Circumferences of 3-connected Tough Graphs with Large Degree Sums

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Abstract

Let G be a 3-connected tough graph of order n with circumference c(G), independence number $\alpha(G)$ and vertex connectivity $\kappa(G)$, such that $d(x)+d(y)+d(z)+d(w)\geq s$ for any independent set { x, y, z, w} of vertices. In [7] we have proved : when $s\geq n+c(G)/2$, every longest cycle of G is a dominating cycle and $c(G)\geq\min\{n, n+s/4-\alpha(G)\}$. This paper improves the results by showing that under the same conditions $c(G)\geq\min\{n, n+s/4-\alpha(G)+1\}$. Furthermore when $s\geq(3n-1)/2+\kappa(G)$, G is hamiltonian .

1 Terminology

All graphs are finite simple graphs. The reader is referred to [3] for undefined terminology. Let G=(V, E) be a graph of order n. For $A\subseteq V$, we use G[A] to denote the subgraph induced by A , while G-A will be used to denote the graph G[V(G)-A]. For a subgraph H of G, G-H=G-V(H). $\kappa(G)$, $\alpha(G)$, $\delta(G)$ and c(G) will denote the vertex connectivity , independence number, minimum degree and circumference of G respectively. The number of components of G will be denoted by $\omega(G)$. We call G a t-tough graph if $|S|\geq t\cdot\omega(G-S)$ for any $S\subseteq V$ such that $\omega(G-S)>1$. The toughness of G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough ($\tau(K_n)=\infty$ for all $n\geq1$). If $\tau(G)\geq 1$ we call G a tough graph . For $u\in V$, we denote the neighborhood of u by N(u), and $d(u)=|N(u)|$. A cycle C of G is called a dominating cycle if every edge of G has at least one of its vertices on C. G is called almost hamiltonian if every longest cycle of G is a dominating cycle. For a cycle C, we denote by $\vec{C}$ the cycle with a fixed cyclic orientation. If u , $v\in V(C)$, then $u\vec{C}v$ denotes consecutive vertices on C from u to v in the orientation specified by $\vec{C}$. The same vertices , in reverse orientation , are given by $v\overleftarrow{C}u$. We use u^+ to denote the successor of vertex u on $\vec{C}$ and u^- the predecessor of u on $\vec{C}$, and $u^{++}=(u^+)^+$, $u^{--}=(u^-)^-$. If $A\subseteq V(C)$, then $A^+=\{v^+|\ v\in A\}$, $A^-=\{v^-|\ v\in A\}$, and $A^{++}=(A^+)^+$. For an integer r , $1\leq r\leq\alpha(G)$, define $\sigma_r(G)=\min\{\sum_{u\in S}d(u)|\ S\subseteq V(G)$ is an independent set of vertices of size r} and $\mu(G)=\max\{d(v)|\ v\in V-V(C'), C'$ is a longest cycle of G} .

2 Main Results

In [7] , we have obtained the following results .

Theorem 1. Let G be a 3-connected tough graph of order n such that $\sigma_4(G) \geq n+c(G)/2$. Then G is almost hamiltonian.

Theorem 2. Let G be a graph of order n such that $\delta(G)\geq 3$ and $\sigma_4(G)\geq n+c(G)/2$. Let G contain a longest cycle C which is a dominating cycle. If $v_0\in V-V(C)$, $A=N(v_0)$, then both $(V-V(C))\cup A^+$ and $(V-V(C))\cup A^-$ are independent sets of vertices.

Theorem 3. Let G be a 3-connected tough graph of order n such that $\sigma_4(G)\geq s\geq n+c(G)/2$. Then $c(G)\geq \min\{n, n+s/4-\alpha(G)\}$; furthermore when $c(G)<n$, there exists a longest cycle C of G and $v_0\in V-V(C)$ such that $\mu(G)=d(v_0)\geq\sigma_4(G)/4$.

This paper improves the above results by showing that:

Theorem 4. Let G be a 3-connected tough graph of order n such that $\sigma_4(G)\geq s\geq n+c(G)/2$. Then G is hamiltonian or there exists a longest cycle C such that $\alpha(G)\geq|V-V(C)|+s/4+1$.

Corollary 5. Let G be a 3-connected tough graph of order n such that $\sigma_4(G)\geq s\geq n+c(G)/2$. Then $c(G)\geq\min\{n, n+s/4-\alpha(G)+1\}$.

Since $\alpha(G)\leq n/(\tau(G)+1)$, $\sigma_4(G)$and $c(G)$ are all integers, Theorem 4 also implies the following corollary.

Corollary 6. Let G be a 3-connected τ-tough graph of order n such that $\tau(G)\geq 1$ and $\sigma_4(G)\geq s\geq n+ c(G)/2$. Then $c(G)\geq\min\{n, n\tau/(\tau+1)+s/4+1\}$; furthermore when $\sigma_4(G)\geq n+(n-1)/2$, if $\tau(G)\geq 5/3$ or $\delta(G)\geq\alpha(G)-1$, then G is hamiltonian.

