Broadcasting in Planar Graphs

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Abstract

For an arbitrary graph on n vertices, the minimum time required to broadcast is $\lceil \log_2 n \rceil$, and for any n, there exist graphs on n vertices with broadcast time equal to $\lceil \log_2 n \rceil$. When restricted to planar graphs, this is generally not the case; however, just one additional time unit is sufficient to allow broadcasting in certain planar graphs. We also show that the maximum number of vertices in a planar graph with broadcast time t is at least $2^{t-1} + 2^{\lfloor 2t/3 \rfloor} + 1$.

1 Introduction

Broadcasting in a communication network is the process of transmitting a message from one vertex of the network to all other vertices of the network by placing a series of calls over the communication lines of the network. The goal is to accomplish this as quickly as possible, subject to the following: (1) each call involves exactly two vertices; (2) each call requires one unit of time; (3) a vertex can participate in only one call per unit of time; (4) a vertex can only call an adjacent vertex.

Let a graph G represent a communication network, and $V(G)$ the set of vertices of G. For each $u \in V(G)$, we define the *broadcast time of vertex* u to be the minimum time required to broadcast from u, and denote this by $b(u)$. Since the number of informed vertices can at most double during each unit of time that elapses, it is clear

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that $b(u) \geq \lceil \log_2 n \rceil$, where n is the number of vertices in G. The *broadcast time of the graph* G, $b(G)$, is the maximum broadcast time for the vertices of G; i.e.,

$$b(G) = \max\{b(u) \ : \ u \in V(G)\}.$$

For the complete graph on $n \geq 2$ vertices, K_n, it is easy to see that $b(K_n) = \lceil \log_2 n \rceil$, so for any value of n, there is a graph on n vertices in which broadcasting can be completed in minimum possible time, $\lceil \log_2 n \rceil$. However, it turns out that it is not necessary to have as dense a graph as K_n to achieve $b(G) = \lceil \log_2 n \rceil$. In general, [4] and [3] consider the problem of finding the minimum number of edges in a graph G on n vertices with $b(G) = \lceil \log_2 n \rceil$. The hypercube on n vertices can easily be seen to have broadcast time $\lceil \log_2 n \rceil$ (simply broadcast in the i^{th} dimension during the i^{th} time unit), and has the minimum possible number of edges required to broadcast in time $\lceil \log_2 n \rceil$. The question of broadcasting in graphs with bounded degree has also been addressed [2].

In many applications, the networks considered are required to be planar. Here, we consider the problem of finding planar graphs G on n vertices for which $b(G) = \lceil \log_2 n \rceil$ if possible, or at least $b(G)$ is "close to" $\lceil \log_2 n \rceil$. Because planar graphs are inherently sparse, we are not concerned with minimizing the number of edges. Notice that the hypercubes are non-planar for large dimensions, so are not of use to us in general.

We define the *planar broadcast time for* n, $b_p(n)$, as the minimum broadcast time for planar graphs on n vertices; i.e.,

$$b_p(n) = \min\{b(G) \ : \ G \text{ planar}, |V(G)| = n\}.$$

Clearly, $b_p(n) \geq \lceil \log_2 n \rceil$. Consider a graph G on $n = 2^m$ vertices, $m \geq 6$, and suppose that $b(G) = \lceil \log_2 n \rceil = m$. In this case, $b(u) = \lceil \log_2 n \rceil = m$ for each vertex $u \in V(G)$, so that during the broadcasting process, the number of informed vertices must double in each unit of time. In particular, the vertex from which the broadcast originates must broadcast m times, and thus has degree at least m. But this is true for every vertex, and so G has minimum degree at least m. Since $m \geq 6$, it is clear that G can not be planar, and thus in general, $b_p(n) > \lceil \log_2 n \rceil$.

