Some New D-Optimal Designs

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Abstract

We construct several new $(v; r, s; \lambda)$ supplementary difference sets with v odd and $r + s = \lambda + (v-1)/2$. They give rise to D-optimal designs of order $2v$. D-optimal designs of orders 158, 194, and 290 are constructed here for the first time. We also give an up to date survey of this class of supplementary difference sets in arbitrary Abelian groups of odd order $v < 100$.

0. Introduction

Supplementary difference sets (SDS) in finite Abelian groups is an active topic of research. Examples of supplementary difference sets were given as early as 1939 in a paper of Bose [1]. More recently, these have been formally defined and popularized in the work of J. Seberry (Wallis) [15, 17, 18, 19].

In this paper we consider only one special class of supplementary difference sets (X, Y) in a finite Abelian group G of order v. This means that every $a \in G$, $a \neq 0$, can be represented as $a = x - y$ with $x, y \in X$ or $x, y \in Y$ in λ ways (in total), where λ is a constant independent of a. If $|X| = r$ and $|Y| = s$, we say that (X, Y) have parameters $(v; r, s; \lambda)$. Furthermore we require that v be odd and $r+s = \lambda+(v-1)/2$. Such supplementary difference sets will be called D-optimal because they give rise to D-optimal designs of order $2v$. D-optimal SDS's have been studied by many authors starting with Ehlich [9]. There is only one infinite series of such sets known at the present time (see [11]).

We present several new D-optimal SDS's with $v = 27, 49, 73, 79, 97, 113$, and 145. In particular, we construct for the first time D-optimal designs of orders 158, 194,

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and 290. For $v = 27, 49$ our SDS's lie in an elementary Abelian group. No such SDS's were known before for these values of v. For $v = 73$ the known SDS has parameters $(73; 36, 28; 28)$ and we construct three non-equivalent SDS's having a different set of parameters, namely $(73; 42, 30; 36)$. For $v = 113$ a D-optimal design of order $2v$ is known, it belongs to an infinite series constructed by A.L. Whiteman [20]. His construction does not use SDS's. The first example of SDS with parameters $(113; 49, 49; 42)$ was found recently (see [12]). Our SDS with the same parameters is not equivalent to that example.

We collect in Table 1 all known results about the existence of D-optimal SDS's, with parameters $(v; r, s; \lambda)$, $v < 100$, that satisfy the known necessary conditions. There are a number of undecided cases indicated by the question mark.

1. D-optimal Supplementary Difference Sets

Let G be a finite Abelian group (written multiplicatively) of order v, and $\boldsymbol{Z}G$ its group ring over $\boldsymbol{Z}$ (the ring of integers). For $X \subset G$ let $X' = G \backslash X$ denote the complement of X in G, and let

$$N(X) = \left(\sum_{x \in X} x\right) \cdot \left(\sum_{x \in X} x^{-1}\right) \in \boldsymbol{Z}G.$$

We also set

$$T = \sum_{x \in G} x.$$

We say that the ordered k-tuple $(X_1, \cdots, X_k)$, with $X_i \subset G$, are *supplementary difference sets* (SDS) with parameters $(v; n_1, \cdots, n_k; \lambda)$ if $|X_i| = n_i$ for $i = 1, \cdots, k$, and

$$\sum_{i=1}^{k} N(X_i) = \left(\sum_{i=1}^{k} n_i - \lambda\right) \cdot 1 + \lambda T,$$

where $1 \in G$ is the identity element. For $k = 1$ we obtain the definition of difference sets.

We now introduce a special class of SDS.

Definition (1.1) We say that $(v; r, s; \lambda)$ supplementary difference sets are *D-optimal* if v is odd and $r + s = \lambda + (v - 1)/2$.

We shall see in the next section why these SDS's are important. The following two propositions are well known.

