.}$ By Corollary 3.2, we only need to show that $8, 9, \ldots, 13 \in E(11)$. Since $T(4,6)$ has a pair of independent vertices, by Lemma 3.3 we can use $T(4,6)$ together with $T(6,7)$ and $T(7,7)$ to construct $T(8,11), T(10,11)$ and $T(11,11)$. The remaining trades, $T(9,11), T(12,11)$ and $T(13,11)$ are given in the Appendix.

$\underline{v = 12.}$ By Corollary 3.2, we only need to show that $8, 10, 11, \ldots, 15 \in E(12)$. Since $T(4,6)$ has a pair of independent vertices, by Lemma 3.3 we can use $T(4,6)$ together with $T(6,7), T(7,7)$ and $T(8,8)$ to construct $T(8,12), T(10,12), T(11,12)$ and $T(12,12)$. Also, by Lemma 3.1 (and the remark immediately following it) there exists a $T(9,9), T(10,9)$ and $T(11,9)$ each with a set of 3 independent vertices. Hence by Lemma 3.3, we can use these together with $T(4,6)$ to construct $T(13,12), T(14,12)$ and $T(15,12)$.

$\underline{v = 13.}$ By Corollary 3.2, we only need to show that $10, 11, \ldots, 20 \in E(13)$. Here, we can use $T(4,6)$ together with $T(6,8), T(7,9), T(8,9), T(9,9), \ldots, T(12,9)$ to construct $T(10,13), T(11,13), \ldots, T(16,13)$. Since there exist $T(10,9)$ and $T(11,9)$, each with sets of 3 independent vertices, we can use them with $T(7,7)$ to construct $T(17,13)$ and $T(18,13)$. The remaining two trades, $T(19,13)$ and $T(20,13)$ are given in the Appendix.

$\underline{v = 14.}$ By Corollary 3.2, we only need to show that $10, 11, \ldots, 22 \in E(14)$. Here, we can use $T(4,6)$ together with $T(6,8), T(7,9), T(8,9), T(9,9), \ldots, T(12,9)$, $T(13,10)$ to construct $T(10,14), T(11,14), \ldots, T(17,14)$. To construct $T(18,14)$, $T(19,14), \ldots, T(22,14)$ we first use trades given by applying Lemma 3.1 to pairs of group divisible triple systems with 3 groups of size 4 and with 2,1 and 0 common triples. This gives us $T(14,12), T(15,12)$ and $T(16,12)$ each with a set of 4 independent vertices. We can then use a $T(4,6)$ together with each of these trades to construct $T(18,14), T(19,14)$ and $T(20,14)$.

To construct a $T(21,14)$ we begin with a $T(13,12)$ that contains two disjoint sets of 4 independent vertices; this trade can be constructed from a pair of group divisible triple systems with 3 groups of size 4 having 3 common triples, by Lemma 3.1. By using this trade together with a $T(4,6)$ we can construct a $T(17,14)$ which contains a set of 6 independent vertices. This is done by ensuring that the two vertices of the $T(4,6)$ which are not identified with vertices of the $T(13,12)$ are independent. This trade can then be used with another $T(4,6)$ to construct a $T(21,14)$. A $T(22,14)$ can be constructed in a similar manner by starting with a $T(14,12)$ that contains two disjoint sets of 4 independent vertices.

$\underline{v=15.}$ By Corollary 3.2, we only need to show that $11, 12, \ldots, 28 \in E(15)$. Here, we can use $T(4,6)$ together with $T(7,9), T(8,11), T(9,11), \ldots, T(17,11)$ to construct $T(11,15), T(12,15), \ldots, T(21,15)$. The trades $T(22,15), T(23,15), T(24,15)$ and $T(25,15)$ exist by Corollary 3.4.

We can construct $T(26,15), T(27,15)$ and $T(28,15)$ using Lemma 3.5 (3) with $s_1 = s_2 = s_3 = 4, x+2 = y = z = 6$ and $I = 2, 1$ and 0 respectively. $\square$

Lemma 3.8 If $v > 15$ and $E(v-6) = P(v-6)$, then there exists a $T(s,v)$ for all $s \in P(v)$ with $s \leq \max(P(v-6)) + 4$.