Using the above results, we obtain another sufficient condition for hamiltonian cycles.

Theorem 7. Let G be a 3-connected tough graph of order n with vertex connectivity $\kappa(G)$ such that $\sigma_4(G)\geq (3n-1)/2+\kappa(G)$. Then G is hamiltonian.

Our proof of Theorem 7 also requires a number of well-known results as follows:

Lemma 8[1]. Let G be a graph of order n and S a vertex cut of G. Suppose some component of G-S is complete and has vertex set B. If u and v are nonadjacent vertices in $V-(S\cup B)$ such that $d(u)+d(v)\geq n-|B|+1$, then G is hamiltonian if and only if G+uv is hamiltonian.

Lemma 9[6]. Let G be a graph of order n such that $\sigma_1(G)=\delta(G)\geq n/2>1$. Then G is hamiltonian.

Lemma 10[4]. Let G be a graph of order n with nonadjacent vertices u and v. If $d(u)+d(v)\geq n$, then G is hamiltonian if and only if G+uv is hamiltonian.

Lemma 11[5]. Let G be such a graph that $\alpha(G)\leq\kappa(G)$. Then G is hamiltonian.

Lemma 12[9]. For any graph G, $\kappa(G)\leq\delta(G)$.

3 The proofs

Preliminaries

Let G be a non-hamiltonian 3-connected tough graph of order n such that $\sigma_4(G)\geq n+c(G)/2$, and C be a longest cycle of G with a fixed cyclic orientation $\vec{C}$. Let $v_0\in V-V(C)$ such that $d(v_0)=\mu(G)$. By theorems 1 and 3, C is a dominating cycle and $d(v_0)\geq\sigma_4(G)/4$. Set $A=N(v_0)=\{v_1, v_2, \ldots, v_k\}(k\geq\kappa(G)\geq 3)$ such that $v_i\in v_{i-1}\vec{C}v_{i+1}$. Set $u_i=v_i^+$, $w_i=v_{i+1}^-$, $L_i=u_i\vec{C}w_i$ for $1\leq i\leq k$ (indices mod k). For $1\leq r< s\leq k$, define

$$R_1(u_r)=\{v\in u_r\vec{C}v_s | u_rv^+\in E\}$$

$S_1(u_s)=\{v\in u_r\bar{C}v_s| u_sv\in E\}$

$R_2(u_r)=\{v\in u_s\bar{C}v_r| u_rv\in E\}$

$S_2(u_s)=\{v\in u_s\bar{C}v_r| u_sv^+\in E\}$

$B(u_r, u_s)=R_1(u_r)\cup S_1(u_s)\cup R_2(u_r)\cup S_2(u_s)$.

The following propositions facilitate the proof of Theorem 4 .

Proposition 1. $A\cap A^+=A\cap A^-=\phi$.

Proof. Since C is a longest cycle , Proposition 1 is obviously true .

Proposition 2. If $v\in u_i\bar{C}u_j$ with $i<j$ and $u_jv\in E$ then $u_iv^+\notin E$.

Proof . Suppose otherwise , then there exists a cycle $u_i\bar{C}vu_j\bar{C}v_iv_0v_j\bar{C}v^+u_i$ longer than C , which is a contradiction . Hence Proposition 2 is true.

Proposition 3. For $1\le r<s\le k$, $d(u_r)+d(u_s)\le|B(u_r, u_s)|\le|V(C)|$.

Proof . Since C is a longest cycle as well as a dominating cycle , by Proposition 2 we have $N(u_r)=R_1^+(u_r)\cup R_2(u_r)$, $N(u_s)=S_1(u_s)\cup S_2^+(u_s)$, and $R_1(u_r)\cap S_1(u_s)=R_2(u_r)\cap S_2(u_s)=\phi$. Thus $d(u_r)+d(u_s)=|N(u_r)|+|N(u_s)|\le|R_1^+(u_r)|+|R_2(u_r)|+|S_1(u_s)|+|S_2^+(u_s)|=|B(u_r, u_s)|\le |V(C)|$ for $1\le r<s\le k$. Hence Proposition 3 is true .

Proposition 4. $A^+\cap A^-\neq\phi$ and if $u\in A^+\cap A^-$ then $d(u)\le d(v_0)$.

Proof . Suppose $A^+\cap A^-=\phi$. Then $c(G)\ge3d(v_0)\ge3\sigma_4(G)/4\ge3(n+c(G)/2)/4$, i.e. $c(G)\ge6n/5>n$. This contradiction shows that $A^+\cap A^-\neq\phi$. If $u\in A^+\cap A^-$, then there exists a longest cycle C', $v_0u^+\bar{C}u^-v_0$, such that $u\in V-V(C')$. By the choice of C and v_0 , we have $d(v_0)\ge d(u)$. Thus Proposition 4 is true.