Proposition (1.2) *If* (X, Y) *are D-optimal* SDS, *then the same is true for* (X', Y), (X, Y'), *and* (X', Y'). ∎

Proposition (1.3) *If* (X, Y) *are D-optimal SDS with parameters* $(v; r, s; \lambda)$, *then*

$$2(2v - 1) = (v - 2r)^2 + (v - 2s)^2. \tag{1.1}$$

∎

We shall say that the parameters $(v; r, s; \lambda)$ are *feasible* if v is odd, $r + s = \lambda + (v-1)/2$, and (1.1) holds. Table 1 contains the list of all feasible parameters with $v < 100$. We shall be mainly concerned with the following question: If the parameters $(v; r, s; \lambda)$ are feasible and G is an Abelian group of order v, does there exist SDS in G with these parameters?

Let (X, Y) be a D-optimal SDS in G. The following *elementary operations* produce again D-optimal SDS:

(i) $(X, Y) \to (X', Y)$;

(ii) $(X, Y) \to (X, Y')$;

(iii) $(X, Y) \to (Y, X)$;

(iv) $(X, Y) \to (aX, Y)$, $a \in G$;

(v) $(X, Y) \to (X, aY)$, $a \in G$;

(vi) $(X, Y) \to (\sigma(X), \sigma(Y))$, where σ is an automorphism of G.

We say that two D-optimal SDS's are *equivalent* if one can be obtained from the other by a finite number of elementary operations.

2. D-optimal Designs

Let H be a $\{\pm 1\}$-matrix of order n. We shall write I_n for the identity matrix of order n, and J_n for the matrix of order n having all entries equal 1. The well known Hadamard inequality implies that

$$\det(H) \leq n^{n/2}. \tag{2.1}$$

Furthermore, if equality holds, then H is a Hadamard matrix, i.e., $HH^T = I_n$, where H^T is the transpose of H.

From now on we assume that $n = 2v$ where v is odd. If $v > 1$, there are no Hadamard matrices of order n, and so the inequality (2.1) is always strict. In fact Ehlich [9] has shown that a stronger inequality is valid:

$$\det(H) \leq 2^v (2v-1)(v-1)^{v-1}. \tag{2.2}$$

Definition (2.1) A $\{\pm 1\}$-matrix H of order $n = 2v$, v odd, for which equality holds in (2.2) is called *maximal.*

If A and B are commuting $\{\pm 1\}$-matrices of order v, such that

$$AA^T + BB^T = 2(v-1)I_v + 2J_v, \tag{2.3}$$

then

$$H = \begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix} \tag{2.4}$$

is a maximal matrix.

Definition (2.2) The maximal matrices of order $n = 2v$, v odd, given by (2.4), where A and B commute and satisfy (2.3), are called *D-optimal designs.*

Now let (X, Y) be a D-optimal SDS in an Abelian group G of order v. Define a $\{\pm 1\}$-matrix $A = (a_{x,y})$ of order v (indexed by elements $x, y \in G$) by

$$a_{x,y} = \begin{cases} -1, & \text{if } xy^{-1} \in X; \\ 1, & \text{otherwise.} \end{cases}$$

Define $B = (b_{x,y})$ similarly by using Y instead of X. The matrices A and B satisfy (2.3), and so by inserting these matrices into the array (2.4) we obtain a D-optimal design. We shall refer to D-optimal designs constructed in this way from D-optimal SDS as being of *Abelian type.* If the group G is cyclic, these designs are known as *D-optimal designs of circular type.*

Koukouvinos, Kounias, and Seberry [11] used a theorem of Spence [16, Theorem 1] to construct an infinite series of D-optimal SDS's.

Theorem (2.3) [16, 11] *If q is a prime power then there exists a D-optimal SDS of circular type with parameters $(v = q^2 + q + 1; q^2, q^2; \lambda = q(q-1)/2)$.* ■

Another important result is due to A.L. Whiteman.

Theorem (2.4) [20] *If q is an odd prime power and $v = 2q^2 + 2q + 1$, then there exist D-optimal designs of order $n = 2v$.* ■

The D-optimal designs constructed by Whiteman do not arise from SDS's. They are derived from the infinite series of SBIBD's constructed by A.E. Brouwer [2], by using an idea of Kharaghani [10]. It is an open question whether D-optimal designs of Abelian type exist for the orders n given in Whiteman's theorem. For some partial results in this direction see [12] and Section 6.

