Odd induced subgraphs in graphs of maximum degree three

David M. Berman, Hong Wang, and Larry Wargo

Department of Mathematics
University of New Orleans
New Orleans, Louisiana, USA 70148

Abstract

A long-standing conjecture asserts the existence of a positive constant c such that every simple graph of order n without isolated vertices contains an induced subgraph of order at least cn such that all degrees in this induced subgraph are odd. Radcliffe and Scott have proved the conjecture for trees, essentially with the constant $c = 2/3$. Scott proved a bound for c depending on the chromatic number. For general graphs it is known only that c, if it exists, is at most $2/7$.

In this paper, we prove that for graphs of maximum degree three, the theorem is true with $c = 2/5$, and that this bound is best possible.

Gallai proved that in any graph there is a partition of the vertices into two sets so that the subgraph induced by each set has each vertex of even degree; also there is a partition so that one induced subgraph has all degrees even and the other has degrees odd. (See [3] problem 17.) Clearly we can not assure a partition in which each subgraph has all degrees odd. The weaker question then arises whether every simple graph contains a "large" induced subgraph with all degrees odd.

We say that an *odd subgraph* of G is an induced subgraph H such that every vertex of H has odd degree in H. We use $f(G)$ to denote the maximum order of an odd subgraph of G. (To avoid trivial cases, we will restrict G to be without isolated vertices.) We may thus state the conjecture in the form that there exists a positive constant c such that for an n-vertex graph G, $f(G) \geq cn$. (This conjecture is cited by Caro [2] as "part of the graph theory folklore".)

Caro [2] proved a weaker conjecture of Alon that for an n-vertex graph G, $f(G) \geq c\sqrt{n}$. Scott [5] improved this, proving that $f(G) \geq cn/log(n)$. Radcliffe and Scott [4] have proved the original conjecture for trees, essentially with the constant $c = 2/3$. In general it is known [2] only that c, if it exists, is at most $2/7$. In [5] Scott proves a bound for c based on the chromatic number of G. It follows immediately from this bound that for a graph of maximum degree three $f(G) \geq n/3$.

Australasian Journal of Combinatorics 15(1997), pp.81–85

In this paper, we prove the best possible bound for graphs of maximum degree three.

THEOREM. Every simple graph G of order n without isolated vertices and with maximum degree at most three has an induced subgraph H of order at least $2n/5$ in which all vertices are of odd degree in H.

Since an odd subgraph must have an even number of vertices, for general n we could write $f(g) \geq 2\lceil n/5 \rceil$. This bound is then sharp for any cycle of length up to nine. For a larger value of n we may get a graph achieving this bound by taking the disjoint union of such cycles. We do not have examples with connected graphs, and make the following strengthening of the original conjecture:

CONJECTURE. Every connected simple graph G of order n (irrespective of its maximum degree) has an induced subgraph H of order at least $2\lfloor n/4 \rfloor$ in which all vertices are of odd degree in H.

We will refer to an odd subgraph having at least two fifths of the vertices of a graph as a *big* odd subgraph. Let $\langle u, v \rangle$ denote the subgraph induced by the vertices u, v; let the *claw at* v denote the induced subgraph consisting of a vertex v of degree three and its neighbors of degree one. Otherwise, our notation follows [1].

To prove the theorem, suppose it is false, and let G be a counter-example with as few vertices as possible. Clearly G is connected. We will obtain a contradiction by showing that it must be 3-regular. We do this in a sequence of three lemmas.

Lemma 1. G has no vertex of degree one.

Proof of Lemma 1. Suppose instead that G has a vertex p of degree one, and let x be its neighbor. Clearly, if x has degree one we are done.

If x has only one other neighbor, call it y, we consider $G' = G - \{p, x, y\}$. But G' can have at most two isolated vertices (the neighbors of y) so deleting them we get a graph G'' with $|G''| \geq |G| - 5$ and no isolated vertices. By induction, G'' has a big odd subgraph H. Then H together with $\langle p, x \rangle$ gives a big subgraph of G.

If x has two additional neighbors y_1 and y_2 adjacent to each other, then let $G' = G - \{p, x, y_1, y_2\}$. If G' has at most one isolated vertex, then deleting it (if it exists) gives a graph G'' which by induction has a big odd subgraph. This subgraph together with $\langle p, x \rangle$ is a big odd subgraph of G and we are done.

Thus G' has two isolated vertices, say z_1 and z_2 adjacent to y_1 and y_2 respectively. If both are isolated then $\{p, x, y_1, y_2, z_1, z_2\}$ is all of G, as G is connected, and this has all degrees odd, so we are done.