Proposition 5. If $u\in A^+\cap A^-$, $uv\in E$ and $v\in V(C)$, then $\{v^+\}\cup(V-V(C))\cup A^+$ is an independent set of vertices .

Proof . By Proposition 2, $\{v^+\}\cup A^+$ is an independent set of vertices . Suppose there exists $v'\in V-V(C)$ such that $v'v^+\in E$ .Clearly $v_0\neq v'$,otherwise there exists the cycle $v'u^+\bar{C}vu\bar{C}v^+v'$ longer than cycle C. And $v'u_i\notin E$ for any $i\in\{1, 2, \dots, k\}$, otherwise when $u_i\in u^+\bar{C}v$, there is a cycle $v'v^+\bar{C}u^-v_0v_i\bar{C}uv\bar{C}u_iv'$ longer than cycle C ; when $u_i\in v^+\bar{C}u$, there is a cycle $v'v^+\bar{C}v_iv_0u^+\bar{C}vu\bar{C}u_iv'$ longer than cycle C. Similarly $v'u_i^+\notin E$ for any $i\in\{1, 2, \dots, k\}$. This is to say ,no edge of G joins v' to the vertex in $A^+\cup A^{++}$ .Since C is a dominating cycle, $N(v')\subseteq V(C)$, and by Proposition 1 , $A^+\cap A^{++}=\phi$. Thus $d(v')\le|V(C)|-2d(v_0)$. Furthermore ,since $k\ge3$, there exists $u_m\in A^+$ such that $\{v_0, v', u, u_m\}$ is an independent set of vertices , and $d(u_m)=\min\{d(u_j)| j\in\{1, 2, \dots, k\}, u_j\neq u\}$. By Proposition 3, $d(u_m)\le|V(C)|/2$. By Proposition 4 , $d(u)\le d(v_0)$. Hence we have $n+c(G)/2\le\sigma_4(G)\le d(v_0)+d(v')+d(u)+d(u_m)\le|V(C)|+c(G)/2\le n-2+c(G)/2$, a contradiction, which shows that $v'v^+\notin E$ for any $v'\in V-V(C)$, i.e. $\{v^+\}\cup(V-V(C))$ is an independent set of vertices. Then by Theorem 2, $\{v^+\}\cup(V-V(C))\cup A^+$ is an independent set of vertices. Thus Proposition 5 is true .

Proposition 6. If $y\in u_i^+\bar{C}w_i^-$, $z\in v_{i+1}\bar{C}v_i$ and u_i, $u_i^+\neq w_i$ such that $yz\in E$, $u_i\bar{C}w_i^-\subseteq N(w_i)$ and $u_i^+\bar{C}w_i\subseteq N(u_i)$ for some $i\in\{1, 2, \dots, k\}$, then $\{z^+\}\cup(V-V(C))\cup A^+$ is an independent set of vertices.

Proof. When $z\in A$, Proposition 6 is true by Theorem 2. When $z\notin A$, suppose $u_jz^+\in E$ for some $j\in\{1,2,\dots,k\}$. If $u_j\in z^+\bar{C}w_i$, then there exists a cycle $u_jz^+\bar{C}v_jv_0v_{i+1}\bar{C}zy\bar{C}wy^-\bar{C}u_j$ longer than C. If $u_j\in v_{i+1}\bar{C}z$, then the cycle $u_jz^+\bar{C}v_iv_0v_j\bar{C}y^+u_i\bar{C}yz\bar{C}u_j$ is longer than C. In either case we reach a contradiction. Hence $u_jz^+\notin E$ for all $j\in\{1, 2, \dots, k\}$, i.e. $\{z^+\}\cup A^+$ is an independent set of vertices. As in the proof of Proposition 5, we have $\{z^+\}\cup(V-V(C))$ is an independent set of vertices. And by Theorem 2, $\{z^+\}\cup(V-V(C))\cup A^+$ is an independent set of vertices. Thus Proposition 6 is true.

Proposition 7. If $v\in u_i\bar{C}w_i$ and $u_iv\in E-E(C)$ for some $i\in\{1, 2, \dots, k\}$, then $\{v^-\}\cup(V-V(C))\cup(A^+-\{u_i\})$ is an independent set of vertices.

Proof. By Proposition 2, $\{v^-\}\cup(A^+-\{u_i\})$ is an independent set of vertices. As in the proof of Proposition 5, we have $\{v^-\}\cup(V-V(C))$ is an independent set of vertices. By Theorem 2, $(V-V(C))\cup(A^--\{u_i\})$ is an independent set of vertices. Hence $\{v^-\}\cup(V-V(C))\cup(A^+-\{u_i\})$ is an independent set of vertices. Thus Proposition 7 is true.

