# Edge Homogeneous Embeddings of Cycles in Graphs

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**Abstract**

Let $n > m \geq 4$ be positive integers. The edge framing number $efr(C_m, C_n)$ of $C_m$ and $C_n$ is defined as the minimum size of a graph every edge of which belongs to an induced $C_m$ and an induced $C_n$. We show that $efr(C_m, C_n) = n+4$ if $n = 2m-4$ and $m \geq 5$, $efr(C_m, C_n) = n+5$ if $n = 2m-6$ and $m \geq 7$ and $efr(C_m, C_n) = n+6$ if $n = 2m-8$ $(m \geq 10)$ or $m = n-1$ (where $n \geq 5$ and $n \notin \{6, 8\}$) or $m = n-2$ ($n = 6$ or $n \geq 9$). It is also shown that $efr(C_m, C_n) \geq n+6$ for $n > m \geq 4$ with $n \neq 2m-4$ or $2m-6$ and $(m, n) \neq (5, 7)$. Furthermore, for the cases $n = 2m-4$ $(m \geq 5)$ and $n = 2m-6$ $(m \geq 7)$ we show that $C_m$ and $C_n$ are uniquely edge framed.

## 1 Introduction

In this paper, we use fairly standard graph theoretic terminology and notation. For example, for a graph $G = (V, E)$ with vertex set $V$ and edge set $E$, $p(G)$ and $q(G)$ will denote, respectively, the number of vertices $|V|$ (also called the order) and the number of edges $|E|$ (also called the size). If $v \in V$, the degree of $v$ in $G$ is written as $deg\, v$ and the minimum degree of $G$ is given by $\delta(G) = \min\{deg\, v : v \in V\}$, whereas the maximum degree of $G$ is given by $\Delta(G) = \max\{deg\, v : v \in V\}$. For other graph theory terminology we follow [3].

Chartrand, Gavlas, and Schultz [1] introduced the framing number of a graph. A graph $G$ is *homogeneously embedded* in a graph $H$ if for every vertex $x$ of $G$ and every vertex $y$ of $H$, there exists an embedding of $G$ in $H$ as an induced subgraph with $x$ at $y$. A graph $F$ of minimum order in which $G$ can be homogeneously embedded is called a *frame* of $G$, and the order of $F$ is called the *framing number* $fr(G)$ of $G$.

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In [1] it is shown that a frame exists for every graph, although a frame need not be unique.

Results involving frames and framing numbers of graphs have been presented by, among others Chartrand, Gavlas, and Schultz [1], Chartrand, Henning, Hevia, and Jarrett [2], Entringer, Goddard, and Henning [4], Gavlas, Henning, and Schultz [5], Goddard, Henning, Oellermann, and Swart [7, 8], and Henning [9].

Maharaj [10] introduced the edge framing number of a graph. A nonempty graph $G$ is said to be *edge homogeneously embedded* in a graph $H$ if for every edge $e$ of $G$ and every edge $f$ of $H$, there exists an edge isomorphism between $G$ and a vertex induced subgraph of $H$ which sends $e$ to $f$. A graph $F$ of minimum size in which $G$ can be edge homogeneously embedded is called an *edge frame* of $G$, and the size of $F$ is called the *edge framing number* $efr(G)$ of $G$. In [10] it is shown that an edge frame exists for every nonempty graph, although an edge frame need not be unique.

**Theorem A** *(Maharaj) Every nonempty graph has an edge frame.*

Maharaj [10] showed that edge homogeneous embedding does not directly imply (vertex) homogeneous embedding in general, and vice versa. Thus the two embedding requirements do not directly imply each other. However they are related in a natural way through line graphs. Maharaj [10] showed that for a large class of graphs, homogeneous embedding reduces to edge homogeneous embedding. In this sense, the edge homogeneous embedding requirement is a stronger embedding requirement than the (vertex) embedding requirement. The following result from [10] will prove useful to us.

**Theorem B** *(Maharaj) Let $G$ be a nonempty graph which is different from $C_3$ and $K_{1,3}$. If $G$ has two adjacent vertices of maximum degrees, and if $G$ can be edge homogeneously embedded in a graph $H$, then $\delta(H) \geq \Delta(G)$.*

For nonempty graphs $G_1$ and $G_2$, the edge framing number $efr(G_1, G_2)$ of $G_1$ and $G_2$ is defined as the minimum size of a graph $F$ such that $G_i$ $(i = 1, 2)$ can be edge homogeneously embedded in $F$. The graph $F$ is called an edge frame of $G_1$ and $G_2$. Then $efr(G_1, G_2)$ exists and, in fact, $efr(G_1, G_2) \leq efr(G_1 \cup G_2)$.

In this paper we investigate the edge framing number $efr(G_1, G_2)$ for several pairs $G_1$, $G_2$ of cycles. It is shown that $efr(C_5, C_7) = 12$. Furthermore, it is established that

$$efr(C_m, C_n) = \begin{cases} n+4 & \text{if } n = 2m-4 \text{ and } m \geq 5 \\ n+5 & \text{if } n = 2m-6 \text{ and } m \geq 7 \\ & \text{if } n = 2m-8 \text{ and } m \geq 10, \text{ or} \\ n+6 & \text{if } n = m+1 \text{ and } n \geq 5 \text{ and } n \notin \{6,8\}, \text{ or} \\ & \text{if } m = n-2 \text{ and } n = 6 \text{ or } n \geq 9 \end{cases}$$

It is also shown that $efr(C_m, C_n) \geq n + 6$ for $n > m \geq 4$ with $n \neq 2m - 4$ or $2m - 6$ and $(m, n) \neq (5, 7)$. Furthermore, for the cases $n = 2m - 4$ $(m \geq 5)$ and $n = 2m - 6$ $(m \geq 7)$ we show that $C_m$ and $C_n$ are uniquely edge framed.