It is somewhat surprising that $b_p(n)$ does not differ by much from $\lceil \log_2 n \rceil$, and that just one extra time unit permits us to broadcast even from low degree vertices in an efficient manner. We will see in the next section that

$$\lceil \log_2 n \rceil \leq b_p(n) \leq \lceil \log_2 n \rceil + 1,$$

and that when $n = 2^t + 1$, $b_p(n) = \lceil \log_2 n \rceil$. We then go on to address the problem of determining the maximum number of vertices in a planar graph with fixed broadcast time t.

2 Binary wheels

We begin with a description of a planar graph that has an efficient broadcasting scheme, and follow this with a description of the broadcasting scheme. The *binary* *wheel*, denoted BW_t, is defined as follows: begin with 2^t vertices $0, 1, \ldots, 2^t - 1$. For every j, $0 \le j \le t-1$, and every vertex x where 2^j divides x, put an edge between x and $x + 2^j$. Finally, add an extra vertex ∞ and an edge from ∞ to each of $0, 1, \ldots, 2^t - 1$. See Figure 1 for an example of BW_4. Notice that the low degree vertices (approximately half the vertices have degree at most five) are nicely distributed among vertices of high degree. The embedding of BW_4 generalizes to BW_t for any t, so it is clear that BW_t is planar.

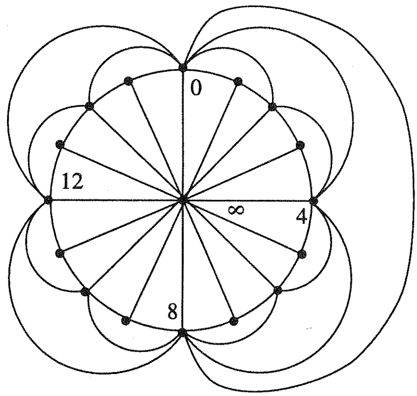

Figure 1.

Theorem 1 $b(BW_t) = t + 1$.

Proof: We will show that in BW_t (on $2^t + 1$ vertices), every vertex can broadcast in time $t + 1 = \lceil \log_2(2^t + 1) \rceil$. We begin with a description of our basic broadcast scheme originating at ∞.

Express each vertex $x \ne \infty$ in binary, as a string of length t. During the first unit of time, ∞ informs 0^t After the k^{th} time unit, $k \ge 1$, the following invariant holds: all vertices $x \ne \infty$ divisible by 2^{t-k} have been informed, except $2^t - 2^{t-k} = 1^k 0^{t-k}$ (and no other vertices have been informed).

During the $(k+1)^{st}$ time unit, the broadcast proceeds as follows: each informed vertex $x \ne \infty$ is divisible by 2^{t-k} and hence can inform $x + 2^{t-k-1}$, and ∞ informs $2^t - 2^{t-k}$; i.e., $x = x_1 x_2 \ldots x_k 0^{t-k}$ informs $x_1 x_2 \ldots x_k 1 0^{t-k-1}$ and ∞ informs $1^k 0^{t-k}$. It is easy to verify that the invariant is maintained. Thus, after t time units, all vertices except $2^t - 1 = 1^t$ have been informed; this vertex is informed by ∞ during the $(t+1)^{st}$ time unit.

Suppose now that the broadcast originates at $2^t - 1 = 1^t$. For convenience, we number the time units starting with 0, and during the 0^{th} time unit, $2^t - 1$ informs ∞. Continue the broadcast from ∞ as before, and notice that all vertices have been informed after time unit t, and thus the total time required to broadcast from $2^t - 1$ is $t + 1$.

To show that we can broadcast in time $t+1$ from an arbitrary vertex x, we need only show that it is possible to originate a broadcast from ∞ so that after t time units, all vertices except x have been informed. Let $x = x_1x_2\dots x_t$, and suppose that 2^{t-s} is the largest power of two that divides x; i.e., $x = x_1x_2\dots x_{s-1}10^{t-s}$. We construct a sequence of vertices $v_1, v_2, \dots v_t$, where $v_1 = x_10^{t-1}$, and for $2 \le k \le t$, $v_k = x_1x_2\dots x_{k-1}10^{t-k}$ (note that $v_s = x$). We will describe a broadcast scheme originating at ∞ so that:

1. for $1 \le k \le s$, after k time units all vertices $y \ne \infty$ divisible by 2^{t-k} have been informed except v_k, and no other vertices have been informed;

2. for $s+1 \le k \le t$, after k time units all vertices $y \ne \infty$ divisible by 2^{t-k} have been informed except for $v_s = x$, and no other vertices have been informed.

