A Sufficient Condition for The Existence of a Spanning Eulerian Subgraph in 2-edge-Connected Graphs *

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Abstract

We prove that if G is a 2-edge-connected graph of order $n \geq 14$ and $\max\{d(u), d(v)\} > \frac{n-5}{3}$ for each pair of nonadjacent vertices u, v of G, then G contains a spanning Eulerian subgraph and hence the line graph of G is Hamiltonian.

1. Introduction

We use [1] for basic terminology and notation not defined here and consider simple graphs only.

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. G is called *Eulerian* if it is connected and every vertex has even degree. For a subgraph H of G, we call H a *spanning Eulerian subgraph* if it is Eulerian and $V(H) = V(G)$; and a *dominating Eulerian subgraph* if it is an Eulerian subgraph and $E(V(G) - V(H)) = \emptyset$. Obviously, any spanning Eulerian subgraph is a dominating Eulerian subgraph. The line graph $L(G)$ of G is a graph which has $E(G)$ as its vertex set and in which two vertices are joined if and only if they are adjacent edges in G. For a vertex v of G, we will denote its degree and neighborhood in G by $d_G(v)$ and $N_G(v)$, respectively.

Several Hamiltonian results about the line graph of the given graph have been discovered based on the typical degree conditions. The following result is proved in [2].

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Theorem 1.1 (Catlin [2]). *Let G be a 2-edge-connected graph of order $n \geq 100$ in which $d_G(u) + d_G(v) > \frac{2n-10}{5}$ for each pair of nonadjacent vertices u, v of G. Then G contains a spanning Eulerian subgraph and hence $L(G)$ is Hamiltonian.*

In present paper we will prove a Fan-like condition for a graph to contain a spanning Eulerian subgraph. Our main result is as follows.

Theorem 1.2. *Let G be a 2-edge-connected graph of order $n \geq 14$ in which $max\{d_G(u), d_G(v)\} > \frac{n-5}{3}$ for each pair of nonadjacent vertices u, v of G. Then G contains a spanning Eulerian subgraph and hence $L(G)$ is Hamiltonian.*

Remarks. Theorem 1.2 is best possible in the sense that the bound on the degree cannot be lowered. To see this, we construct the graph G_0 from the disjoint union of three copies of the complete graph $K_{\frac{n-2}{3}}$ ($n \equiv 2 \bmod 3$) and two copies of K_1, set $H_1 \cong H_2 \cong K_1$ and $H_3 \cong H_4 \cong H_5 \cong K_{\frac{n-2}{3}}$, by adding an edge between each H_i ($i = 1, 3, 4$) and each H_j ($j = 2, 5$). It is seen easily that G_0 is 2-edge-connected graph and there exists nonadjacent vertices u, v of G with $\max\{d_G(u), d_G(v)\} = \frac{n-5}{3}$. However, G_0 contains no spanning Eulerian subgraph since $G_o/\sim \cong K_{2,3}$.

Let H_1, H_2, H_3 be three disjoint copies of the complete graph $K_{\frac{n-1}{3}}$ ($n \equiv 1 \bmod 3$) and let H_4 be a copy of K_1, disjoint from H_1, H_2, H_3. The graph G_n consists of the union of H_1, H_2, H_3 and H_4 by adding an edge between H_i and H_{i+1}, ($i = 1, 2, 3, 4$, the indices taken modulo 4). It is easy to check that the graph G_n satisfies the conditions of Theorem 1.2, but not that of Theorem 1.1 when $n \geq 40$. In fact, it is shown easily that Theorem 1.2 and Theorems 1.1 are incomparable.

2. Preliminaries

In [2] Catlin introduced the following concept. A graph G is collapsible if for every subset S of $V(G)$ of even cardinality there is a subgraph H of G such that $G - E(H)$ is connected and $d_H(v)$ is odd for each vertex v of S. It is clear that K_n is collapsible if and only if $n \neq 2$

We now define an equivalence relation $\sim$ on $V(G)$ by setting $u \sim v$ if and only if there is a collapsible subgraph H of G which contains both u and v. By Catlin's Theorem [2] the union of two collapsible subgraphs with non-null intersection is collapsible, we see that an equivalence class under $\sim$ induce a maximal collapsible subgraph of G.