That leaves the case that x has additional neighbors y_1 and y_2 which are not adjacent. Let $G' = G - \{p, x, y_1, y_2\}$. Suppose G' has at most one isolated vertex. Delete the isolated vertex (if there is one) to get G" with no isolated vertices. So by induction, G'' has a big odd subgraph which together with $\langle p, x \rangle$ is a big odd subgraph of G. (Note that the vertices of $\langle p, x \rangle$ are not adjacent to any vertices of G''.) Therefore G' must have at least two isolated vertices, or we are done. Since G is connected, each isolated vertex of G' must be adjacent in G to at least one of y_1 or y_2.

We must consider two cases here:

Case 1. y_1 or y_2 has two neighbors (in G) that are isolated in G'. Say y_1 is adjacent (in G) to z_1 and z_2, isolated vertices of G'.

Now $G - \{p, x, y_1, y_2, z_1, z_2\}$ has at most two isolated vertices. Delete these; then by induction the resulting graph has a big odd subgraph. This odd subgraph together with the claw at y_1 gives a big odd subgraph of G.

Case 2. Each of y_1 and y_2 has one neighbor (in G) that is isolated in G', say z_1 and z_2, respectively.

We may assume that $d(y_i) = 3$ for $i = 1, 2$; otherwise we may use z_i in place of p, and have a vertex of degree one whose neighbor has degree two, a case we already dealt with. Let $G_1 = G - \{p, x, z_1, z_2\}$. This has no isolated vertices so by induction it has a big odd subgraph H. We get a big odd subgraph of G in one of three ways:

i. If neither y_1 nor y_2 is in H, then take H together with $\langle p, x\rangle$.

ii. If both y_1 and y_2 are in H, then take the subgraph induced by the vertices of H together with the vertices $\{p, x, z_1, z_2\}$.

iii. If y_1 but not y_2 is in H, then take H together with $\langle x, z_1\rangle$.

This completes the proof of Lemma 1.

Lemma 2. G has no vertex of degree two whose neighbors are adjacent.

Proof of Lemma 2. Suppose to the contrary that G has a vertex p of degree two with adjacent neighbors x_1 and x_2.

Then, since $\Delta(G) \leq 3$, for each $i \in \{1, 2\}$, x_i has at most one additional neighbor in G, call it y_i (if it exists).

Let $G_1 = G - \{p, x_i, y_i : (i = 1, 2)\}$. If G_1 has no isolated vertices, then by induction it has a big odd subgraph, which together with $\langle p, x_1\rangle$ is a big odd subgraph of G. Thus G_1 has at least one isolated vertex. Each isolated vertex of G_1 must be adjacent to both y_1 and y_2 (which must therefore be distinct) as by Lemma 1 G has no vertex of degree one. Let $G_2 = G - \{p, x_1, x_2, y_1\}$. G_2 has no isolated vertex, so by induction, it has a big odd subgraph, which together with $\langle p, x\rangle$ is a big odd subgraph of G, completing the proof of Lemma 2.

Lemma 3. G has no vertex of degree two.

Proof of Lemma 3. Suppose to the contrary that G has a vertex p with non-adjacent neighbors x_1 and x_2.

Since G has minimum degree two, let y_1 and y_2 (not necessarily distinct) be the other neighbors (in G) of x_1. Let $G' = G - \{p, x_i, y_i : (i = 1, 2)\}$. If G' has no isolated vertex, then by induction it has a big odd subgraph, which together with $\langle p, x_1\rangle$ is a big odd subgraph of G, a contradiction. Thus G' must have at least one isolated vertex.

Note that G' has at most three isolated vertices, as each must be adjacent, in G, to at least two of the vertices x_2,y_1,y_2. But each of these can have at most two edges to vertices other than p and x_1, thus allowing no more than three isolated vertices in G'.

In fact we claim that G' must have exactly one isolated vertex. If there are as many as two, then by the pigeon-hole principle one of x_2, y_1, y_2 must be adjacent to two of them. Say y_1 is adjacent to z_1 and z_2, where z_1 and z_2 are isolated vertices in G'. (The proof proceeds similarly if y_1 is replaced by y_2 or x_2.)

So let $G_2 = G' - \{z_1, z_2, z_3\}$ where z_3 is an isolated vertex in G' (possibly the same as z_1 or z_2). Then G_2 has no isolated vertices, so by induction it has a big odd subgraph, which together with the claw at y_1 is a big odd subgraph of G, a contradiction. At this point we have in G the vertex p, its neighbors x_1 and x_2, and vertices y_1 and y_2 adjacent to x_1. We have shown that in $G - \{p, x_i, y_i : (i = 1, 2)\}$ there is exactly one isolated vertex, say z.

We now must consider three cases, depending on which of the vertices x_2,y_1,y_2 are adjacent to z.

Case 1. z is adjacent to x_2 and to exactly one of y_1 or y_2, say without loss of generality to y_2.