Proof Since $x \in P(v)$ implies $x \in \{y+4 : y \in P(v-6)\}$ or $x > \max(P(v-6))+4$, we need only observe that we can construct (using Lemma 3.3 with $n = 0$) a $T(s+4, v)$ from a $T(4,6)$ and a $T(s, v-6)$. $\square$

Lemma 3.9 If $E(u) = P(u)$ for all $u \leq v$ then there exists a $T(s,v)$ for all s and v as given by Table 3.4.

v	s
$v = 6x \geq 18$	$4x+6 \leq s \leq 6x^2 - 2x - 1$
$v = 6x+1 \geq 19$	$4x+6 \leq s \leq 6x^2 + x - 4$
$v = 6x+2 \geq 14$	$4x+10 \leq s \leq 6x^2 + 2x - 1$
$v = 6x+3 \geq 21$	$4x+6 \leq s \leq 6x^2 + 5x - 3$
$v = 6x+4 \geq 16$	$4x+10 \leq s \leq 6x^2 + 6x$
$v = 6x+5 \geq 17$	$4x+10 \leq s \leq 6x^2 + 7x$

Table 3.4

Proof

We consider the cases $v \equiv 0, 1, \ldots 5 \pmod 6$ one at a time.

$\underline{v = 6x.}$ By Lemma 3.7, we can assume that $x \geq 3$. Hence $2x \geq 6$ and so there exists a pair of group divisible triple systems with 3 groups of size $2x$ and with intersection of size I for each $I \in \{0, 1, 2, \ldots, 4x^2-6, 4x^2-4, 4x^2\}$.

By Lemma 3.5 (1), when $x = 3$ we can use a $T(4,6)$ together with the group divisible triple systems with $I = 1, 2, \ldots, 30$ to construct $T(47,18), T(46,18), \ldots, T(18,18)$. When $x = 4$, we can use the group divisible triple systems with $I = 1, 2, \ldots, 58, 60, 64$ together with $T(6,8)$ and $T(8,8)$ to construct $T(22,24), T(24,24), T(25,24), \ldots, T(87,24)$. A $T(23,24)$ can be constructed from $T(11,12)$ and $T(12,12)$.

When $x \geq 5$, there exists a $T(s, 2x)$ for all integers s with $\lfloor (4x+6)/3 \rfloor \leq s \leq \lceil (2x^2 - 2x - 1)/3 \rceil$. Since $2x^2 - 2x - 1 - (4x+6) > 6$ (when $x \geq 5$), these trades can be used together with the group divisible triple systems with $I = 0, 1, \ldots, x^2-6, x^2$ to construct $T(4x+6, 6x), T(4x+7, 6x), \ldots, T(6x^2-2x-1, 6x)$.

$\underline{v = 6x+1.}$ By Lemma 3.7, we can assume that $x \geq 3$. Hence $2x \geq 6$ and so there exists a pair of group divisible triple systems with 3 groups of size $2x$ and with intersection of size I for each $I \in \{0, 1, 2, \ldots, 4x^2-6, 4x^2-4, 4x^2\}$.

By Lemma 3.5 (2), when $x = 3$ we can use $T(6,7)$ and $T(7,7)$ together with the group divisible triple systems with $I = 1, 2, \ldots, 30, 32, 36$ to construct $T(18, 19)$, $T(19, 19), \ldots, T(53, 19)$.

When $x \geq 4$, there exists $T(s, 2x+1)$ for all integers s with $\lfloor (4x+6)/3 \rfloor \leq s \leq \lceil (2x^2+x-4)/3 \rceil$. Since $2x^2 + x - 4 - (4x+6) > 6$ (when $x \geq 4$), these trades can be used together with the group divisible triple systems with $I = 0, 1, \ldots, x^2 - 6, x^2$ to construct $T(4x+6, 6x+1), T(4x+7, 6x+1), \ldots, T(6x^2+x-4, 6x+1)$.

$\underline{v = 6x+2.}$ By Lemma 3.7, we can assume that $x \geq 3$. Hence $2x \geq 6$ and so there exists a pair of group divisible triple systems with 3 groups of size $2x$ and with intersection of size I for each $I \in \{0, 1, 2, \ldots, 4x^2-6, 4x^2-4, 4x^2\}$.

By Lemma 3.5 (3), when $x = 3$ we can use $T(6,8), T(7,7)$ and $T(8,8)$ together with the group divisible triple systems with $I = 1, 2, \ldots, 30, 32, 36$ to construct $T(22, 20), T(23, 20), \ldots, T(59, 20)$.

When $x \geq 4$, there exists $T(s, 2x+2)$ with a pair of independent vertices (since $2x+2$ is even) for all integers s with $\lfloor (4x+10)/3 \rfloor \leq s \leq \lceil (2x^2+2x-1)/3 \rceil$. Since $2x^2 + 2x - 1 - (4x+10) > 6$ (when $x \geq 4$), these trades can be used together with the group divisible triple systems with $I = 0, 1, \ldots, x^2-6, x^2$ to construct $T(4x+10, 6x+2), T(4x+11, 6x+2), \ldots, T(6x^2+2x-1, 6x+2)$.