# 2 The framing number of pairs of cycles

For integers $n > m \geq 3$, the framing number $fr(C_m, C_n)$ of a cycle $C_m$ of length $m$ and a cycle $C_n$ of length $n$ is defined as the minimum order of a graph every vertex of which belongs to an induced $C_m$ and an induced $C_n$. In [6] the framing number $fr(G_1, G_2)$ for several pairs $G_1$, $G_2$ of cycles is investigated. We will need the following result in [6].

**Lemma A** *For integers $n > m \geq 3$, $fr(C_m, C_n) \geq n + 2$.*

In [6], the class of frames for all those pairs of cycles $C_m$ and $C_n$ ($m < n$) for which $fr(C_m, C_n) = n + 2$ is completely characterized. In order to state this result neatly, we define certain sets of graphs. Let $S = \{(3,5), (3,6)\} \cup \{(m,n) \mid n = m+1$ and $m \geq 3\} \cup \{(m,n) \mid n = 2m-4$ and $m \geq 6\} \cup \{(m,n) \mid n = 2m-3$ and $m \geq 5\} \cup \{(m,n) \mid n = 2m-2$ and $m \geq 4\}$.

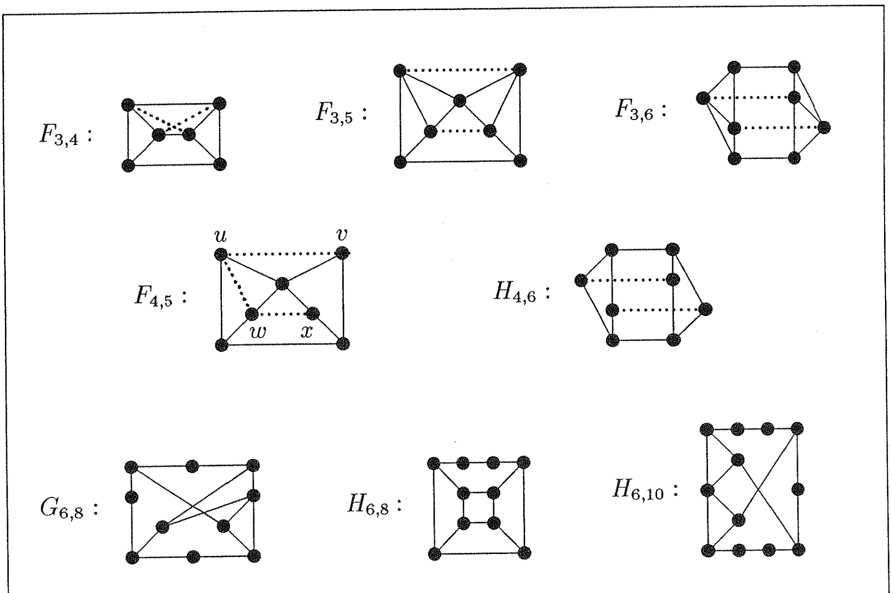


Figure 1:

For each $(m,n) \in S$, we define a set $\Phi_{m,n}$ of graphs as follows. For $m = 3$ and for $i \in \{4,5,6\}$, or for $m = 4$ and for $i = 5$, let $\Phi_{m,i}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{m,i}$ in Figure 1 by adding any combination (the presence or absence) of the dotted edges, provided that if $uw$ is an edge of $F_{4,5}$, then so too are $uv$ and $wx$. Let $\Phi_{4,6}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{4,6}$ or $G_{4,6}$ in Figure 2 or the graph $H_{4,6}$ in Figure 1 by adding any combination (the presence or absence) of the dotted edges. Let $\Phi_{6,8}$ be the set of all nonisomorphic graphs obtainable from the graph $G_{6,8}$ or $H_{6,8}$ in Figure 1 or the graph $F_{6,8}$ in Figure 2 by adding any combination (the presence or absence) of the dotted edges. For $m \geq 5$ and $i = m+1$, or for $m = 5$ or $m \geq 7$ and $i = 2m-3$, or $m \geq 7$ and $i = 2m-4$, let $\Phi_{m,i}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{m,i}$ in Figure 2 by adding any combination (the presence or absence) of the dotted edges, provided that if $uw$ is an edge of $F_{m,2m-3}$, then so too is $vw$.