During the first time unit, ∞ informs the vertex $(1-x_1)2^{t-1} = (1-x_1)0^{t-1}$. Notice that after the first time unit, $v_1 = x_10^{t-1}$ is the only vertex divisible by 2^{t-1} that has not been informed.

Suppose that $1 \le k \le s-1$, and that after k time units all vertices y ($y \ne \infty$) divisible by 2^{t-k} have been informed, except for v_k. During the $(k+1)^{st}$ time unit the broadcast proceeds as follows: if $x_k = 1$, then every informed vertex $y \ne \infty$ is divisible by 2^{t-k}, and informs $y + 2^{t-k-1}$; ∞ informs v_k. On the other hand, if $x_k = 0$, then $x_1x_2\dots x_{k-1}0^{t-k+1}$ informs $v_k = x_1x_2\dots x_{k-1}10^{t-k}$. Every other informed vertex $y \ne \infty$ is divisible by 2^{t-k} and informs $y + 2^{t-k-1}$; ∞ informs $v_k + 2^{t-k-1} = x_1x_2\dots x_{k-1}110^{t-k-1}$.

We will now verify that (after $k+1 \le s$ time units) all vertices divisible by 2^{t-k-1} have been informed, except for v_{k+1}. To do this, it is sufficient to show that after $(k+1)$ time units, (i) v_k is informed, and (ii) every vertex $y = y_1\dots y_k10^{t-k-1}$, $y \ne v_{k+1}$ is informed. If $x_k = 1$, then v_k was informed by ∞, and every vertex $y = y_1\dots y_k10^{t-k-1}$ was informed by $y_1\dots y_k0^{t-k}$, except when $y = v_{k+1}$, in which case $y_1\dots y_k0^{t-k} = x_1\dots x_{k-1}10^{t-k} = v_k$, which was not previously informed. If $x_k = 0$, then v_k was informed by $x_1\dots x_{k-1}0^{t-k+1}$ (this vertex would normally inform $x_1\dots x_{k-1}010^{t-k-1} = v_{k+1}$, which need not be informed). Vertex $y = y_1\dots y_k10^{t-k-1}$ was informed by $y_1\dots y_k0^{t-k}$, except in the case when $y = x_1\dots x_{k-1}110^{t-k-1}$ (v_k could not inform this vertex). In this case, $y = x_1\dots x_{k-1}110^{t-k-1}$ was informed by ∞.

Thus, after s time units, every vertex $y \ne \infty$ that is divisible by 2^{t-s} has been informed, except for v_s. During the $(k+1)^{st}$ time unit, $s \le k \le t-1$ every informed vertex $y \ne \infty$ is divisible by 2^{t-k-1}, and hence informs $y + 2^{t-k-1}$; ∞ informs v_{k+1}. What remains is to verify that (after $k+1$ time units), all vertices divisible by 2^{t-k-1}, except for $v_s = x$, have been informed. To do this, we need only verify that after $(k+1)$ time units, every $y = y_1\dots y_k10^{t-k-1}$ is informed. The vertex $y = y_1\dots y_k10^{t-k-1}$ was informed by $y_1\dots y_k0^{t-k}$, except when $y = v_{k+1}$, in which case y was informed by ∞.

This shows that for any $x \in V(BW_t)$, $x \ne \infty$, it is possible to originate a broadcast from ∞ so that after t time units x is the only uninformed vertex. Since ∞ is adjacent to every other vertex of BW_t, this implies that for any $x \in V(G)$,

$x \neq \infty$, it is possible to originate a broadcast at x that is complete after $t+1$ time units. On the other hand, $\lceil \log_2(2^t+1) \rceil = t+1$, and thus $b(BW_t) = t+1$. ∎

Corollary 2 *If* $n = 2^t + 1$, *then* $b_p(n) = \lceil \log_2 n \rceil$.