Let us say that two D-optimal designs of Abelian type are *equivalent* if the corresponding D-optimal SDS's are equivalent in the sense defined in Section 1.

The equivalence classes of circular D-optimal designs of order $n = 2v$ were enumerated by Yang [23] for $v \leq 9$ and by Kounias, Koukouvinos, Nikolaou and Kakos [13, 14] for $v \leq 27$ and $v = 33, 45$. For the number of equivalence classes of arbitrary D-optimal designs of order $2v$ with $v \leq 9$ see the recent paper of Cohn [7].

3. Some New Cyclic D-optimal SDS's

We give the first example of D-optimal SDS with parameters $(79; 37, 31; 29)$. At the same time this provides the first example of a D-optimal design of order 158.

Let $F = Z_{79}$ be the finite field of order 79, and F^* its multiplicative group. Denote by H the subgroup of order 3 of F^*. Thus $H = \{1, 23, 55\}$. The 26 cosets of H in F^* are enumerated as follows:

$$\begin{aligned}
&\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 4H,\ \alpha_8 = 5H, \\
&\alpha_{10} = 6H,\ \alpha_{12} = 8H,\ \alpha_{14} = 9H,\ \alpha_{16} = 11H, \\
&\alpha_{18} = 12H,\ \alpha_{20} = 15H,\ \alpha_{22} = 18H,\ \alpha_{24} = 22H,
\end{aligned}$$

and $\alpha_{2i+1} = -\alpha_{2i}$ for $0 \le i \le 12$.

We now list two non-equivalent SDS's (X, Y), with parameters $(79; 48, 42; 51)$, in the additive group of F. In both cases X and Y are unions of some cosets α_i, i.e.,

$$X = \bigcup_{i \in J} \alpha_i, \quad Y = \bigcup_{i \in K} \alpha_i, \tag{3.1}$$

and so it suffices to list the index sets J and K. For the first SDS we have:

$$\begin{aligned} J &= \{0, 2, 4, 6, 7, 9, 10, 11, 12, 13, 17, 19, 20, 21, 22, 23\}, \\ K &= \{0, 4, 5, 6, 7, 11, 12, 14, 15, 16, 18, 23, 24, 25\}, \end{aligned}$$

and for the second:

$$\begin{aligned} J &= \{0, 1, 2, 4, 6, 7, 12, 13, 14, 15, 17, 20, 21, 22, 24, 25\}, \\ K &= \{0, 3, 4, 8, 11, 12, 13, 17, 18, 20, 21, 22, 24, 25\}. \end{aligned}$$

By replacing X and Y by their complements in F, we obtain SDS (Y', X') with parameters $(79; 37, 31; 29)$.

Next we give the first example of a D-optimal SDS with parameters (97;46,39;37), and also the first example of a D-optimal design of order 194. Let $F = \mathbb{Z}_{97}$ be the finite field of order 97, and $H = \{1, 35, 61\}$ the subgroup of order 3 of F^*. We enumerate the 32 cosets of H in F^* as follows:

$$\begin{aligned} &\alpha_0 = H, \ \alpha_2 = 2H, \ \alpha_4 = 3H, \ \alpha_6 = 4H, \\ &\alpha_8 = 5H, \ \alpha_{10} = 6H, \ \alpha_{12} = 7H, \ \alpha_{14} = 9H, \\ &\alpha_{16} = 10H, \ \alpha_{18} = 12H, \ \alpha_{20} = 13H, \ \alpha_{22} = 15H, \\ &\alpha_{24} = 18H, \ \alpha_{26} = 20H, \ \alpha_{28} = 23H, \ \alpha_{30} = 26H, \end{aligned}$$

and $\alpha_{2i+1} = -\alpha_{2i}$ for $0 \le i \le 30$. We have found (by using a computer search) an SDS (X, Y) with parameters (97; 51, 39; 42) in the additive group of F. Again the sets X and Y have the form (3.1) where now

$$\begin{aligned} J &= \{2, 4, 6, 7, 11, 13, 16, 20, 21, 22, 24, 25, 27, 28, 29, 30, 31\}, \\ K &= \{0, 1, 11, 12, 14, 18, 20, 21, 23, 25, 26, 28, 31\}. \end{aligned}$$

The SDS (X', Y) has parameters (97; 46, 39; 37).