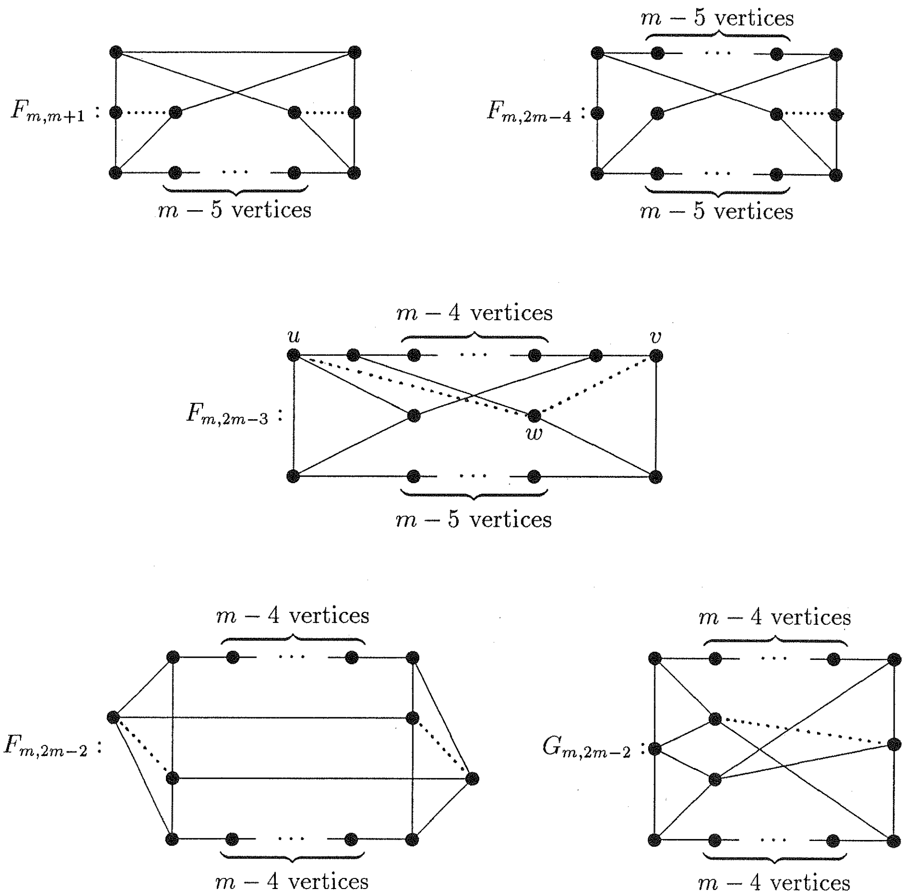


Figure 2:

Let $\Phi_{6,9}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{6,9}$ in Figure 2 by adding any combination (the presence or absence) of the dotted edges. For $m = 5$ or $m \geq 7$, let $\Phi_{m,2m-2}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{m,2m-2}$ or $G_{m,2m-2}$ in Figure 2 by adding any combination (the presence or absence) of the dotted edges. Let $\Phi_{6,10}$ be the set of all nonisomorphic graphs obtainable from the graph $H_{6,10}$ in Figure 1 or the graph $F_{6,10}$ or $G_{6,10}$ in Figure 2 by adding any combination (the presence or absence) of the dotted edges.

**Theorem C** *For integers $n > m \geq 3$, $fr(C_m, C_n) = n+2$ if and only if $(m,n) \in S$. Furthermore, if $(m,n) \in S$, then the set of all nonisomorphic frames of $C_m$ and $C_n$ is given by $\Phi_{m,n}$.*

# 3 The edge framing number of pairs of cycles

For $n > m = 3$, the edge framing number $efr(m,n)$ has been determined by Maharaj [10].

**Proposition A** *For any integer $n > 3$,*

$$efr(C_3, C_n) = \begin{cases} \lceil \frac{n}{2} \rceil + n & \text{if } n \equiv 0\,,\, 2 \text{ or } 3\,(mod\,4) \\ \lceil \frac{n}{2} \rceil + n + 1 & \text{if } n \equiv 1\,(mod\,4) \end{cases}$$

Hence in this section we consider integers $n > m \geq 4$. For such integers, every graph that edge homogeneously embeds $C_n$ and $C_m$ also vertex homogeneously embeds $C_n$ and $C_m$. Hence we have the following corollary of Lemma $A$.

**Corollary 1** *For integers $n > m \geq 4$, if $H$ is a graph that edge homogeneously embeds $C_n$ and $C_m$, then $p(H) \geq n+2$.*

The following lemmas will prove to be useful.

**Lemma 1** *Let $G$ and $H$ be graphs with no induced $C_4$, and let $F$ be an edge frame of $G$ and $H$. If $u$ and $v$ are two vertices of degree 2 in $F$, then $N(u) \neq N(v)$.*

**Proof.** Assume, to the contrary, that $N(u) = N(v)$. We show then that $F - u$ edge homogeneously embeds $G$ and $H$. Let $e \in E(G)$ and let $f \in E(F-u)$. Let $G_e$ be an edge embedding of $G$ in $F$ with $e$ at $f$. If $u \notin V(G_e)$, then $G_e$ is in $F-u$. If $u \in V(G_e)$, then, since $C_4 \not\prec G$, $v \notin V(G_e)$ and therefore $\langle (V(G_e) - \{u\}) \cup \{v\} \rangle$ is an edge embedding of $G$ in $F-u$ with $e$ at $f$. Hence $F-u$ edge homogeneously embeds $G$. Similarly, $F-u$ edge homogeneously embeds $H$. This, however, contradicts the fact that $F$ is an edge frame of $G$ and $H$. □

**Lemma 2** *For integers $n > m \geq 4$, if $H$ is a graph that edge homogeneously embeds $C_n$ and $C_m$, then $H$ contains at least three vertices of degree at least 3.*

**Proof.** Let $C' : v_0, v_1, \ldots, v_{m-1}, v_0$ be an induced $C_m$ in $H$, and let $C''$ be an induced $C_n$ in $H$ which contains the edge $v_0v_1$. Further, let $v_i, v_{i+1}, \ldots, v_0, v_1, \ldots, v_{j-1}, v_j$ $(j < i)$ where addition is taken modulo $m$, be a longest path common to $C'$ and $C''$ that contains the edge $v_0v_1$. Since $v_{i-1}$ and $v_{j+1}$ do not belong to $C''$, it follows that each of $v_i$ and $v_j$ has degree at least 3. We deduce, therefore, that every induced $C_m$ and $C_n$ contains at least two vertices of degree at least 3.

Suppose that $H$ has exactly two vertices, $a$ and $b$ say, of degree at least 3. Since every induced $C_m$ and $C_n$ contains at least two vertices of degree at least 3, the vertices $a$ and $b$ must lie on every induced $C_m$ and $C_n$ in $H$. Consequently, the graph $H$ consists of the vertices $a$ and $b$ and a set $S$ of internally disjoint paths joining $a$ and $b$. Observe that any induced cycle containing an edge of a path from $S$ must contain all the edges of this path. Hence we may denote an induced $C_m$ or $C_n$ containing a path $P \in S$ by $C_m(P)$ or $C_n(P)$, respectively. Let $P'$ be a shortest $a$-$b$ path, and let $P^{(1)}$ denote the $a$-$b$ path of length $n - d(a,b)$ on $C_n(P')$ which is disjoint from $P'$. Furthermore, let $P^{(2)}$ denote the $a$-$b$ path of length $m - (n - d(a,b))$ on $C_m(P^{(1)})$ which is disjoint from $P^{(1)}$. Then $P^{(2)}$ is an $a$-$b$ path of length less than $d(a,b)$, which is impossible. The desired result now follows. □