For $2^{t-1}+1 < n < 2^t+1$, it is possible to modify BW_t and the broadcast scheme described in the proof to obtain a graph on n vertices with broadcast time $t+1 = \lceil \log_2 n \rceil + 1$. Note that the odd vertices in BW_t (i.e., those with $x_t = 1$) are the last to be informed in the broadcast scheme just described, and that they are never used to inform any other vertices. There are 2^{t-1} odd vertices in BW_t, and deleting any subset of these results in a graph in which the broadcast scheme for BW_t can be used to inform all vertices in time at most $t+1$. This leads to the following theorem.

Theorem 3 *For any* n, $\lceil \log_2 n \rceil \leq b_p(n) \leq \lceil \log_2 n \rceil + 1$.

The authors wish to acknowledge D. West for pointing out the following alternate proof of Corollary 2. Define a *broadcast tree* to be a rooted tree on n vertices with root u such that $b(u) = \lceil \log_2 n \rceil$. If $n = 2^t$ for some $t \geq 1$, then there is a unique (up to isomorphism) broadcast tree on n vertices. Let G be the graph obtained from the broadcast tree on 2^t vertices by adding an extra vertex, along with edges to the 2^t vertices. Clearly, G is a planar graph, and G has 2^t+1 vertices. One can verify by arguments similar to those above, that $b(G) = t+1$. Thus, this graph also admits a broadcast scheme in which each vertex broadcasts in time $t+1$, implying that $b_p(2^t+1) = \lceil \log_2(2^t+1) \rceil$. We note that the graph G has fewer edges (only $2^{t+1}-1$ edges) than the graph BW_t (which has $3 \cdot 2^t - 3$ edges). However, the graphs BW_t have the advantage that we can use them to obtain better bounds for the maximum number of vertices in a planar graph with broadcast time t, as we shall see in the next section.

3 Extended binary wheels and q-pods

For fixed t, we define $B_p(t)$ to be the maximum number of vertices in a planar graph with broadcast time t. Note that $B_p(t) \geq n$ if and only if $b_p(n) \leq t$. When $t \leq 3$, the hypercube on 2^t vertices is planar, and thus $B_p(t) = 2^t$ for these values of t. The graph in Figure 2 has 16 vertices, and it is easy to verify that the broadcast time for this graph is four, thus showing that $B_p(4) = 16$. When $t = 5$, the largest planar graph with broadcast time five that we know of contains 30 vertices (see Figure 3), and so $30 \leq B_p(5) \leq 32$.

For values of $t \geq 6$, it seems much more difficult to determine exact values for $B_p(t)$. It is obvious that $B_p(t) \leq 2^t$, and the comments in the previous section show that for $t \geq 6$, $B_p(t) \leq 2^t - 1$. Also, the fact that $b(BW_{t-1}) = t$ shows that $B_p(t) \geq 2^{t-1}+1$. We now proceed to improve this lower bound.

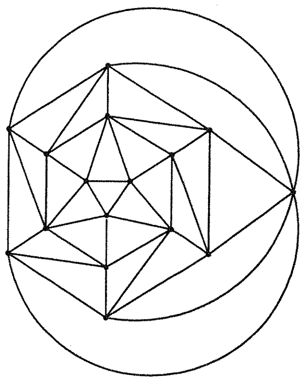

Figure 2.

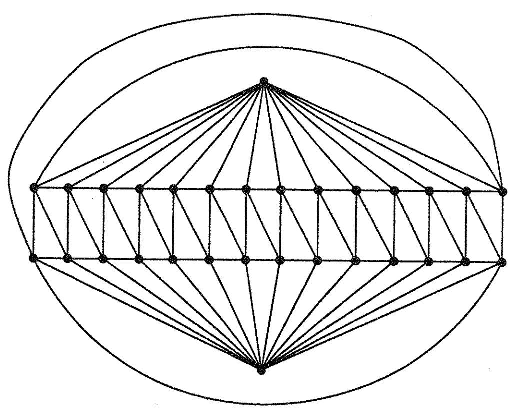

Figure 3.

