The chromaticity of a generalized wheel graph*

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Abstract

In this paper we determine the graphs chromatically equivalent to the generalized wheel $C_5 + K_n$.

1 Introduction

All graphs considered here are simple, undirected and finite. For a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set; $|V(G)|$ is the *order* and $\chi(G)$ is the *chromatic number* of G. For a graph G and a vertex x of G, we denote by $d_G(x)$ the degree of x and by $N_G(x)$ the set of its neighbors. If a graph G_1 is isomorphic to another graph G_2, we write $G_1 \cong G_2$. $G_1 + G_2$ represents the join of two disjoint graphs G_1 and G_2, i.e., the graph obtained from $G_1 \cup G_2$ by adding an edge between each vertex of G_1 and each vertex of G_2.

A graph G is said to be *k-chromatic* if its chromatic number is k; it is *k-critical* if it is k-chromatic and $\chi(H) < \chi(G)$, for any proper subgraph H of G. Let $P(G, \lambda)$ be the chromatic polynomial of G. Two graphs G and H are called *chromatically equivalent* (or, for short, χ-equivalent) and we write $G \sim H$ if $P(G, \lambda) = P(H, \lambda)$ as polynomials in λ. A graph G is said to be *chromatically unique* (or χ-unique) if, from $H \sim G$, it follows that $H \cong G$. For an introduction to chromatic polynomials and for all notation and terms not explained here, we suggest the excellent papers [8], [9], [10].

If n, k are positive integers, $[n]$ is the set $\{1, \ldots, n\}$ and $(n)_k$ is the falling factorial $n(n-1)\ldots(n-k+1)$.

Let $t \geq 3, n \geq 1$ be two integers. We denote by W_t^n the graph $C_t + K_n$. Note that for $n = 1$, W_t^1 is just the wheel W_{t+1} and that is why W_t^n is called a *generalized wheel.* Dong [6] proved that, for $n \geq 1$ and even $t \geq 4$, W_t^n is χ-unique. In our paper, we will consider the case $t = 5$.

We will make use of the notion of *critical graph* to establish our result. This approach for studying chromaticity of graphs was initiated by Koh and Goh [7]. We

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first list some results related to critical graphs we will need. For their proofs and an introduction to the theory of critical graphs, we refer to the papers [3], [4], [5] and to the related chapters in [1], [2].

Proposition 1 *Any k-chromatic graph contains a k-critical graph.*

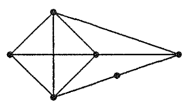
Proposition 2 *There exists no k-critical graph of order* $k+1$.

The following result is also known (see [11]) and can be proved without difficulty.

Proposition 3 $C_5 + K_{k-3}$ *is the only k-critical graph of order* $k+2$.

2 Main Result

It is known that W_5^1 is not χ-unique, because it is χ-equivalent to the graph R below, both having the chromatic polynomial $(\lambda)_4(\lambda^2 - 4\lambda + 5)$.



R

It follows that $W_5^n \sim R + K_{n-1}$, so none of the graphs W_5^n is χ-unique for $n \geq 2$. We will prove that the chromatic equivalence class of W_5^n contains only one graph non-isomorphic to W_5^n.

Lemma 1 *Let n, p, q be positive integers such that $p + q = 2n + 2$. Then*

$$\binom{p}{2} + \binom{q}{2} \geq 2\binom{n}{2} + 2n.$$

Proof: It is not difficult to check that the above inequality is equivalent to $(p-q)^2 \geq 0$. □

Theorem 1 *Let $n \geq 1$ and G be a graph of order $n+5$ such that $G \sim W_5^n$ and $G \ncong W_5^n$. Then $G \cong R + K_{n-1}$.*

Proof: Note that G has $\binom{n}{2} + 5n + 5$ edges, $\binom{n}{3} + 5\binom{n}{2} + 5n$ triangles and $\chi(G) = n+3$. Let H be a $(n+3)$-critical subgraph of G. If the order of H is $n+5$, then, by proposition 3, we get that $H \cong W_5^n$ and hence $G \cong W_5^n$, which is not allowed by the hypothesis. By proposition 2, we deduce that H must be K_{n+3}. Denote by x and y the vertices of G that do not belong to H. It follows that G contains $2n+2$ edges incident with x or y and the number t of the triangles that have at least one vertex in $\{x, y\}$ is $t = 2\binom{n}{2} + 2n - 1$. Let $p = |N_G(x) \cap V(H)|$ and $q = |N_G(y) \cap V(H)|$ and assume $p \geq q$.

We shall first prove that $xy \in E(G)$. Suppose $xy \notin E(G)$; we then have $t = \binom{p}{2} + \binom{q}{2}$. On the other hand, in this case, $p + q$ represents the number of edges incident with x or y and hence $p + q = 2n + 2$. By lemma 1, we deduce

$$t > 2\binom{n}{2} + 2n - 1,$$

which is absurd. So $xy \in E(G)$ and we get that $p + q = 2n + 1$. Note now that $p \leq n + 2$; otherwise, G would contain a complete graph of order $n + 4$. As $p \geq q$, we have $p \geq n + 1$. It follows that there are two cases to be taken into account:

1. $p = n+2$ and $q = n-1$. In this case, H contains at least $n-2$ vertices adjacent to both x and y. Thus:

$$t \geq \binom{p}{2} + \binom{q}{2} + n - 2 = 2\binom{n}{2} + 2n$$

and we have derived a contradiction.

2. $p = n + 1$ and $q = n$. Let r be the number of vertices in H adjacent to both x and y. Then

$$t = \binom{n+1}{2} + \binom{n}{2} + r$$

which implies that $r = n-1$. It follows that H contains $n-1$ vertices adjacent to both x and y, 2 vertices (say z_1, z_2) adjacent only to x, 1 vertex (say z_3) adjacent only to y and 1 vertex (say z_4) adjacent neither to x, nor to y. This implies that the subgraph induced by $\{x, y, z_1, z_2, z_3, z_4\}$ in G contains R and hence G contains a subgraph isomorphic to $R + K_{n-1}$. As G contains exactly $\binom{n}{2} + 5n + 5$ edges, it follows that $G \cong R + K_{n-1}$. □

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