**Proposition 1** *For $m \geq 5$, $efr(C_m, C_{2m-4}) = 2m$. Furthermore, $C_m$ and $C_{2m-4}$ are uniquely edge framed by the graph shown in Figure 3.*

**Proof.** Since $C_m$ and $C_{2m-4}$ can be edge homogeneously embedded in the graph of size $2m$ shown in Figure 3, it follows that $efr(C_m, C_{2m-4}) \leq 2m$. Now let $F$ be an edge frame for $C_{2m-4}$ and $C_m$. By Corollary 1, $p(F) \geq 2m-2$. Applying Theorem B, we have $\delta(F) \geq 2$. Let $k$ be the number of vertices of $H$ of degree at least 3. By Lemma 2, $k \geq 3$. Hence $2(2m) \geq 2q(F) \geq 3k + 2(p(F) - k) = 2p(F) + k \geq 2p(F) + 3$ whence $p(F) \leq 2m - 2$. Thus $p(F) = 2m - 2 = fr(C_m, C_{2m-4})$. By Theorem C, the only graph of order $2m-2$ which both frames $C_m$ and $C_{2m-4}$ and edge homogeneously embeds $C_m$ and $C_{2m-4}$ is the graph shown in shown in Figure 3. Consequently, $efr(C_m, C_{2m-4}) = 2m$, and $C_m$ and $C_{2m-4}$ are uniquely edge framed by the graph shown in Figure 3. □

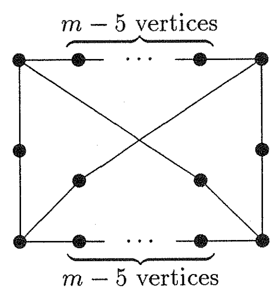


Figure 3: An edge frame for $C_m$ and $C_{2m-4}$ for $m \geq 5$.

**Lemma 3** *Let $n > m \geq 4$ where $n \neq 2m - 4$ and $(m,n) \neq (5,7)$. If a graph $H$ edge homogeneously embeds $C_m$ and $C_n$, then $p(H) \geq n + 3$.*

**Proof.** Let $H$ be a graph which edge homogeneously embeds $C_m$ and $C_n$. By Corollary 1, $p(H) \geq n+2$. Suppose that $p(H) = n+2$. Then by Lemma A we deduce that $H$ frames $C_m$ and $C_n$. By Theorem C it follows that $(m,n) \in S$, where $S$ is the set of ordered pairs defined in Section 2. For $(m,n) \in S$ the frames for $C_m$ and $C_n$ have been completely determined in Theorem C and in each case it is easily checked that $H$ does not edge homogeneously embed $C_m$ and $C_n$ unless $n = 2m-4$ (in which case $H$ is the graph shown in Figure 3) or $n = 2m-3$ and $m = 5$ (in which case $H$ is the graph shown in Figure 4). This produces a contradiction and we deduce that $p(H) \geq n+3$. □

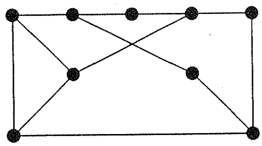

Figure 4: An edge frame for $C_5$ and $C_7$.

**Proposition 2** *For $m \geq 7$, $efr(C_m, C_{2m-6}) = 2m-1$.*

**Proof.** Since $C_m$ and $C_{2m-6}$ can be edge homogeneously embedded in the graph of size $2m-1$ shown in Figure 5, it follows that $efr(C_m, C_{2m-6}) \leq 2m-1$. We show that $efr(C_m, C_{2m-6}) = 2m-1$ by verifying that there is no graph of size $2m-2$ or less which edge homogeneously embeds $C_m$ and $C_{2m-6}$. Suppose, to the contrary, that such a graph $H$ exists. By Lemma 3, $p(H) \geq 2m-3$. Applying Theorem B, we have $\delta(H) \geq 2$. Let $k$ be the number of vertices of $H$ of degree at least 3. By Lemma 2, $k \geq 3$. Hence $4m-4 \geq 2q(H) \geq 3k + 2(p(H)-k) = 2p(H) + k \geq 2(2m-3)+3 = 4m-3$, which is impossible. □

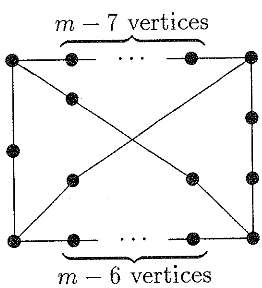


Figure 5: An edge frame for $C_m$ and $C_{2m-6}$ for $m \geq 7$.

**Lemma 4** *For $n > m \geq 4$ where $n \neq 2m-4$ or $2m-6$, there is no graph of order $n+3$ and size at most $n+5$ that edge homogeneously embeds $C_m$ and $C_n$.*

**Proof.** Assume, to the contrary, that such a graph $H$ exists. Applying Theorem B, we have $\delta(H) \geq 2$. Let $k$ be the number of vertices of $H$ of degree at least 3. Hence $2n + 10 \geq 2q(H) \geq 3k + 2(p(H) - k) = 2p(H) + k = 2n + 6 + k$, so $k \leq 4$. By Lemma 2, $k \geq 3$. Thus $k = 3$ or 4.