The broadcast scheme described for the binary wheel, BW_{t-1}, in the previous section is efficient in the sense that broadcasting is completed in time t, the minimum possible time. However, observe that when broadcasting from ∞, only one call is made during the last time unit (all vertices except ∞ are idle during the last time unit), and when broadcasting from $x \neq \infty$, x makes the initial call to ∞, but then is idle for the rest of the time. By making use of these idle vertices, we can increase the number of vertices that can be informed in t time units.

The general structure of the binary wheel can be used to construct a graph with more than $2^{t-1}+1$ vertices having broadcast time t. We first define a *q-pod* as follows: begin with $2^q + 1$ vertices $0, 1, 2, \ldots, 2^q$. For every j, $0 \le j \le q$, and every vertex x where 2^j divides x, put an edge between x and $x+2^j$. The vertices 0 and 2^q are called the *endpoints* of the q-pod. It is easy to see that q-pods are planar. Observe that the graph BW_{t-1} can be obtained by taking two $(t-2)$-pods, identifying the two 0 vertices and the two 2^{t-2} vertices, removing any multiple edges, and then adding a vertex ∞ joined to all the other vertices of the graph.

An *extended binary wheel*, denoted E_{t-1}, is the graph on $2^{t-1} + 2^{\lfloor 2t/3 \rfloor} + 1$ vertices constructed as follows: begin with two $(t-2)$-pods, P_1 and P_2, and one $2^{\lfloor 2t/3 \rfloor}$-pod, P_3. Relabel the vertices of P_2 and P_3 by changing the label of every $x \in V(P_2)$ to $x + 2^{t-2}$, the label of every $y \in V(P_3), y \neq 2^{\lfloor 2t/3 \rfloor}$, to $y + 2^{t-1}$, and the label of $2^{\lfloor 2t/3 \rfloor} \in V(P_3)$ to 0. Next, identify vertices with identical labels (i.e., $0 \in V(P_1)$ is identified with $0 \in V(P_3)$, $2^{t-2} \in V(P_1)$ is identified with $2^{t-2} \in V(P_2)$, and $2^{t-1} \in V(P_2)$ is identified with $2^{t-1} \in V(P_3)$). Finally, add another vertex ∞ and an edge from ∞ to each of $0, 1, 2, \ldots, (2^{t-1} + 2^{\lfloor 2t/3 \rfloor} - 1)$. It is easy to verify that E_{t-1} is planar. We will now show that $B(E_{t-1}) = t$, thus proving the following theorem.

Theorem 4 *For $t \ge 6$, $B_p(t) \ge 2^{t-1} + 2^{\lfloor 2t/3 \rfloor} + 1$.*

The proof of this requires two preliminary results. We first define a *q^*-pod* to be a q-pod with one of its endpoints deleted; a q^*-pod has precisely one endpoint.

Lemma 5 *If x is the endpoint of a q^*-pod, then $b(x) = q$.*

Proof: The q^*-pod contains the broadcast tree on 2^q vertices, rooted at x, as a spanning subgraph, and hence $b(x) = q$. ■

Lemma 6 *For every integer $q \geq 3$, the broadcast time for the q-pod is at most $\lceil 3q/2 \rceil - 1$.*

Proof: Let $q \geq 3$, and let D_q denote the q-pod on vertices $0, 1, 2, \ldots 2^q$. We need to show that for any vertex $x \in V(D_q)$, $b(x) \leq \lceil 3q/2 \rceil - 1$. The proof is by induction on q. For $q = 3$, it is routine to verify that every vertex in D_3 has broadcast time $4 = \lceil 3q/2 \rceil - 1$; similarly, for $q = 4$, one can check that every vertex in D_4 has broadcast time $5 = \lceil 3q/2 \rceil - 1$.