$\underline{v = 6x+3.}$ By Lemma 3.7, we can assume that $x \geq 3$. Hence $2x+1 > 6$ and so there exists a pair of group divisible triple systems with 3 groups of size $2x+1$ and with intersection of size I for each $I \in \{0, 1, 2, \ldots, 4x^2+4x+1-6, 4x^2+4x+1-4, 4x^2+4x+1\}$.

By Lemma 3.5 (1), when $x = 3$ we can use $T(6,7)$ and $T(7,7)$ together with the group divisible triple systems with $I = 1, 2, \ldots, 30, 32, 36$ to construct $T(18, 21)$, $T(19, 21), \ldots, T(66, 21)$.

When $x \geq 4$, there exists $T(s, 2x+1)$ for all integers s with $\lfloor (4x+6)/3 \rfloor \leq s \leq \lceil (2x^2+x-4)/3 \rceil$. Since $2x^2 + x - 4 - (4x+6) > 6$ (when $x \geq 4$), these trades can be used together with the group divisible triple systems with $I = 0, 1, \ldots, x^2-6, x^2$ to construct $T(4x+6, 6x+3), T(4x+7, 6x+3), \ldots, T(6x^2+5x-3, 6x+3)$.

$\underline{v = 6x+4.}$ By Lemma 3.7, we can assume that $x \geq 2$. Hence $2x+1 \geq 5$ and so there exists a pair of group divisible triple systems with 3 groups of size $2x+1$ and with intersection of size I for each $I \in \{0, 1, 2, \ldots, 4x^2+4x+1-6, 4x^2+4x+1-4, 4x^2+4x+1\}$.

By Lemma 3.5 (2), when $x = 2$ we can use $T(4,6)$ together with the group divisible triple systems with $I = 1, 2, \ldots, 19$ to construct $T(18, 16), T(19, 16), \ldots,$ $T(36, 16)$.

By Lemma 3.5 (2), when $x = 3$ we can use $T(6,8), T(7,7)$ and $T(8,8)$ together with the group divisible triple systems with $I = 1, 2, \ldots, 19, 25$ to construct $T(22, 22), T(23, 22), \ldots, T(72, 22)$.

When $x \geq 4$, there exists $T(s, 2x+2)$ for all integers s with $\lfloor (4x+10)/3 \rfloor \leq s \leq \lceil (2x^2+2x-1)/3 \rceil$. Since $2x^2+2x-1-(4x+10) > 6$ (when $x \geq 4$), these trades can be used together with the group divisible triple systems with $I = 0, 1, \ldots, x^2-6, x^2$ to construct $T(4x+10, 6x+4), T(4x+11, 6x+4), \ldots, T(6x^2+6x, 6x+4)$.

$v = 6x + 5.$ By Lemma 3.7, we can assume that $x \geq 2$. Hence $2x + 1 \geq 5$ and so there exists a pair of group divisible triple systems with 3 groups of size $2x+1$ and with intersection of size I for each $I \in \{0, 1, 2, \dots, 4x^2 + 4x + 1 - 6, 4x^2 + 4x + 1 - 4, 4x^2 + 4x + 1\}$.

By Lemma 3.5 (3), when $x = 2$ we can use $T(4,6), T(6,7)$ and $T(7,7)$ together with the group divisible triple systems with $I = 1, 2, \dots, 19, 21, 25$ to construct $T(18,17), T(19,17), \dots, T(38,17)$.

By Lemma 3.5 (3), when $x = 3$ we can use $T(6,8), T(8,8), T(8,9), T(9,9)$ and $T(10,9)$ together with the group divisible triple systems with $I = 1, 2, \dots, 19, 21, 25$ to construct $T(22,22), T(23,22), \dots, T(75,22)$.

When $x \geq 4$, there exists $T(s, 2x+3)$ for all integers s with $\lfloor (4x+10)/3 \rfloor \leq s \leq \lceil (2x^2 + 5x - 1)/3 \rceil$. However, in this case we cannot guarantee that all of these trades will have a pair of independent vertices and so we also may need some trades of order $2x+2$. There exists $T(s, 2x+2)$ for all integers s with $\lfloor (4x+10)/3 \rfloor \leq s \leq \lceil (2x^2 + 2x - 1)/3 \rceil$. The largest volume trade we can construct with $I = (2x+1)^2$ is of volume $\lceil 2\frac{2x^2+2x-1}{3} \rceil + \lceil \frac{2x^2+5x-1}{3} \rceil$ which is at least $2x^2 + 3x - 1$. Since $2x^2 + 3x - 1 - (4x + 10) > 6$ (when $x \geq 4$), these trades can be used together with the group divisible triple systems with $I = 0, 1, \dots, x^2 - 6, x^2$ to construct $T(4x+10, 6x+5), T(4x+11, 6x+5), \dots, T(6x^2+7x, 6x+5)$. □

We are now ready to prove the main theorem.