**Case 1.** $k = 3$.

Since every graph contains an even number of vertices of odd degree, at least one vertex of $H$ has degree 4 or more. Thus $2n + 10 \geq 2q(H) \geq 10 + 2(p(H) - 3) = 2p(H) + 4 = 2n + 10$. Since all these inequalities must be equalities, it follows that $q(H) = n + 5$ and $H$ contains two vertices of degree 3, one of degree 4, and $n$ of degree 2. Let $w$ denote the vertex of degree 4. Since no vertex of degree 2 in $H$ can lie on a $K_3$, and since $q(H) = n + 5$ and $\delta(H) = 2$, it follows that every induced $C_n$ in $H$ must contain the vertex $w$. Let $C_w : w = w_1, w_2, \ldots, w_n, w_1$ be an induced $C_n$ containing $w$, and let $a$, $b$, and $c$ be the names of the three vertices of $H$ not in $C_w$. Without loss of generality, we may assume that $w$ is adjacent to $a$ and $b$. Since $q(H) = n + 5$ and $\delta(H) = 2$, at most one of $a$ and $b$ is adjacent to a vertex of $C_w$ different from $w$. Without loss of generality, we may assume that $b$ is adjacent to no vertex of $C_w$ other than $w$. Since no vertex of degree 2 in $H$ can lie on a $K_3$, and since $q(H) = n + 5$, the vertices $a$ and $b$ cannot be adjacent. Hence $b$ is adjacent only to $c$ and $w$.

Suppose firstly that $a$ is adjacent to $c$. If $deg\, c = 2$, then $c$ belongs to no induced $C_\ell$ for $\ell \geq 5$. Hence $deg\, c = 3$. Then $a$ and $b$ are vertices of degree 2 with $N(a) = N(b)$. Thus we must have $m = 4$ otherwise by Lemma 1 we have a contradiction. Now $c$ is adjacent with $w_j$ for some $j\,(2 \leq j \leq n)$. Thus $H$ is the graph shown in Figure 6.

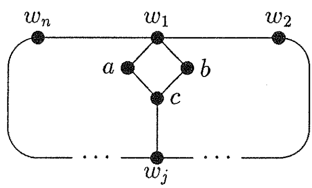


Figure 6: The graph $H$.

Then $deg\, w_j = deg\, c = 3$, $deg\, w_1 = 4$, and the remaining vertices of $H$ have degree 2. Thus any induce $C_4$ containing the edge $w_1w_2$ must contain the vertices $w_1$, $w_j$, $c$ and either $a$ or $b$. Consequently $j = 2$. Similarly, by considering the edge $w_1w_n$ we get $j = n$. Thus $n = 2$, a contradiction. Thus $a$ and $c$ are not adjacent. Since $q(H) = n + 5$, $deg\, a = deg\, c = 2$. Since no vertex of degree 2 belongs to a $K_3$, the vertex $a$ is not adjacent to $w_2$ or $w_n$. Furthermore, the vertex $c$ is not adjacent to $w_2$ or $w_n$, for otherwise $c$ belongs to no induced $C_n$ for $n \geq 5$. Without loss of generality, we may assume that $a$ is adjacent to $w_r$ and $c$ is adjacent to $w_s$ where $3 \leq s < r \leq n - 1$. The graph $H$ is shown in Figure 7.

Since the vertex $b$ belongs to no $C_4$, we must have $m \geq 5$. If $r = n - 1$, then $a$ and $w_n$ are vertices of degree 2 with $N(a) = N(w_n)$ which contradicts Lemma 1. Hence $r \leq n - 2$. We now consider the vertex $a$. The vertex $a$ belongs to three cycles,

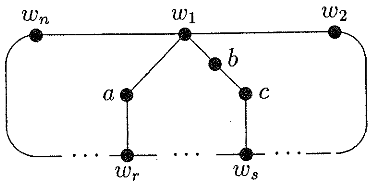


Figure 7: The graph $H$.

namely, $C^{(1)}: a, w_r, w_{r+1}, \ldots, w_n, w_1, a$ (of length $n-r+3$), $C^{(2)}: a, w_1, w_2, \ldots, w_r, a$ (of length $r+1$) and $C^{(3)}: a, w_1, b, c, w_s, w_{s+1}, \ldots, w_r, a$ (of length $r-s+5$). At least one of these cycles is of length $n$. If $C^{(1)}$ has length $n$, then $r = s = 3$ contradicting $r > s$. If $C^{(2)}$ has length $n$, then $r = n-1$ contradicting $r \leq n-2$. Therefore $C^{(3)}$ must be of length $n$, implying that $n - 2 \geq r = n + s - 5$, so $s \leq 3$. Thus $s = 3$ and $r = n - 2$. But then the vertex $w_n$ belongs to three cycles of lengths 5, $n$ and $n+1$. Hence $m = 5$. However the edge $w_3w_4$ then belongs to no $C_5$, a contradiction. Hence Case 1 produces a contradiction.

**Case 2.** $k = 4$.
Then $2n + 10 \geq 2q(H) \geq 2n + 6 + k = 2n + 10$. Since all these inequalities must be equalities, it follows that $q(H) = n + 5$ and $H$ contains four vertices of degree 3 and $n - 1$ vertices of degree 2. The following claim will prove to be useful.

**Claim 1** *If $C'$ is an induced $C_n$ in $H$ and $U$ the set of three vertices of $H$ that do not belong to $C'$, then $\langle U \rangle \cong K_1 \cup K_2$ or $P_3$. Furthermore, if $\langle U \rangle \cong K_1 \cup K_2$, then each vertex of $U$ has degree 2 in $H$. If $\langle U \rangle \cong P_3$, then the central vertex of this $P_3$ has degree 3 in $H$ and the two end-vertices have degree 2 in $H$.*

**Proof.** Since $q(H) = n + 5$, there are exactly five edges incident with the vertices of $U$. Since $\delta(H) = 2$, and no vertex of degree 2 belongs to a $K_3$, a simple counting argument shows that $q(\langle U \rangle) = 1$ or 2. Hence $\langle U \rangle \cong K_1 \cup K_2$ or $P_3$. If $\langle U \rangle \cong K_1 \cup K_2$, then, since $q(H) = n + 5$, each vertex of $U$ has degree 2 in $H$. If $\langle U \rangle \cong P_3$, then three of the five edges incident with vertices of $U$ are also incident with vertices of $C'$. It follows that exactly three of the four vertices of degree 3 belong to $C'$ and the remaining vertex of degree 3 is in $U$. Hence one vertex of $U$ has degree 3 and the remaining two vertices have degree 2. Suppose $\langle U \rangle$ is the path $a, b, c$, and $C'$ is the (induced) cycle $v_1, v_2, \ldots, v_n, v_1$. We show that $deg\, b = 3$. If this is not the case, then we may assume that $deg\, a = 3$ and $deg\, b = deg\, c = 2$. Without loss of generality, we may assume $av_1$, $av_i$ and $cv_j$ are edges of $H$ where $2 \leq i < j \leq n$. The graph $H$ is shown in Figure 8.