We partition the vertices of D_q into levels as follows: a vertex v is in level zero if $v \equiv 1 \pmod 2$; v is in level one if $v \equiv 2 \pmod 4$; v is in level two if $v \equiv 0 \pmod 4$. Since every vertex v is adjacent to $v-2, v-1, v+1$ and $v+2$, it is clear that every vertex at level zero and every vertex at level one is adjacent to a vertex at level two.

Let x be any vertex in D_q, and consider the broadcast originating at x. If x is a level zero or level one vertex, then during the first time unit, x informs a level two neighbour, n_x; otherwise, x is a level two vertex, and we let $n_x = x$. A $(q-2)$-pod, D_{q-2}, can be obtained from D_q by suppressing all vertices at levels zero and one, and deleting the resulting loops and multiple edges. Then n_x is a vertex in D_{q-2}, and by the induction hypothesis, $b(n_x) \leq \lceil 3(q-2)/2 \rceil - 1$ in D_{q-2}. This means that in D_q, all level two vertices can be informed in at most $\lceil 3(q-2)/2 \rceil - 1$ time units from the broadcast originating at n_x. Therefore, in $\lceil 3(q-2)/2 \rceil$ time units, all level two vertices have been informed. At this point, it is easy to see that the level one and level zero vertices can all be informed in just two more time units. Therefore, $b(x) \leq \lceil 3(q-2)/2 \rceil + 2 = \lceil 3q/2 \rceil - 1$. The result now follows by induction. ■

The authors are grateful to B. Bauslaugh for simplifying the original proof of Lemma 6.

Proof of Theorem 4: We will describe a broadcast scheme for E_{t-1}, showing that each vertex has broadcast time t. There are a number of cases to consider.

Case 1: First, suppose that the broadcast originates at ∞. During the first unit of time, ∞ informs 0, and during the second unit of time 0 informs 2^{t-2} and ∞ informs 2^{t-1}. By Lemma 5, vertex 0 can originate a broadcast that informs $1, 2, \ldots 2^{t-2} - 1$ in the remaining $t-2$ time units; Similarly, vertex 2^{t-2} can originate a broadcast that informs $(2^{t-2}+1), (2^{t-2}+2), \ldots (2^{t-1}-1)$ in the remaining $t-2$ time units. As well, since $\lfloor 2t/3 \rfloor \leq t-2$, vertex 2^{t-1} can originate a broadcast to inform $(2^{t-1} + 1), (2^{t-1}+2), \ldots, (2^{t-1} + 2^{\lfloor 2t/3 \rfloor} - 1)$ in the remaining $t-2$ time units. Therefore, $b(\infty) = t$.

Case 2: Now suppose the broadcast originates at one of vertices 0, 2^{t-2} or 2^{t-1}. If the broadcast originates at vertex 0, then the first call is from 0 to ∞, and the broadcast then proceeds as in Case 1. By the symmetry of E_{t-1}, a broadcast originating at 2^{t-1} is identical to one originating at vertex 0, and so can be completed in t time units. Finally, suppose the broadcast originates at 2^{t-2}; the first call is from 2^{t-2} to ∞. During the second unit of time, 2^{t-2} informs 0 and ∞ informs 2^{t-1}. Thus after the first two time units, vertices ∞, 0, 2^{t-2} and 2^{t-1} are the only informed vertices, as in Case 1, and thus the broadcast can proceed in the same fashion. Therefore, $b(0) = b(2^{t-2}) = b(2^{t-1}) = t$.