We now consider the case $v = 73$. D-optimal SDS (X, Y) with parameters (73; 37, 28; 29) are known, they belong to the infinite series mentioned in Theorem (2.3). By replacing X with X' the parameters are replaced with (73; 36, 28; 28). We have constructed several non-equivalent D-optimal SDS's (X, Y) with parameters (73; 42, 30; 36). No SDS's with these parameters were known before.

Let $F = \mathbb{Z}_{73}$ be the finite field of order 73, and H the subgroup of order 3 of F^*. Thus $H = \{1, 8, 64\}$. We enumerate the 24 cosets of H in F^* as follows:

$$\begin{aligned} &\alpha_0 = H, \ \alpha_2 = 2H, \ \alpha_4 = 3H, \ \alpha_6 = 4H, \\ &\alpha_8 = 5H, \ \alpha_{10} = 6H, \ \alpha_{12} = 7H, \ \alpha_{14} = 11H \\ &\alpha_{16} = 12H, \ \alpha_{18} = 13H, \ \alpha_{20} = 14H, \ \alpha_{22} = 21H, \end{aligned}$$

and $\alpha_{2+i+1} = -\alpha_{2i}$ for $0 \leq i \leq 11$.

We shall list three non-equivalent SDS's (X, Y) with parameters (73; 42, 30; 36). In all three cases X and Y have the form (3.1). For the first SDS we have:

$$\begin{aligned} J &= \{0,2,3,4,5,6,12,13,14,15,16,18,20,21\}, \\ K &= \{2,5,9,11,12,13,16,18,22,23\}; \end{aligned}$$

for the second:

$$\begin{aligned} J &= \{2,4,7,8,9,10,12,14,15,16,18,19,20,21\}, \\ K &= \{0,2,5,8,9,13,15,19,22,23\}; \end{aligned}$$

and for the third:

$$\begin{aligned} J &= \{0,1,2,3,6,9,12,14,16,17,18,20,22,23\}, \\ K &= \{4,7,10,13,14,15,16,17,22,23\}. \end{aligned}$$

4. Some New Non-Cyclic D-optimal SDS's

D-optimal SDS's having parameters (27; 11, 9; 7) are known in an Abelian group G of order $v = 27$ when G is either cyclic (see [22]) or of type 3×9 (see [4]). We now give an example of an SDS (X, Y), having the same parameters, in the elementary Abelian group G (written additively, with generators $1, a, b$). They are given by:

$$\begin{aligned} X &= \{-1, a, -a, -b, 1+a, 1-a, a-b, b-a, -a-b, 1-a+b, b-1-a\}, \\ Y &= \{1, -a, 1-b, -1-b, a+b, -a-b, 1+a-b, a-1-b, b-1-a\}. \end{aligned}$$

Next we consider D-optimal SDS with parameters (49; 22, 18; 16). Such SDS of circular type was found first by Cohn [5]. We present now the first example of such SDS in the elementary Abelian group G (written additively, with generators 1 and a). They are:

$$\begin{aligned} X = \{&0,1,2,3,4,5,6,a,2a,3a,4a,5a,6a,1+5a,2+3a, \\ &3+a,3+3a,4+6a,5+4a,5+5a,6+2a,6+6a\}, \end{aligned}$$

and

$$\begin{aligned} Y = \{&3,5,6,a,2a,4a,1+a,1+4a,1+5a,2+a,2+2a, \\ &2+3a,3+3a,4+2a,4+4a,4+6a,5+5a,6+6a\}. \end{aligned}$$

The first paper investigating D-optimal SDS in non-cyclic Abelian groups is [4]. Several such SDS's are constructed in that paper. More precisely the authors have constructed D-optimal designs of the form (2.4) where the blocks A and B are block-circulant matrices with circulant blocks of size 3 or 5.