Consider the case $d(x_2) = 2$. Then $G - \{p, x_2, y_2, z\}$ has no isolated vertices so by induction it has a big odd subgraph which together with $\langle x_2, z \rangle$ is a big odd subgraph of G and we are done.

So we may assume $d(x_2) = 3$. Let w be the third neighbor of x_2 and let $G_2 = G - \{p, x_2, y_2, z, w\}$. If G_2 has no isolated vertices, then it has a big odd subgraph which together with $\langle x_2, z \rangle$ is a big odd subgraph of G, a contradiction. So G_2 must have at least one isolated vertex, which could arise in one of two ways: either $w = y_1$, isolating x_1; or there is a vertex w' in G_2 which is adjacent (in G) to w and y_2.

In the latter case, let $G_3 = G - \{x_2, y_2, z, w, w'\}$. Then G_3 has no isolated vertex, so it has a big odd subgraph H_3. If $p \notin v(H_3)$, let $H = H_3 \bigcup \langle x_2, z \rangle$. If instead $p \in v(H_3)$, then $\langle p, x_1 \rangle$ is a component of H_3. So $y_1 \notin v(H_3)$, as x_1 can not have degree two in the odd subgraph H_3. Then let H be $H_3 - p$ together with the claw at y_2. Either way, H is a big odd subgraph of G, a contradiction.

Now consider the case that $w = y_1$. We may assume that y_1 and y_2 are not adjacent and that at least one of them has degree three in G. Otherwise we are done immediately.

In fact, each has degree three in G. Say y_1 has degree two; let y be the third neighbor of y_2. (The proof is the same if y_2 has degree two.). Let $G_4 = G - \{p, x_1, x_2, y_1, y_2, y, z\}$. This has no isolated vertex, as G has no vertex of degree one. So it has a big odd subgraph, which together with the claw at x_2 is a big subgraph of G and we are done.

Thus $d(y_1) = d(y_2) = 3$ in G.

If there is a vertex w adjacent to both y_1 and y_2 then $G - \{p, x_i, y_i, z, w : (i = 1, 2)\}$ has no isolated vertex. So it has a big odd subgraph, which together with the claw at x_2 gives a big odd subgraph of G.

Consequently, let w_1 and w_2 be the remaining neighbors in G of y_1 and y_2, respectively. Now let $G_5 = G - \{p, x_i, y_i, z, w_i : (i = 1, 2)\}$. Since G has no vertex of degree one, G_5 has at most two isolated vertices which must be adjacent to both w_1 and w_2. Deleting from G_5 the isolated vertices (if they exist) thus yields a subgraph G_5' of G of order at least $n - 10$. By induction, this has a big odd subgraph which together with the claw at x_2 gives a big odd subgraph of G, a contradiction.

Case 2. z is adjacent just to y_1 and y_2.

Let $G_6 = G - \{x_1, y_1, y_2, z, u_1\}$ where u_1 is the remaining neighbor in G of y_1, if any. Then G_6 must have an isolated vertex, or else by induction it has a big odd subgraph which together with $\langle y_1, z \rangle$ is a big odd subgraph of G, a contradiction. However, by the same argument used before, G_6 has at most one isolated vertex, say u_2, which must be adjacent to y_1 and y_2. Then $G_6 - \{u_2, p\}$ has no isolated vertices, and therefore has a big odd subgraph, which together with the claw at y_2 is a big odd subgraph of G, and we are done.

Case 3. z is adjacent to y_1, y_2 and x_2.

Let $G_7 = G - \{p, x_i, y_i, z, u_i : (i = 1, 2)\}$, where u_1 and u_2 are the (possible) remaining neighbors (in G) of y_1 and y_2, respectively.

There can be at most two isolated vertices of G_7, as each must be adjacent, in G, to at least two of the vertices u_1, u_2, and x_2, which among them have at most five available edges. Deleting the isolated vertices (if any) from G_7 yields a connected graph of order at least $n - 10$. By induction this has a big odd subgraph which together with the claw at x_1 is a big odd subgraph of G and we are done.

ACKNOWLEDGMENT

The authors thank the referee for several helpful suggestions.

REFERENCES

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, North-Holland, New York (1976).
[2] Y. Caro, On induced subgraphs with odd degrees, *Discrete Math.*, 132 (1994), 23-28.
[3] L. Lovasz, *Combinatorial Problems and Exercises*, North-Holland, 1979.
[4] J. Radcliffe and A.D. Scott, Every tree contains a large induced subgraph with all degrees odd, *Discrete Math.*, 140 (1994), 275-279.
[5] A.D. Scott, Large induced subgraphs with all degrees odd, *Combin. Probab. Comput.* (1992), 335-349.

(Received 10/4/96)