Case 3: We now consider the case where the broadcast originates at vertex x, where $2^{t-1} + 1 \le x \le 2^{t-1} + 2^{\lfloor 2t/3 \rfloor} - 1$. The first call is from x to ∞. During the second time unit, x can originate a broadcast to inform all the vertices in the $\lfloor 2t/3 \rfloor$-pod on vertices $2^{t-1}, (2^{t-1}+1), \ldots, (2^{t-1} + 2^{\lfloor 2t/3 \rfloor} - 1), 0$. By Lemma 6, this requires $\lceil 3\lfloor 2t/3 \rfloor /2 \rceil$ time units, and $\lceil 3\lfloor 2t/3 \rfloor /2 \rceil \le t-1$, so this can be accomplished in the remaining $t-1$ time units. All remaining vertices can all be informed by a broadcast originating at ∞ during the second unit of time. For $2 \le k \le t$, ∞ informs 2^{t-k} during the k^{th} time unit. Notice that 2^{t-k} is the endpoint of a $(t-k)^*$-pod on vertices $2^{t-k}, (2^{t-k}+1), \ldots, (2^{t-k+1}-1), 2^{t-k+1}-1$, and by Lemma 5, vertex 2^{t-k} can originate a broadcast to inform $(2^{t-k}+1), (2^{t-k}+2), \ldots, (2^{t-k+1}-1)$ in $(t-k)$ time units. After t time units, all vertices have been informed, so $b(x) = t$.

Case 4: The final case is when the broadcast originates at a vertex x not covered by one of the previous three cases. If $t = 6$, then $t-2 = \lfloor 2t/3 \rfloor$, and E_{t-1} has threefold symmetry; it follows from the previous cases that $b(x) = t$ for every vertex x, and we may therefore assume that $t \ge 7$. In this case, $\lfloor 2t/3 \rfloor = t - m$ for some $m \ge 3$. The symmetry of E_{t-1} allows us to assume, without loss of generality, that $0 < x < 2^{t-2}$. Express each vertex x in binary, as a string of length $t-2$; i.e., $x = x_1 x_2 \ldots x_{t-2}$.

During the first time unit, x informs ∞, and during the remaining $t-1$ time units, the broadcast originating at x can inform all vertices in the $\lfloor 2t/3 \rfloor$-pod with endpoints $x_1 x_2 \ldots x_{m-2} 0^{t-m}$ and $x_1 x_2 \ldots x_{m-2} 0^{t-m} + 2^{\lfloor 2t/3 \rfloor}$ (by Lemma 6). We will now describe the calls that ∞ makes in the remaining $t-1$ time units. There are two cases to consider, according as $x_{m-2} = 0$ or $x_{m-2} = 1$.

First suppose that $x_{m-2} = 0$. Then during the second time unit, ∞ informs vertex 2^{t-2}; during the k^{th} time unit, $3 \le k \le (m-1)$, ∞ informs vertex

$$x_1 x_2 \ldots x_{k-3}(1 - x_{k-2})0^{t-k}.$$

During the m^{th} time unit, ∞ informs 2^{t-1}. The vertex informed by ∞ during time k, $2 \le k \le m$, is the endpoint of a $(t-k)^*$-pod, and by Lemma 5, this vertex can inform all vertices in the $(t-k)^*$-pod in the remaining $t-k$ time units. The only vertices that will not be informed are the vertices inside the $(t-m)$-pod with endpoints $x_1 x_2 \ldots x_{m-2} 0^{t-m} + 2^{t-m}$ and $x_1 x_2 \ldots x_{m-2} 0^{t-m} + 2^{t-m+1}$ (the endpoints of this $(t-m)$-pod are already taken care of). These vertices can be informed by a broadcast, originating at infinity, during times $(m+1), (m+2), \ldots (t-1), t$ as follows: during time k, $m+1 \le k \le t$, ∞ informs vertex

$$x_1 x_2 \ldots x_{m-3} 1^{k-m+1} 0^{t-k}.$$

Each of these vertices is the endpoint of a $(t-k)^*$-pod, all of whose vertices can be informed in the remaining $t-k$ time units.