Proof of Theorem 4

When $n\leq 11$, it is easy to verify Theorem 4. Hence we may assume that $n\geq 12$.

If G is hamiltonian, there is nothing to prove. Otherwise, choose C, v_0 and A as above. By theorems 1,2 and 3, $\alpha(G)\geq|V-V(C)|+|A^+|=|V-V(C)|+d(v_0)$ and $d(v_0)\geq\sigma_4(G)/4$. If $d(v_0)\geq\sigma_4(G)/4+1$ then $\alpha(G)\geq|V-V(C)|+\sigma_4(G)/4+1$ so that Theorem 4 holds. Thus we may assume that $\sigma_4(G)/4\leq d(v_0)\leq(\sigma_4(G)+3)/4$. Suppose $\alpha(G)\leq|V-V(C)|+|A^+|$.

Claim 1. If $u\in A^+\cap A^-$ then $N(u)\subseteq N(v_0)$.

Otherwise, suppose there exists $v\in V(C)$ such that $uv\in E$ and $v\notin N(v_0)$. Then by Proposition 5, $\{v^+\}\cup(V-V(C))\cup A^+$ is an independent set of vertices, so that $|V-(C)|+|A^+|<\alpha(G)$, a contradiction.

Claim 2. There exist vertices $u_i\in A^+$ and $w_j\in A^-$ with $i\neq j$ such that $u_iw_j\in E$.

Since G is a tough graph, G-A has at most k components, one of which has vertex set $\{v_0\}$. Hence there exist integers $i, j\in\{1, 2, \dots, k\}$ with $i\neq j$ such that some vertex in L_i is joined to some vertex in L_j by either an edge e or a path of length 2 with its internal vertex in $V-V(C)-\{v_0\}$. Since the arguments for these two cases are completely analogous, we will assume that the first case applies. If e joins a vertex in A^+ to a vertex in A^-, the claim is established. Otherwise, let e=yz, where $y\in L_j$ and $y\neq u_j$, w_j. Suppose $u_jw_j\in E$. Then by Proposition 7, $\{w_j^-\}\cup(V-V(C))\cup(A^+-\{u_j\})$ is an independent set of vertices. Thus $u_jw_j^{-+}\in E$. By repeating the above argument we conclude that each vertex in $L_j-\{u_j,w_j\}$ is adjacent to u_j. Similarly, each vertex in $L_j-\{u_j, w_j\}$ is adjacent to w_j. Now by Proposition 6, $\{z^+\}\cup(V-V(C))\cup A^+$ is an independent set of vertices so that $\alpha(G)>|V-V(C)|+|A^+|$. This contradiction shows that $u_jw_j\notin E$. By Theorem 2, $\{w_j\}\cup(V-V(C))$ is an independent set of vertices. Since $\alpha(G)\leq|V-V(C)|+|A^+|$, w_j must be adjacent to some vertex in $A^+-\{u_j\}$. Thus claim 2 holds.

Claim 3. There exist vertices $x_1, x_2, x_3\in A^+\cap A^-$ such that $N(x_1)=N(x_2)=N(v_0)$.

Since $n-1\geq|V(C)|\geq 3(d(v_0)-|A^+\cap A^-|)+2|A^+\cap A^-|$, $|A^+\cap A^-|\geq 3d(v_0)-|V(C)|$. Note that $n\geq 12$. When $d(v_0)=\sigma_4(G)/4$, we have $|A^+\cap A^-|\geq 3(n+c(G)/2)/4-c(G)\geq(n+5)/8$, so that $|A^+\cap A^-|\geq 3$. Let $\{x_1, x_2, x_3\}\subseteq A^+\cap A^-$. Then $\{v_0, x_1, x_2, x_3\}$ is an independent set, and $\max\{d(x_1), d(x_2), d(x_3)\}\leq d(v_0)$ by Claim 1. Suppose $d(x_i)<d(v_0)$ for some $i\in\{1, 2, 3\}$. Then we have $\sigma_4(G)\leq d(v_0)+d(x_1)+d(x_2)+d(x_3)<4d(v_0)=\sigma_4(G)$, a contradiction. Hence $d(x_1)=d(x_2)=d(x_3)=d(v_0)$. By Claim 1, $N(x_1)=N(x_2)=N(v_0)$. When $d(v_0)=(\sigma_4(G)+1)/4$, we have $|A^+\cap A^-|\geq 3$ too. Let $\{x_1, x_2, x_3\}\subseteq A^+\cap A^-$ such that $d(x_3)\leq d(x_2)\leq d(x_1)\leq d(v_0)$. Suppose $d(x_2)\leq d(v_0)-1$. Then we have $\sigma_4(G)\leq d(v_0)+d(x_1)+d(x_2)+d(x_3)\leq 2(\sigma_4(G)+1)/4+2((\sigma_4(G)+1)/4-1)=\sigma_4(G)-1$, a contradiction, which shows that $d(x_1)=d(x_2)=d(v_0)$. Then by Claim 1, $N(x_1)=N(x_2)=N(v_0)$. Similarly when $d(v_0)=(\sigma_4(G)+2)/4$ or $d(v_0)=(\sigma_4(G)+3)/4$, there exist $x_1, x_2, x_3\in A^+\cap A^-$ such that $N(x_1)=N(x_2)=N(v_0)$. Thus Claim 3 holds.