The claim made there that there are no D-optimal designs (2.4) of order 30 with A and B multi-circulants is certainly in error. Indeed there is only one type of Abelian group of order 15. Consequently the well known D-optimal design (2.4) of order 30 (see [9]), in which A and B are circulants, can be rewritten so that A and B become multi-circulants. Explicitly we can take:

$$A = \mathrm{circ}(A_0, A_1, A_2, A_3, A_4),$$
$$B = \mathrm{circ}(B_0, B_1, B_2, B_3, B_4),$$

where

$$A_0 = \mathrm{circ}(-,-,+),\ A_1 = A_0,\ A_2 = -A_0,$$
$$A_3 = \mathrm{circ}(+,-,+),\ A_4 = \mathrm{circ}(+,+,+),$$
$$B_0 = A_0,\ B_1 = A_3,\ B_2 = A_4,$$
$$B_3 = \mathrm{circ}(-,+,+),\ B_4 = A_4.$$

By $\mathrm{circ}(x_1, \cdots, x_m)$ we denote the circulant matrix whose first row is $(x_1, \cdots, x_m)$, and $+$ and $-$ stand for $+1$ and -1, respectively.

The error appears to be in their claim [4, p. 129] that it suffices to consider the case where all the blocks D_i or G_j are $\pm A$ or $\pm B$.

Similarly, their claim that there are no D-optimal designs (2.4) of order 90 where A and B are block-circulant matrices with circulant blocks of size 5 is in error. Indeed Cohn [5] has constructed a D-optimal SDS with $v = 45$. Since the direct product of cyclic groups of order 5 and 9 is again cyclic, Cohn's D-optimal design of circular type can be rewritten as (2.4) with A and B block-circulants having circulant blocks of size 5.

The analogous claim made in [4] regarding the non-existence of D-optimal designs (2.4) of order 90 in which A and B are block-circulant matrices, with circulant blocks of size 3, remains in doubt.

5. D-optimal SDS's with $v < 100$

On the next page we present the table of feasible parameters $(v; r, s; \lambda)$ with $v < 100$. For each Abelian group of order v we indicate whether or not D-optimal SDS with these parameters are known.

6. SDS Substitutes for Whiteman Designs

In connection with Theorem (2.4), the authors of [12] have raised the following question: Is there an infinite series of cyclic D-optimal SDS's with parameters

$$(v = 2q^2 + 2q + 1; q^2, q^2; \lambda = q(q-1)), \tag{6.1}$$

where q is an odd prime power? They point out that they may exist also when q is a power of 2. They show that such SDS (X, Y) indeed exist for $q = 2, 3, 4, 5, 7$, and 9. Moreover in all their examples $Y = (2q+1)X$ and $G = \boldsymbol{Z}_v$.

We have constructed (independently) such SDS's for $q = 7, 8$. It turns out that our SDS for $q = 7$ is not equivalent to the one given in [12]. Our SDS for $q = 8$ provides the first example of a D-optimal design of order 290. We conclude with the description of our SDS's having parameters (6.1) for $q = 7, 8$.