Since the vertex $b$ belongs to no 4-cycle, we may assume here that $n > m \geq 5$. Now there are only two induced cycles containing the edge $v_1v_2$, namely $C'$ and the cycle $C'': v_1, v_2, \ldots, v_i, a, v_1$. Since $C'$ has length $n$, $C''$ must have length $m$ so that $i = m - 1$. We now consider the edge $av_1$. The edge $av_1$ belongs to three induced cycles, namely, $C''$ (of length $m$), $v_1, a, v_{m-1}, v_m, v_{m+1}, \ldots, v_n, v_1$ (of length $n-m+4$)

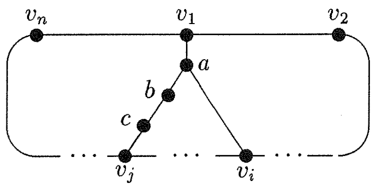

Figure 8: The graph $H$.

and $C''' : v_1, a, b, c, v_j, \ldots, v_n, v_1$ (of length $n - j + 5$). Thus $n = n - m + 4$ or $n = n - j + 5$. If $n = n - m + 4$, then $m = 4$ contradicting $m \geq 5$. Thus $C'''$ has length $n$ and $j = 5$. Hence $m - 1 = i \leq j - 1 = 4$, so $m \leq 5$, i.e., $m = 5$. But then the edge $av_4$ belongs to no $C_n$, a contradiction. We deduce, therefore, that $deg\, b = 3$ and $deg\, a = deg\, c = 2$. This completes the proof of the claim. □

We now return to the proof of Case 2. Let $u$ and $v$ be two (distinct) vertices of degree 3 for which $d(u, v)$ is a *minimum*, and let $P$ be a shortest $u$-$v$ path. Then all interior vertices (if any) of $P$ have degree 2. Let $C_P : v_1, v_2, \ldots, v_n, v_1$ be an induced $C_n$ containing an edge of $P$. Necessarily, $C_P$ contains all edges of $P$. Let $a, b, c$ be the three vertices of $H$ that do not belong to $C_P$. By Claim 1, $\langle\{a, b, c\}\rangle \cong K_1 \cup K_2$ or $P_3$. We consider the two possibilities in turn.

**Case 2.1** $\langle\{a, b, c\}\rangle \cong P_3$.

Without loss of generality, we may assume that $a, b, c$ is a path. By Claim 1, $deg\, b = 3$ and $deg\, a = deg\, c = 2$. Since $b$ is adjacent to a vertex of degree 3 of $C_P$, our choice of $u$ and $v$ implies that $d(u, v) = 1$, so $u$ and $v$ are adjacent vertices on $C_P$. Without loss of generality, we may assume that $u = v_1$ and $v = v_2$. If $b$ is adjacent to either $u$ or $v$, then, without loss of generality, $H$ is then the graph shown in Figure 9($i$). Since the vertex $a$ belongs to induced cycles of only two possible lengths, namely, 4 and $n$, we must have $m = 4$. But then the edge $v_1v_n$ belongs to no $C_m$, a contradiction. Hence $b$ is adjacent to neither $u$ nor $v$, so $bv_i$ is an edge for some $i$ $(3 \leq i \leq n)$.

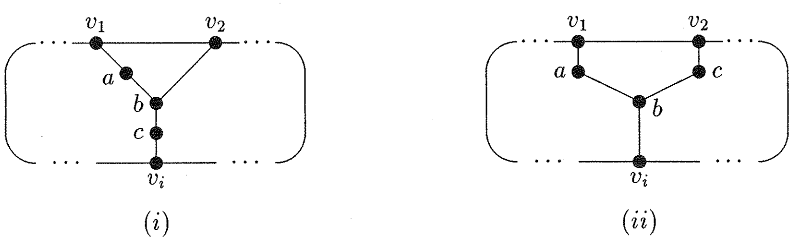

Figure 9: The graph $H$.

Without loss of generality, $H$ is then the graph shown in Figure 9($ii$). Since the edge $v_1v_2$ belongs to no 4-cycle, we must have $m \geq 5$. The edge $bc$ belongs to three cycles, namely $b, c, v_2, v_1, a, b$ (of length 5), $b, c, v_2, v_3, \ldots, v_i, b$ (of length $i + 1$)

and $c, b, v_i, v_{i+1}, \ldots, v_n, v_1, v_2, c$ (of length $n-i+5$). Since $n > 5$, we must have $n = i+1$ or $n-i+5$. Suppose $n = n-i+5$. Then $i = 5$ and the edge $v_1v_n$ lies on cycles of only two possible lengths, namely, $n-1$ and $n$. Hence $m = n-1$. Now the edge $v_1v_2$ ($v_2v_3$) lies on cycles of length $5, 7$ and $n$ ($6, 7$ and $n$, respectively). We deduce that $m = 7$ and $n = 8$. However, then, $n = 2m-6$ which is contrary to our choice of $m$ and $n$. Thus $n = i+1$, i.e., $i = n-1$. The edge $v_2v_3$ then lies only on cycles of length $n$ and $n+1$ so that $v_2v_3$ does not lie on any cycle of length $m$. This produces a contradiction.