We now assume that $x_{m-2} = 1$. Then during the second time unit, ∞ informs vertex 2^{t-1}; during the k^{th} time unit, $3 \le k \le (m-1)$, ∞ informs vertex

$$\begin{array}{ll} x_1 x_2 \ldots x_{k-2} 0^{t-k} + 2^{t-k-1} & \text{if } x_k = 0 \\ x_1 x_2 \ldots x_{k-2} 0^{t-k} & \text{if } x_k = 1. \end{array}$$

During the m^{th} time unit, ∞ informs vertex 0. The vertex informed by ∞ during time k, $2 \le k \le m$, is the endpoint of a $(t-k)^*$-pod, and by Lemma 5, this vertex can inform all vertices in the $(t-k)$-pod in the remaining $t-k$ time units. The only vertices that will not be informed are the vertices inside the $(t-m)$-pod with endpoints $x_1 x_2 \ldots x_{m-3} 0^{t-m+1}$ and $x_1 x_2 \ldots x_{m-3} 1 0^{t-m}$ (the endpoints of this $(t-m)$-pod are already taken care of). These vertices can be informed by a broadcast, originating at infinity, during times $(m+1), (m+2), \ldots (t-1), t$ as follows: during time k, $m+1 \le k \le t$, ∞ informs vertex

$$x_1 x_2 \ldots x_{m-3} 0 1^{k-m} 0^{t-k}.$$

Again, each of these vertices is the endpoint of a $(t-k)^*$-pod, all of whose vertices can be informed in the remaining $t-k$ time units. Therefore, $b(x) = t$, and this completes the proof of the theorem. ∎

4 Future directions

For $t \ge 6$, we have shown that

$$2^{t-1} + 2^{\lfloor 2t/3 \rfloor} + 1 \le B_p(t) \le 2^t - 1.$$

The upper bound for $B_p(t)$ can be improved to

$$2^{t-1} + 2^{t-2} + 2^{t-3} + 2^{t-4} + 2^{t-5},$$

by simply using that fact that every planar graph contains a vertex of degree at most five. However, further improvements on this bound should be possible. The planar graph constructed to demonstrate the fact that $B_p(t) \ge 2^{t-1} + 2^{\lfloor 2t/3 \rfloor} + 1$ raises a number of interesting questions. The graph E_{t-1} contains a universal vertex; i.e., a vertex adjacent to all other vertices of the graph.

Question 1 *What is the maximum number of vertices in a planar graph G with a universal vertex and having broadcast time t?*

Question 2 *If G is a planar graph with broadcast time t and at least $2^{t-1} + 2^{t-2}$ vertices, does this imply that G contains a universal vertex?*

Answers to these questions would enable us to improve the upper bound on $B_p(t)$.

The existence of the universal vertex in E_{t-1} leads in another direction as well: that of determining the effect of restricting the maximum degree of the graph.

Problem 3 *Find large planar graphs with degree bounded by Δ in which all vertices can originate a broadcast that completes in time at most t.*

Finally, we relate our topic to the popular subject of large graphs with given diameter (cf. [1]). Suppose we change the rules for broadcasting so that a vertex is allowed to inform all its neighbours in a single time unit. Then the broadcast time from a vertex v is simply the maximum distance from v to any other vertex of the graph. To ensure that the broadcast time of the graph is at most k, it is necessary for the graph to have diameter at most k. This gives rise to the following problem.

Problem 4 *Find large planar graphs with diameter k and degree at most Δ.*

Let $p(\Delta, t)$ denote the maximum number of vertices in a planar graph with diameter t and maximum degree Δ. We have shown [5] that, for $\Delta \geq 8$, $p(\Delta, t) = \lfloor 3\Delta/2 \rfloor + 1$, and along with Fellows [6] have shown that $p(\Delta, 3)$ is roughly between 4.5Δ and 8Δ. As well, asymptotic results have been obtained for larger values of t (see [6], [7]). In general, $p(\Delta, t)$ is between a constant times $\Delta^{\lfloor t/2 \rfloor}$ and a constant times $t\Delta^{\lfloor t/2 \rfloor}$; in particular, it is of the order $\Theta(\Delta^{\lfloor t/2 \rfloor})$ for any fixed value of t. Recently, S. Kwek [8] has improved the upper bound on $p(\Delta, t)/\Delta^{\lfloor t/2 \rfloor}$ from $O(t)$ to $O(t/\Delta + \sqrt{t})$, and hopes that a further improvement to $O(1)$ may be possible.

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