By Claim 3, without loss of generality, let $u_1, u_r\in A^+\cap A^-$ with $r\neq 1$ such that $N(u_1)=N(u_r)=N(v_0)$. Let j be the maximum index such that $u_iw_j\in E$ for some i with $i\neq j$. By Theorem 2, $j\neq 1, r$. Since $u_1v_{j+1}, u_rv_{j+1}\in E$, by Proposition 2 we have $1<r<i<j$. Now we consider u_j.

Claim 4. The vertex u_j is not adjacent to any vertex in $V(C)-(L_j\cup A)$.

Suppose $u_jy\in E$, where $y\in L_t$ and $t>j$. By theorems 1 and 2, $\{w_t\}\cup(V-V(C))$ is an independent set of vertices, and $y, w_t\notin A^+$. Thus w_t must be adjacent to some vertex in A^+. By the choice of the index j we conclude that $u_tw_t\in E$ and $y\neq u_t, w_t$. But then by Proposition 7, $\{w_t^-\}\cup(A^+-\{u_t\})\cup(V-V(C))$ is an independent set of vertices. Thus $u_tw_t^-\in E$. By repeating the above argument we have $N(u_t)\supseteq L_t-\{u_t\}$. Similarly $N(w_t)\supseteq L_t-\{w_t\}$. But then by Proposition 6, $\{u_j^+\}\cup(V-V(C))\cup A^+$ is an independent set of vertices, a contradiction. Hence we suppose $u_jy\in E$, where $y\in L_t$ and $j>t$. Since $N(u_1)=A$, $u_1v_{t+1}\in E$. By Proposition 2, $y\neq w_t$. Furthermore $\{y^+\}\cup(V-V(C))$ is an independent set of vertices. To see this, suppose $y^+v_0'\in E$, where $v_0'\in V-V(C)$. Since $L_t\cap A=\phi$, $v_0'\neq v_0$. If $v_0'u_p\in E$, where either $p>j$ or $p\leq t$, then there exists a cycle $v_0'y^+\vec{C}v_jv_0v_p\vec{C}u_jy\vec{C}u_pv_0'$ longer than cycle C. If $v_0'u_p\in E$ where $t<p<j$, then there exists another cycle $v_0'y^+\vec{C}v_pu_1\vec{C}yu_j\vec{C}v_1v_0v_j\vec{C}u_pv_0'$ longer than cycle C too. Thus v_0' is not adjacent to any vertex in $A^+-\{u_j\}$. Similarly, v_0' is not adjacent to any vertex in $A^{++}-\{u_j^+\}$. By Proposition 1, $A^+\cap A^{++}=\phi$. By theorems 1,2 and 3, $v_0'u_j, v_0'w_j\notin E$, and v_0' is also not adjacent to any vertex in V-V(C), so that $d(v_0')\leq|V(C)|-2(d(v_0)-1)-2=|V(C)|-2d(v_0)$. Let $d(u_m)=\min\{d(u_i)|2\leq i\leq k\}$. Then $\{v_0, v_0', u_1, u_m\}$ is an independent set of vertices. Thus $n+c(G)/2\leq\sigma_4(G)\leq d(v_0)+d(v_0')+d(u_1)+d(u_m)\leq|V(C)|+|V(C)|/2<n-1+c(G)/2$, a contradiction. Hence $y^+v_0'\notin E$ for all $v_0'\in V-V(C)$, i.e. $\{y^+\}\cup(V-V(C))$ is an independent set of vertices. But since $\alpha(G)\leq|V-V(C)|+|A^+|$, y^+ must be adjacent to some vertex $u_s\in A^+$. Since $u_jy\in E$, by Proposition 2, it must be the case that $t<s\leq j$. Now we apply the above argument to y^+, y^{++}, etc. It follows that there exists an integer b such that $t<b\leq j$ and

$u_b w_t \in E$. But since $u_1 v_{t+1} \in E$, we reach a contradiction with Proposition 3 . Thus Claim 4 holds.

Claim 5. There exists a vertex $u_a \in A^+ \cap A^-$ with $a \neq i, j$, such that $u_j v_{a+1} \in E$.