Table 1
D-optimal SDS's for $v < 100$

v	G	r	s	λ	Existence	v	G	r	s	λ	Existence
1	1	0	0	0	Yes [9]	49	49	22	18	16	Yes [5]
3	3	1	0	0	Yes [9]		7×7	22	18	16	Yes *
5	5	1	1	0	Yes [9]	51	3×17	21	20	16	Yes [6]
7	7	3	1	1	Yes [9]	55	5×11	24	21	18	?
9	9	3	2	1	Yes [9]	57	3×19	28	21	21	Yes [11]
	3×3	3	2	1	Yes [4]	59	59	28	22	21	?
13	13	6	3	3	Yes [9]	61	61	25	25	20	Yes [8]
		4	4	2	Yes [9]	63	7×9	29	24	22	?
15	3×5	6	4	3	Yes [9]			27	25	21	Yes [8]
19	19	7	6	4	Yes [9]		$3 \times 3 \times 7$	29	24	22	?
21	3×7	10	6	6	Yes [21]			27	25	21	?
23	23	10	7	6	Yes [21]	69	3×23	31	27	24	?
25	25	9	9	6	Yes [24]	73	73	36	28	28	Yes [11]
	5×5	9	9	6	Yes [4]			31	30	25	Yes *
27	27	11	9	7	Yes [22]	75	3×25	36	29	28	?
	3×9	11	9	7	Yes [4]		$3 \times 5 \times 5$	36	29	28	?
	$3 \times 3 \times 3$	11	9	7	Yes *	77	7×11	34	31	27	?
31	31	15	10	10	Yes [24]	79	79	37	31	29	Yes *
33	3×11	15	11	10	Yes [4]	85	5×17	39	34	31	?
		13	12	9	Yes [25]			36	36	30	?
37	37	16	13	11	Yes [5]	87	3×29	38	36	31	?
41	41	16	16	12	Yes [5]	91	7×13	45	36	36	Yes [11]
43	43	21	15	15	Yes [3]	93	3×31	45	37	36	?
		18	16	13	Yes [3]			42	38	34	Yes [8]
45	5×9	21	16	15	Yes [5]	97	97	46	39	37	Yes *
	$3 \times 3 \times 5$	21	16	15	?	99	9×11	43	42	37	?
							$3 \times 3 \times 11$	43	42	37	?

The question mark in the last column means that the existence question has not been resolved so far. The asterisk in the last column means that such SDS is given in the previous two sections of this paper.

Case $q = 7$. Let F be the finite field $\boldsymbol{Z}_{113}$, and let H be the subgroup of F^* of order 7. Explicitly we have

$$H = \{1, 16, 28, 30, 49, 106, 109\}.$$

We enumerate the cosets α_i, $0 \leq i \leq 15$, of H in F^* as follows:

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 5H,$$
$$\alpha_8 = 6H,\ \alpha_{10} = 9H,\ \alpha_{12} = 10H,\ \alpha_{14} = 13H,$$

and $\alpha_{2i+1} = -\alpha_{2i}$ for $0 \leq i \leq 7$. Then

$$X = \bigcup_{i \in J} \alpha_i, \quad Y = \bigcup_{i \in K} \alpha_i, \tag{6.2}$$

where

$$J = \{0, 2, 3, 5, 7, 9, 13\}, \quad K = \{0, 1, 3, 5, 7, 8, 12\}.$$

Furthermore we have $Y = 15X$, and note that $15 = 2q + 1$.

Case $q = 8$. Let $R = \boldsymbol{Z}_{145}$ be the ring of integers modulo 145. Its group of units (i.e., invertible elements) has order 112. Let H be the subgroup of order 7 of this group of units. Explicitly we have

$$H = \{1, 16, 36, 81, 111, 136, 141\}.$$

The group H acts on the additive group $(R, +)$ by multiplication. We enumerate the orbits of H for this action as follows:

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 5H,$$
$$\alpha_8 = 6H,\ \alpha_{10} = 7H,\ \alpha_{12} = 10H,\ \alpha_{14} = 11H,$$
$$\alpha_{16} = 14H,\ \alpha_{18} = 22H,\ \alpha_{20} = \{29\},\ \alpha_{22} = \{58\},$$

and $\alpha_{2i+1} = -\alpha_{2i}$ for $0 \leq i \leq 11$. The SDS (X, Y) again has the form (6.2) with

$$J = \{0, 1, 3, 4, 5, 10, 12, 13, 17, 20\}, \quad K = \{0, 2, 6, 7, 10, 11, 14, 15, 19, 20\}.$$

In this case we have $Y = 11X$ in $(R, +)$. Note that $11 \neq 2q + 1$.

References

[1] R.C. Bose, On the construction of balanced incomplete block designs, *Ann. Eugenics* 9 (1939), 353-399.