Let $\{H_i | i = 1, 2, \ldots, k\}$ be the collection of all maximal collapsible subgraphs of G. Write $G/\sim$ for the graph obtained from G by deleting $\bigcup_{i=1}^{k} E(H_i)$ and then contracting in turn $H_1, H_2, \ldots, H_k$ to k new vertices $v_1, v_2, \ldots, v_k$ so that no edge of $E(G) - \bigcup_{i=1}^{k} E(H_i)$ is lost. We call $G/\sim$ the quotient graph of G and call H_i the preimages of v_i in G. Catlin showed the following result.

Theorem 2.1 (Catlin[2]). *Let G be a graph and let $G/\sim$ be its quotient graph. Then*

(a). $G/\sim$ is simple graph and K_3-free;

(b). G has a spanning Eulerian subgraph if and only if $G/\sim$ has a spanning Eulerian subgraph;

(c). Let $V_4 = \{v \in V(G/\sim) | d_{G/\sim}(v) < 4\}$. If $G/\sim$ is nontrivial and 2-edge-connected, then $|V_4| \geq 4$, and if equality holds, then $G/\sim$ is Eulerian.

The next theorem will be needed in the proof of our main result.

Theorem 2.2 (Harary and Nash-Williams[3]). *The line graph $L(G)$ of a simple graph G with at least three edges is Hamiltonian if and only if G has a dominating Eulerian subgraph.*

3. The Proof of Theorem 1.2

Proof. Let $H_1, H_2, \ldots, H_k$ denote all the maximal collapsible subgraphs of G. Set Set $G^* = G/\sim$ and $V(G^*) = \{v_1, v_2, \ldots, v_k\}$, where H_i is the preimage of v_i for $i = 1, 2, \ldots, k$. We may assume without loss of generality that

$$d_{G^*}(v_1) \leq d_{G^*}(v_2) \leq \cdots \leq d_{G^*}(v_k).$$

Now we assume that G contains no the spanning Eulerian subgraph. Then Theorem 2.1(b) implies that G^* contains also no spanning Eulerian subgraph. Since G is 2-edge-connected, G^* is 2−edge-connected. Therefore, Theorem 2.1(c) asserts $d_{G^*}(v_5) \leq 3$.

Claim 1. If $|V(H_i)| \geq 4$ for $1 \leq i \leq 5$, then there is a vertex $h_i \in V(H_i)$ such that $N_G(h_i) \subseteq V(H_i)$.

Since G^* is simple graph and $d_{G^*}(v_i) \leq 5$ for $1 \leq i \leq 5$, the claim follows.

Claim 2. If $H_i \cong H_j \cong K_1$ with $1 \leq i < j \leq 5$, then v_i and v_j is adjacent in G^*.

Assume the contrary. Since the preimages of v_i and v_j in G are themselves, $v_iv_j \notin E(G^*)$ implies $v_iv_j \notin E(G)$ and so $\max\{d_G(v_i), d_G(v_j)\} > \frac{n-5}{3}$. Since $\max\{d_G(v_i), d_G(v_j)\} = \max\{d_{G^*}(v_i), d_{G^*}(v_j)\} \leq 3$, it follow that $n < 14$, contradicting the assumption $n \geq 14$.

We now consider two cases.

Case 1. $H_1 \cong K_1$ for some $1 \leq i \leq 5$.