Suppose otherwise. Consider any vertex $v_{m+1} \in A$ with $m \neq j$. If $u_m \notin A^+ \cap A^-$, by Claim 4 , $u_j v_{m+1}^-$, $u_j v_m^- \notin E$. Similarly , if $u_m \in A^+ \cap A^-$, then $u_j v_{m+1}$, $u_j v_{m+1}^- \notin E$. Furthermore since $u_i w_j \in E$, by Proposition 2 we have $u_j v_{j+1} \notin E$. Thus $d(u_j) \leq |V(C)|-2d(v_0)$. Let $d(u_m)= \min \{ d(u_i) | u_i \in A^+ \cap A^-, i \neq 1, j\}$. By propositions 2 and 3 , $d(u_m) \leq |V(C)|/2$. Now $\{v_0 , u_1 , u_j , u_m\}$ is an independent set of vertices , we have $n+ c(G)/2 \leq \sigma_4(G) \leq d(v_0)+d(u_1)+d(u_j)+d(u_m) \leq n-1 + c(G)/2$, a contradiction . Thus there exists a vertex $u_a \in A^+ \cap A^-$ with $a \neq j$, such that $u_j v_{a+1} \in E$. Since $u_i w_j \in E$, by Theorem 2, $u_i \notin A^+ \cap A^-$ and hence $i \neq a$. Thus Claim 5 holds .

Claim 6. $u_a v_i \notin E$.

Suppose otherwise . Then when $i<a<j$, we have $u_a v_i$, $u_i w_j \in E$, contradicting Proposition 2 .When $a<i$ or $a>j$ then the cycle $u_j v_{a+1} \bar{C} v_i u_a \bar{C} v_{j+1} v_0 v_j \bar{C} u_i w_j \bar{C} u_j$ is longer than cycle C ,which is a contradiction. Thus Claim 6 holds .

Claim 7. If $i<a<j$ then $u_a v_{j+1} \notin E$. If $i>a$ or $a>j$ then $u_a v_j \notin E$.

Suppose otherwise . Then when $i<a<j$,we have $u_a v_{j+1}$, $u_i w_j \in E$, a contradiction with Proposition 2 . When $i>a$ or $a>j$, $u_a v_j \in E$, then the cycle $u_j v_{a+1} \bar{C} v_i v_0 v_{j+1} \bar{C} u_a v_j \bar{C} u_i w_j \bar{C} u_j$ is longer than cycle C , a contradiction . Thus Claim 7 holds .

Note that $\{v_0 , u_1 , u_r , u_a\}$ is an independent set of vertices . By claims 1 and 7, we have $d(u_a)<d(u_1)=d(u_r)=d(v_0) \leq (\sigma_4(G)+3)/4$, so that $\sigma_4(G) \leq d(v_0)+d(u_1)+d(u_r) +d(u_a) \leq d(u_a) + 3 (\sigma_4(G) +3)/4$. Hence $d(u_a) \geq (\sigma_4(G)+3)/4-3 \geq d(v_0)-3$, so that $|N(v_0)-N(u_a)| \leq 3$. By claims 6 and 7 , we have $2 \leq |N(v_0)-N(u_a)| \leq 3$. In the following arguments let C(s) denote the cycle $u_j v_{a+1} \bar{C} v_s u_a \bar{C} v_{j+1} v_0 v_i \bar{C} u_s v_j \bar{C} u_i w_j \bar{C} u_j$ with length longer than cycle C , where s=1 or r . We distinguish different cases below .

Case 1. $1<a<i$ or $a>j$.

In this case , $N(v_0)-N(u_a)=\{v_i , v_j\} \cup \{x\}$, where $x \in \{\phi , v_1 , v_r\}$. If $j<a \leq k$, then when $x \neq v_r$, G contains cycle C(r) ; when $x \neq v_1$, G contains cycle C(1) . Similarly, if $i>a>1$, G also contains cycle either C(r) or C(1) .

Case 2. $i<a<j$.

In this case $N(v_0)-N(u_a)=\{v_i , v_{j+1}\} \cup \{x\}$, where $x \in \{\phi, v_1 , v_r\}$. But then when $x \neq v_r$, G contains cycle $u_a v_r \bar{C} u_1 v_i \bar{C} u_r v_1 \bar{C} v_{j+1} v_0 v_{a+1} \bar{C} w_j u_i \bar{C} u_a$ with length longer than cycle C, and when $x \neq v_1$, G contains cycle $u_a v_1 \bar{C} v_{j+1} v_0 v_r \bar{C} u_1 v_i \bar{C} u_r v_a \bar{C} u_i w_j \bar{C} u_a$ longer than cycle C too.

This final contradiction shows that the hypothesis $\alpha(G) \leq |V-V(C)|+|A^+|$ is not true . Thus we have $\alpha(G) \geq |V-V(C)|+|A^+|+1 \geq |V-V(C)|+\sigma_4(G)/4+1$, i.e. Theorem 4 holds .