[2] A.E. Brouwer, An infinite series of symmetric designs, *Math. Centrum Amsterdam Report* Z W 202/83.

[3] T. Chadjipantelis and S. Kounias, Supplementary difference sets and D-optimal designs for $n \equiv 2 \bmod 4$, *Discrete Math.* 57 (1985), 211-216.

[4] T. Chadjipantelis, S. Kounias, and C. Moyssiadis, Construction of D-optimal designs for $N \equiv 2 \bmod 4$ using block-circulant matrices, *J. Combin. Theory* Ser. A 40 (1985), 125-135.

[5] J.H.E. Cohn, On determinants with elements ± 1, II, *Bull. London Math. Soc.* 21 (1989), 36-42.

[6] J.H.E. Cohn, A D-optimal design of order 102, *Discrete Math.* 102 (1992), 61-65.

[7] J.H.E. Cohn, On the number of D-optimal designs, *J. Combin. Theory* Ser. A 66 (1994), 214-225.

[8] D.Ž. Đoković, On maximal $(1,-1)$-matrices of order $2n, n$ odd, *Radovi Matematički* 7 (1991), 371-378.

[9] H. Ehlich, Determinantenabschätzungen für binäre Matrizen, *Math. Z.* 83 (1964), 123-132.

[10] H. Kharaghani, A Construction of D-optimal designs for $n \equiv 2 \bmod 4$, *J. Combin. Theory*, Ser. A 46 (1987), 156-158.

[11] C. Koukouvinous, S. Kounias, and J. Seberry, Supplementary difference sets and optimal designs, *Discrete Math.* 88 (1991), 49-58.

[12] C. Koukouvinos, J. Seberry, A.L. Whiteman, and Ming-yuan Xia, Optimal designs, supplementary difference sets and multipliers (preprint).

[13] S. Kounias, C. Koukouvinos, N. Nikolaou, and A. Kakos, The non-equivalent circulant D-optimal designs for $n \equiv 2 \bmod 4, n \leq 54, n = 66$, *J. Combin. Theory Ser.* A 65(1994), 26-38.

[14] S. Kounias, C. Koukouvinos, N. Nikolaou, and A. Kakos, The non-equivalent circulant D-optimal designs for $n = 90$ (preprint).

[15] J. Seberry Wallis, Some remarks on supplementary difference sets, *Colloquia Mathematica Societatis Janos Bolyai* 10(1973), 1503-1526.

[16] E. Spence, Skew-Hadamard matrices of the Goethals-Seidel type, *Canad. J. Math.* 27 (1975), 555-560.

[17] J. Wallis, On supplementary difference sets, *Aequationes Math.* 8 (1972), 242-257.

[18] J. Wallis, A note on supplementary difference sets, *Aequationes Math.* 10 (1974), 46-49.

[19] W.D. Wallis, Anne Penfold Street, and Jennifer Seberry Wallis, *Combinatorics: Room Squares, Sum-free Sets, Hadamard Matrices*, vol. 292, Lecture Notes in Math. Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[20] A.L. Whiteman, A Family of D-optimal designs, *Ars Combinatoria* 30 (1990), 23-26.

[21] C.H. Yang, Some designs for maximal $(+1,-1)$-determinant of order $n \equiv 2\,(\mathrm{mod}4)$, *Math. Comp.* 20 (1966), 147-148.

[22] C.H. Yang, A construction for maximal $(+1,-1)-$ matrix of order 54, *Bull. Amer. Math. Soc.* 72 (1966), 293.

[23] C.H. Yang, On designs of Maximal $(+1,-1)-$matrices of order $n \equiv 2\,(\mathrm{mod}4)$, *Math. Comp.* 22 (1968), 174-180.

[24] C.H. Yang, On designs of maximal $(+1,-1)-$ Matrices of Order $n \equiv 2\,(\mathrm{mod}4)$ II, *Math. Comp.* 23 (1969), 201-205.

[25] C.H. Yang, Maximal binary matrices and sum of two squares, *Math. Comp.* 30 (1976), 148-153.

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