Suppose that $H_a = K_1$, $1 \leq a \leq 5$. In this case, we can first conclude that $H_i \ncong K_3, 1 \leq i \leq 5$. Otherwise set $H_b = K_3$, $1 \leq b \neq a \leq 5$. Since G^* is simple graph and $d_{G^*}(v_a) \leq 3$, there exists a vertex h in H_b with $d_G(h) \leq 3$ such that h and v_a is nonadjacent in G. Hence, it follows from the degree condition that

$$\frac{n-5}{3} < \max\{d_G(h), d_G(v_a)\} \leq 3,$$

which gives that $n < 14$ contradicting the assumption $n \geq 14$. Moreover, since G^* is K_3-free, Claim 2 implies that the set $\{H_i | i = 1, 2, \ldots, 5\}$ contains at most two elements being isomorphic to K_1. Notice that K_2 is not collapsible and the collapsible graphs with three vertices are isomorphic to K_3. We have that $|V(H_j)| \geq 4$ for each $H_i \ncong K_1, 1 \leq i \leq 5$. Hence, Claim 1 asserts that there is a vertex $h_j \in H_j$ so that $N_G(h_j) \subseteq V(H_j)$. This implies that $v_a h_j \notin E(G)$ and so $\max\{d_G(v_a), d_G(h_j)\} > \frac{n-5}{3}$. Since $d_G(v_a) = d_{G^*}(v_a) \leq 3 \leq \frac{n-5}{3}$, it follows that

$$d_G(h_j) > \frac{n-5}{3} \text{ for each } h_j \in H_i \ncong K_1, 1 \leq i \leq 5.$$

Thus, we obtain

$$\begin{aligned} n &= |V(G)| \geq \textstyle\sum_{i=1}^{5} |V(H_i)| \\ &\geq \textstyle\sum_{\substack{H_i \ncong K_1 \\ 1 \leq i \leq 5}} |V(H_i)| + 2 \\ &\geq \textstyle\sum_{\substack{H_i \ncong K_1 \\ 1 \leq i \leq 5}} (d_G(h_i) + 1) + 2 \\ &> 3(\tfrac{n-5}{3} + 1) + 2 = n, \end{aligned}$$

a contradiction, which completes the proof of Case 1.

Case 2. $H_i \ncong K_1$ for all $1 \leq i \leq 5$.

In this case, we first can conclude that the set $\{H_i | i = 1, 2, \ldots, 5\}$ contains at most one graph isomorphic to K_3. Otherwise assume that $H_a \cong H_b \cong K_3$, $1 \leq a < b \leq 5$. Since G^* is simple and $d_{G^*}(v_a) \leq d_{G^*}(v_b) \leq 3$, there exists the vertex $h_a \in H_a$ and the vertex $h_b \in H_b$ such that $\max\{d_G(h_a), d_G(h_b)\} \leq 3$ and $h_a h_b \notin E(G)$. Thus we obtain

$$\frac{n-5}{3} < \max\{d_G(h_a), d_G(h_b)\} \leq 3,$$

which implies $n < 14$. This contradicts the assumption $N \geq 14$. Again because K_2 is not collapsible and the collapsible graphs with three vertices are isomorphic to K_3, we have that $|V(H_j)| \geq 4$ for each $H_j \ncong K_3$, $1 \leq j \leq 5$. Hence Claim 1 asserts that there is a vertex $h_j \in H_j \ncong K_3, 1 \leq j \leq 5$, such that $N_G(h_j) \subseteq V(H_j)$ and so $\{h_j \in H_j \ncong K_3 | 1 \leq j \leq 5\}$ is a independent set of G. By the hypothesis of the theorem, we see that there are at least three such vertices h_j whose degree of each vertex is greater than $\frac{n-5}{3}$. Therefore, we obtain

$$\begin{aligned} n &= |V(G)| \geq \textstyle\sum_{i=1}^{5} |V(H_i)| \\ &\geq \textstyle\sum_{\substack{H_i \ncong K_3 \\ 1 \leq i \leq 5}} |V(H_i)| + 3 \\ &\geq \textstyle\sum_{\substack{H_i \ncong K_1 \\ 1 \leq i \leq 5}} (d_G(h_i) + 1) + 3 \\ &> 3(\tfrac{n-5}{3} + 1) + 3 = n + 1, \end{aligned}$$

a contradiction, which completes the proof of this case and Theorem 1.2. □

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