Proof of Theorem 7

Suppose there exists a non-hamiltonian 3-connected tough graph H of order n such that $\sigma_4(H) \geq (3n-1)/2+\kappa(H)$. Let G be such a graph with a maximum number of edges .Note that

$\sigma_4(G)\geq n+n/2-1/2+\kappa(G)\geq n+n/2-1/2+3=n+(n+5)/2\geq n+c(G)/2$. By Corollary 5, G is almost hamiltonian and $c(G)\geq \min\{n,n+\sigma_4(G)/4+1-\alpha(G)\}$.

If $\alpha(G)\leq\sigma_4(G)/4+1$, then $\sigma_4(G)/4+1-\alpha(G)\geq 0$, so that $n+\sigma_4(G)/4+1-\alpha(G)\geq n$. Hence, $c(G)\geq n$, so that G is hamiltonian, which is a contradiction. We assume, therefore, that $\alpha(G)>\sigma_4(G)/4+1$. But $\sigma_4(G)/4+1\geq((3n-1)/2+\kappa(G))/4+1=(3n+2\kappa(G)+7)/8$. By Lemma 11, $\alpha(G)\geq\kappa(G)+1$.

Case 1. $\alpha(G)=\kappa(G)+1$. Since $\kappa(G)+1>(3n+2\kappa(G)+7)/8$, we have $\kappa(G)>n/2-1/6$. But $n/2-1/6>(n-1)/2$ and by Lemma 12, $\delta(G)=\sigma_1(G)\geq\kappa(G)>(n-1)/2$. Hence $\kappa(G)\geq n/2$. This implies that $\delta(G)\geq n/2$. By Lemma 9, G is hamiltonian, which is a contradiction.

Case 2. $\alpha(G)=\kappa(G)+2$. Since $\kappa(G)+2>(3n+2\kappa(G)+7)/8$, we have $8\kappa(G)+16>3n+2\kappa(G)+7$, i.e. $6\kappa(G)>3n-9$, i.e. $\kappa(G)>(n-3)/2$. Since G is non-hamiltonian, Lemma 9 implies that $n/2>\delta(G)=\sigma_1(G)\geq\kappa(G)>(n-3)/2$. But then $n+\sigma_4(G)/4+1-\alpha(G)\geq n+(3n+2\kappa(G)+7)/8-\kappa(G)-2$ $=n+(3n-6\kappa(G)+7)/8-2$. We have two cases to consider.

Case 2.1. $n\equiv 1 \pmod 2$. Then $n/2>\delta(G)=\sigma_1(G)\geq\kappa(G)>(n-3)/2$, which implies that $\delta(G)=\kappa(G)=(n-1)/2$. But then $n+(3n-6\kappa(G)+7)/8-2=n-3/4$, so that $c(G)\geq\min\{n,n+\sigma_4(G)/4+1-\alpha(G)\}\geq n-3/4$. It follows that $c(G)\geq n$, implying that G is hamiltonian, which is a contradiction.

Case 2.2. $n\equiv 0 \pmod 2$. Then $n/2>\delta(G)=\sigma_1(G)\geq\kappa(G)>(n-3)/2$, i.e. $n/2-1\geq\delta(G)=\sigma_1(G)\geq\kappa(G)\geq n/2-1$, ie. $\delta(G)=\kappa(G)=n/2-1$. But then $n+(3n-6\kappa(G)+7)/8-2=n-3/8$, so that $c(G)\geq\min\{n,n+\sigma_4(G)/4+1-\alpha(G)\}\geq n-3/8$. It follows that $c(G)\geq n$, implying that G is hamiltonian, which is a contradiction.

Case 3. $\alpha(G)=\kappa(G)+3$. Since $\kappa(G)+3>(3n+2\kappa(G)+7)/8$, we have $\kappa(G)>n/2-17/6$. Hence $\alpha(G)>(3n+7)/8+(n/2-17/6)/4=n/2+1/6>n/2$. Let A be any independent set of G of size at least $n/2+1$ and let $A'=V(G)-A$. Then $\omega(G-A')=|A|>n/2>|A'|$, which contradicts the fact that G is tough.