Theorem 3.10 For all integers v, $E(v) = P(v)$.

Proof We proceed by induction on v. As well as the result that $E(v) = P(v)$ for $v \leq 15$ (see Lemma 3.7) we use the three existence results of Lemmas 3.8 and 3.9 and Corollary 3.2. We refer to these as the *small, middle* and *large volume constructions* respectively. All we need to do is show that:

(1) the small volume construction gives trades of all possible volumes $\leq s_1$ say;

(2) the large volume construction gives trades of all possible volumes $\geq s_2$ say;

(3) the middle volume construction gives trades of all possible volumes s with $s_1 < s < s_2$.

It is straightforward to check that these conditions are satisfied by considering trades of orders congruent to each of the residues modulo 6 individually. □

4 Appendix

In this section we give several trades which have been referred to earlier in the paper.

T(7,9)

T_1			T_2		
0	1	2	0	1	5
0	3	4	0	2	4
0	5	6	0	3	6
1	3	7	1	2	7
1	5	8	1	3	8
2	4	7	3	4	7
3	6	8	5	6	8

T(8,9)

T_1			T_2		
0	1	2	0	1	4
0	3	4	0	2	7
0	5	6	0	3	5
0	7	8	0	6	8
1	3	5	1	2	3
1	4	6	1	5	6
2	3	7	3	4	6
3	6	8	3	7	8

T(9,10)

T_1			T_2		
0	1	2	0	1	7
0	3	4	0	2	6
0	5	6	0	3	5
0	7	8	0	4	8
1	3	5	1	2	3
1	4	6	1	4	9
1	7	9	1	5	6
2	3	6	3	4	6
4	8	9	7	8	9

T(10,10)

T_1			T_2		
0	1	2	0	1	7
0	3	4	0	2	5
0	5	6	0	3	8
0	7	8	0	4	6
1	3	5	1	2	6
1	4	6	1	3	9
1	7	9	1	4	5
2	3	6	2	3	4
2	4	5	3	5	6
3	8	9	7	8	9

T(9,11)

T_1			T_2		
0	1	2	0	1	5
0	3	4	0	2	4
0	5	6	0	3	7
0	7	8	0	6	8
1	3	9	1	2	9
1	5	10	1	3	10
2	4	9	3	4	9
3	7	10	5	6	10
6	8	10	7	8	10

T(12,11)

T_1			T_2		
0	1	2	0	1	7
0	3	4	0	2	9
0	5	6	0	3	5
0	7	8	0	4	6
0	9	10	0	8	10
1	3	5	1	2	8
1	4	6	1	3	6
1	7	9	1	4	5
1	8	10	1	9	10
2	3	6	2	3	4
2	4	5	2	5	6
2	8	9	7	8	9

T(13,11)

T_1			T_2		
0	1	2	0	1	5
0	3	4	0	2	7
0	5	6	0	3	9
0	7	8	0	4	6
0	9	10	0	8	10
1	3	5	1	2	8
1	4	6	1	3	6
1	7	9	1	4	7
1	8	10	1	9	10
2	3	6	2	3	4
2	4	7	2	5	6
2	5	8	3	5	8
3	8	9	7	8	9

T(19,13)

T_1			T_2		
0	1	2	0	1	3
0	3	4	0	2	5
0	5	6	0	4	6
0	7	8	0	7	11
0	9	10	0	8	10
0	11	12	0	9	12
1	3	5	1	2	6
1	4	6	1	4	8
1	7	9	1	5	9
1	8	10	1	7	10
2	3	6	2	3	8
2	4	5	2	4	7
2	7	10	2	9	10
2	8	9	3	4	11
3	7	11	3	5	6
3	8	12	3	7	12
4	7	12	4	5	12
4	8	11	7	8	9
5	9	12	8	11	12

T(20,13)

T_1			T_2		
0	1	2	0	1	3
0	3	4	0	2	4
0	5	6	0	5	9
0	7	8	0	6	12
0	9	10	0	7	10
0	11	12	0	8	11
1	3	5	1	2	10
1	4	6	1	4	5
1	7	9	1	6	9
1	8	10	1	7	8
2	3	6	2	3	8
2	4	5	2	5	6
2	7	10	2	7	9
2	8	9	3	4	6
3	7	11	3	5	11
3	8	12	3	7	12
4	7	12	4	7	11
4	8	11	4	8	12
5	9	11	8	9	10
6	9	12	9	11	12

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