**Case 2.2** $\langle\{a, b, c\}\rangle \cong K_1 \cup K_2$.
Without loss of generality, we may assume that $a$ is the isolated vertex in $\langle U\rangle$, so $bc$ is an edge. By Lemma 1, each of $a, b$ and $c$ has degree 2. Let $C_a$ be an induced $C_n$ containing the vertex $a$. We show that the edge $bc$ belongs to $C_a$. If this is not the case, then, without loss of generality, we may assume that $C_a$ is $a, v_2, v_3, \ldots, v_n, a$. By Claim 1, the three vertices $v_1, b$ and $c$ that do not belong to $C_a$ induce either a $P_3$ or $K_1 \cup K_2$. If $\langle\{v_1, b, c\}\rangle \cong P_3$, then, since $\Delta(H) = 3$, the vertex $v_1$ must be an end-vertex of $\langle\{v_1, b, c\}\rangle \cong P_3$. But then $v_1$ has degree 3 in $H$ which contradicts Claim 1. Thus $\langle\{v_1, b, c\}\rangle \cong K_1 \cup K_2$ and $v_1$ has degree 2 in $H$. Hence $a$ and $v_1$ are two nonadjacent vertices of degree 2 in $H$ with $N(a) = N(v_1)$. This, however, contradicts Lemma 1 if $m \geq 5$. Hence $m = 4$. Without loss of generality, we may assume that the vertex $b$ ($c$) is adjacent with the vertex $v_i$ ($v_j$, respectively) where $3 \leq i < j \leq n-1$. Since the edge $bc$ must lie on an induced $C_4$, it follows that $j = i+1$. However the edge $bc$ then belongs to no cycle of length 5 or more. This produces a contradiction. We deduce, therefore, that the edge $bc$ must belong to $C_a$.

Let $S$ be the set of three vertices of $C_P$ that do not belong to $C_a$. By Claim 1, $\langle S\rangle \cong K_1 \cup K_2$ or $P_3$. Clearly, $\langle S\rangle \cong K_1 \cup K_2$. Without loss of generality, we may assume that $S = \{v_2, v_i, v_{i+1}\}$ where $5 \leq i \leq n-2$. Hence $n \geq 7$, and $v_1, v_3, v_{i-1}$ and $v_{i+2}$ are the four vertices of degree 3 in $H$. If $N(a) = N(v_2)$, then, since the edge $bc$ belongs to cycles only of length 6 and $n$, it follows that $m = 6$. However, the vertex $a$ belongs to cycles only of length 4 and $n$, so $m = 4$, a contradiction. Hence $N(a) \neq N(v_2)$.

If $C_a$ is given by $v_1, b, c, v_3, v_4 \ldots, v_{i-1}, a, v_{i+2}, \ldots, v_n, v_1$, then $H$ is the graph shown in Figure 10($i$). Now the edge $v_1v_n$ belongs to cycles of length $n-1, n, n+1$. Thus $m = n-1$. However, the edge $bc$ belongs to no induced $C_{n-1}$ ($n \geq 7$). Hence we may assume, without loss of generality, that $C_a$ is given by either $C_a^{(1)}$: $v_1, a, v_{i-1}, v_{i-2}, \ldots, v_3, b, c, v_{i+2}, \ldots, v_n, v_1$, in which case $H$ is the graph shown in Figure 10($ii$), or $C_a^{(1)}$: $v_1, b, c, v_{i-1}, v_{i-2}, \ldots, v_3, a, v_{i+2}, \ldots, v_n, v_1$, in which case $H$ is the graph shown in Figure 10($iii$). If $C_a$ is $C_a^{(1)}$, then the edge $v_1v_n$ belongs to cycles of length $n-i+4$ and $n$. Thus $m = n-i+4$. Furthermore, the edge $v_3v_4$ belongs to cycles of length $i, i+2$ and $n$. Thus $m = i$ or $i+2$. If $m = i$, then $n = 2m-4$ and if $m = i+2$, then $n = 2m-6$. In either case we contradict our choice of $m$ and $n$. A similar argument shows that $C_a$ cannot be $C_a^{(2)}$. This completes the proof of Case 2.2, and therefore of Lemma 4. □

**Corollary 2** *For $n > m \geq 4$ with $n \neq 2m-4$ or $2m-6$ and $(m, n) \neq (5, 7)$,*

$$efr(C_m, C_n) \geq n+6.$$

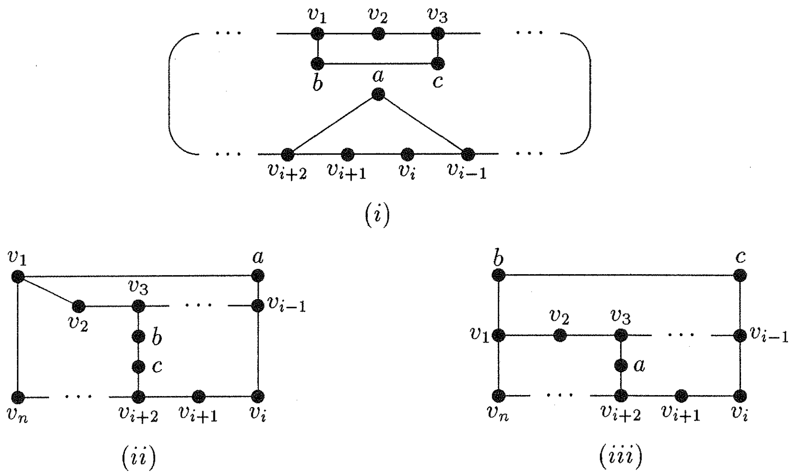


Figure 10:

**Proof.** We show that $efr(C_m, C_n) \geq n + 6$ by verifying that there is no graph of size $n + 5$ or less which edge homogeneously embeds $C_m$ and $C_n$. Suppose, to the contrary, that such a graph $H$ exists. By Lemma 3, $p(H) \geq n + 3$, and by Lemma 4, $p(H) \neq n + 3$; consequently, $p(H) \geq n + 4$. Applying Theorem B, we have $\delta(H) \geq 2$. Let $k$ be the number of vertices of $H$ of degree at least 3. By Lemma 2, $k \geq 3$. Hence $2n + 10 \geq 2q(H) \geq 3k + 2(p(H) - k) = 2p(H) + k \geq 2n + 11$, which is impossible. □

**Corollary 3** *For $m \geq 7$, $C_m$ and $C_{2m-6}$ are uniquely edge framed by the the graph of size $2m - 1$ shown in Figure 5.*

**Proof.** Let $F$ be an edge frame for $C_m$ and $C_{2m-6}$. Then by Proposition 2, $q(F) = 2m - 1$. By Corollary 1, $p(F) \geq 2m - 4$. Applying Theorem B, we have $\delta(F) \geq 2$. Let $k$ be the number of vertices of $F$ of degree at least 3. By Lemma 2, $k \geq 3$. Hence $4m - 2 = 2q(F) \geq 3k + 2(p(F) - k) = 2p(F) + k \geq 2p(F) + 3$, whence $p(F) \leq 2m - 3$. Thus $2m - 4 \leq p(F) \leq 2m - 3$. If $p(F) = 2m - 4$, then $p(F) = fr(C_m, C_{2m-6})$ and so $F$ frames $C_m$ and $C_{2m-6}$. However, by Theorem C, there is no graph of order $2m - 4$ which edge homogeneously embeds $C_m$ and $C_{2m-6}$ for $m \geq 7$. Thus $p(F) = 2m - 3 = (2m - 6) + 3$. From the proof of Lemma 4 we deduce that $C_m$ and $C_{2m-6}$ have at most one edge frame. We conclude that $C_m$ and $C_{2m-6}$ are uniquely edge framed. □

**Proposition 3** *For $m \geq 4$ and $m \notin \{5, 7\}$, $efr(C_m, C_{m+1}) = m + 7$.*

**Proof.** Since $C_m$ and $C_{m+1}$ can be edge homogeneously embedded in the graph of size $m + 7$ shown in Figure 11$(i)$ for $m = 4$ and in Figure 11$(ii)$ for $m = 6$ or $m \geq 8$, it follows that $efr(C_m, C_{m+1}) \leq m + 7$. By Corollary 2, $efr(C_m, C_{m+1}) \geq m + 7$. Consequently $efr(C_m, C_{m+1}) = m + 7$ as required. □


(i) (ii)

Figure 11:

**Proposition 4** *For* $m \geq 10$, $efr(C_m, C_{2m-8}) = 2m - 2$.

**Proof.** Since $C_{2m-8}$ and $C_m$ can be edge homogeneously embedded in the graph of size $2m-2$ shown in Figure 12, it follows that $efr(C_{2m-8}, C_m) \leq 2m-2$. By Corollary 2, $efr(C_{2m-8}, C_m) \geq 2m-2$. Consequently $efr(C_{2m-8}, C_m) = 2m-2$ as required. □

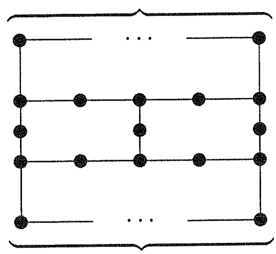


Figure 12: An edge frame for $C_m$ and $C_{2m-8}$ for $m \geq 10$.

**Proposition 5** $efr(C_5, C_7) = 12$.

**Proof.** Since $C_5$ and $C_7$ can be edge homogeneously embedded in the graph $F_{5,7}$ (without the dotted edges) of size 12 shown in Figure 2, it follows that $efr(C_5, C_7) \leq 12$. We show that $efr(C_5, C_7) = 12$ by verifying that there is no graph of size at most 11 which edge homogeneously embeds $C_5$ and $C_7$. Suppose, to the contrary, that such a graph $H$ exists. By Corollary 1, $p(H) \geq 9$. Applying Theorem B, we have $\delta(H) \geq 2$. Let $k$ be the number of vertices of $H$ of degree at least 3. By Lemma 2, $k \geq 3$. Hence $22 \geq 2q(H) \geq 3k + 2(p(H) - k) = 2p(H) + k \geq 2p(H) + 3$ whence $p(H) \leq 9$. Consequently, $p(H) = 9 = fr(C_5, C_7)$ and so $H$ frames $C_5$ and $C_7$. However, from Theorem C, the frames for $C_5$ and $C_7$ all have sizes greater than 11. This produces a contradiction. □

**Proposition 6** *For* $m = 4$ *or* $m \geq 7$, $efr(C_m, C_{m+2}) = m + 8$.

**Proof.** Since $C_m$ and $C_{m+2}$ can be edge homogeneously embedded in the graph of size $m+8$ shown in Figure 13(*i*) for $m = 4$ and in Figure 13(*ii*) for $m \geq 7$, it follows that $efr(C_m, C_{m+2}) \leq m+8$. By Corollary 2, $efr(C_m, C_{m+2}) \geq m+8$. Consequently $efr(C_m, C_{m+2}) = m + 8$ as required. □

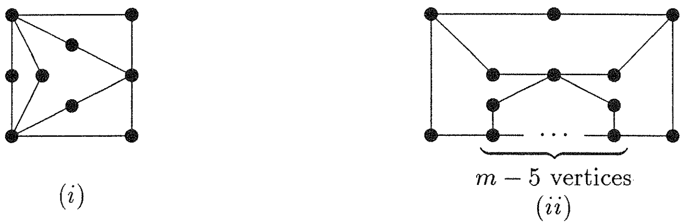


Figure 13:

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