Case 4. $\alpha(G)\geq\kappa(G)+4$. Let T be an independent set of vertices such that $|T|=\alpha(G)$, S be a vertex cut such that $|S|=\kappa(G)$ and let $G_1, G_2, \ldots, G_t$ be the components of G-S. Choose w_1, $w_2\in T$ such that $d(x)\geq\max\{d(w_1), d(w_2)\}$ for all $x\in T-\{w_1, w_2\}$. Consider any pair v_1, v_2 of distinct vertices in $T-\{w_1, w_2\}$. Since $\{v_1, v_2, w_1, w_2\}$ is an independent set of vertices in G, we have $2(d(v_1)+d(v_2))\geq d(v_1)+d(v_2)+d(w_1)+d(w_2)\geq\sigma_4(G)\geq(3n-1)/2+\kappa(G)$. Hence $d(v_1)+d(v_2)\geq(3n-1)/4+\kappa(G)/2$. Since, by the inclusion-exclusion principle, $|N(v_1)\cap N(v_2)|=d(v_1)+d(v_2)-|N(v_1)\cup N(v_2)|$ and $|N(v_1)\cup N(v_2)|\leq n-\alpha(G)$, it follows that $|N(v_1)\cap N(v_2)|\geq(3n-1)/4+\kappa(G)/2-n+\alpha(G)=\alpha(G)-n/4-1/4+\kappa(G)/2>(3n+2\kappa(G)+7)/8+\kappa(G)/2-(n+1)/4=(n+6\kappa(G)+5)/8>\kappa(G)$. (To see that $(n+6\kappa(G)+5)/8>\kappa(G)$, suppose, to the contrary that $(n+6\kappa(G)+5)/8\leq\kappa(G)$. Then $n+6\kappa(G)+5\leq 8\kappa(G)$, so that $(n+5)/2\leq\kappa(G)\leq\alpha(G)-4$. Hence $\alpha(G)\geq(n+5)/2+4$ which, as before, contradicts the fact that G is tough.) It follows that any pair of distinct vertices in $T-\{w_1, w_2\}$ cannot be in different components of G-S. Assume, without loss of generality, that $T-\{w_1, w_2\}\subseteq S\cup V(G_1)$. Set $B=V-(S\cup V(G_1))$. We now prove that G[B] is complete. Suppose otherwise. Let $x_1, x_2\in B$ such that $x_1\neq x_2$ and $x_1x_2\notin E$. Recall that $\alpha(G)\geq\kappa(G)+4$, so that $|T\cap V(G_1)|\geq 2$. Assume $\{y_1, y_2\}\subseteq T\cap V(G_1)$ with $y_1\neq y_2$.

Then $\{y_1, y_2, x_1, x_2\}$ is an independent set of vertices and we have $(3n-1)/2 + \kappa(G) \le \sigma_4(G) \le d(y_1)+d(y_2)+d(x_1)+d(x_2) \le 2(|V(G_1)|+\kappa(G)-\alpha(G)+2)+2(|B|+\kappa(G)-2)=2(n-\alpha(G)+\kappa(G))$, since $n=|V(G_1)|+|B|+\kappa(G)$. But then $\alpha(G)\le(n+2\kappa(G)+1)/4 < (3n+2\kappa(G)+7)/8 < \alpha(G)$, which is a contradiction .(If $(n+2\kappa(G)+1)/4 \ge (3n+2\kappa(G)+7)/8$, then one can show that $\alpha(G)\ge (n+5)/4+4$, which, as before, would contradict the toughness of the graph.) This contradiction shows that G[B] is complete. Since T is an independent set of vertices, it follows that $|T\cap B|\le 1$ and $|T\cap V(G_1)|\ge 3$. Without loss of generality assume that $\{y_1, y_2, y_3\}\subseteq T\cap V(G_1)$. Let $i, j\in\{1, 2, 3\}$ such that $i\ne j$. If $d(y_i)+d(y_j)\ge n$, then by lemma 10, the graph $G+y_i y_j$ is hamiltonian if and only if G is hamiltonian. By definition, $G+y_i y_j$ is also a 3-connected tough graph of order n with $\sigma_4(G+y_i y_j)\ge \sigma_4(G)$. Recalling that $\omega(G+y_i y_j-S) = \omega(G-S)>1$, we have $\kappa(G+y_i y_j)=\kappa(G) = |S|$ so that $\sigma_4(G+y_i y_j)\ge(3n-1)/2+\kappa(G+y_i y_j)$. By our choice of G, $G+y_i y_j$ is hamiltonian. But then G is also hamiltonian, which is a contradiction.We conclude that $d(y_i)+d(y_j)\le n-1$ for all $i, j\in\{1, 2, 3\}$with $i\ne j$. Assume , without loss of generality , that $d(y_3)=\min\{d(y_1), d(y_2), d(y_3)\}$. Then $d(y_3)\le(n-1)/2$. Let $v\in B$. Then $\{y_1, y_2, y_3, v\}$ is an independent set of vertices, so that $d(y_1)+d(y_2)\ge\sigma_4(G)-d(y_3)-d(v)\ge(3n-1)/2+\kappa(G)-d(y_3)-(|B|+\kappa(G)-1)=n-|B|+1+(n-1)/2-d(y_3)\ge n-|B|+1$. By Lemma 8, G is hamiltonian if and only if $G+y_1y_2$ is hamiltonian . But now by the choice of G, $G+y_1y_2$ is hamiltonian .But then G is also hamiltonian . This final contradiction completes the proof of Theorem 7 .

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