Domination critical graphs with respect to relative complements

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Abstract

Let G be a spanning subgraph of $K_{s,s}$ and let H be the complement of G relative to $K_{s,s}$; that is, $K_{s,s} = G \oplus H$ is a factorization of $K_{s,s}$. The graph G is *γ-critical relative to $K_{s,s}$* if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for all $e \in E(H)$, where $\gamma(G)$ denotes the domination number of G. We investigate γ-critical graphs for small values of γ. The 2-critical graphs and 3-critical graphs are characterized. A characterization of disconnected 4-critical graphs is presented. We show that the diameter of a connected 4-critical graph is at most 5 and that this bound is sharp. The diameter of a connected γ-critical graph, $\gamma \geq 4$, is shown to be at most $3\gamma - 6$.

1 Introduction

A set D of vertices of a graph $G = (V, E)$ is a *dominating set* if every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. The graph G is said to be *γ-domination critical*, or just *γ-critical*, if $\gamma(G) = \gamma$ and $\gamma(G + e) = \gamma - 1$ for every edge e in the complement $\bar{G}$ of G. This concept of γ-critical graphs has been studied by, among others, Blitch [1], Sumner [5], Sumner and Blitch [4], and Wojcicka [6].

If G is a spanning subgraph of F, then the graph $F - E(G)$ is the *complement of G relative to F* with respect to a fixed embedding of G into F. The idea of

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a relative complement of a graph was suggested by Cockayne [2] and is studied in [3]. In this paper, we investigate domination critical graphs with respect to relative complements.

We shall assume that the complete bipartite graph $K_{s,s}$ has partite sets $\mathcal{L}$ and $\mathcal{R}$ (representing "left" and "right"), and that $G \oplus H = K_{s,s}$ is a factorization of $K_{s,s}$. (If G and H are graphs on the same vertex set but with disjoint edge sets, then $G \oplus H$ denotes the graph whose edge set is the union of their edge sets.) Notice that if there is a unique (proper) 2-coloring of the vertices of G with each color coloring s vertices, then the graph H is unique. That is, if G is uniquely embeddable in $K_{s,s}$, then H is unique. We henceforth consider only spanning subgraphs G of $K_{s,s}$ such that G is uniquely embeddable in $K_{s,s}$.

We say that G is *γ-critical relative to $K_{s,s}$* if $\gamma(G) = \gamma$ and $\gamma(G + e) = \gamma - 1$ for all $e \in E(H)$. We denote the relative complement H of G by $\overline{G}$. (The rest of this paper deals only with relative complements with the exception of $\bar{K}_2$, so confusion with complements in the ordinary sense is unlikely.) Obviously, $\gamma(G) \geq \gamma(K_{s,s})$, so if $s \geq 2$, then $\gamma(G) \geq 2$. The only 1-critical graph is therefore $G \cong K_2$ (vacuously) with $s = 1$. It is also a simple matter to characterize the 2-critical graphs. If $s \geq 2$ and G is a proper subgraph of $K_{s,s}$, then $\gamma(G + e) = 1$ for all $e \in E(\overline{G}) \neq \emptyset$, which is impossible. Hence for $s \geq 2$, the only 2-critical graph relative to $K_{s,s}$ is $K_{s,s}$.

The structure of γ-critical graphs for $\gamma \geq 3$ is more complex. For $\gamma \geq 3$, assume that G is a γ-critical graph relative to $K_{s,s}$. If u and v are non-adjacent vertices in different partite sets of G, then $\gamma(G + uv) = \gamma - 1$ and so there exists a set W of cardinality $\gamma - 1$ that dominates $G + uv$. Since W does not dominate G, it must be that exactly one of u and v, say v, belongs to W and that W dominates all of G except u. Thus $S = W - \{v\}$ is a set of cardinality $\gamma - 2$ such that $S \cup \{v\}$ dominates $G - u$ and we write $[v, S] \to u$. In particular, when we write $[v, S] \to u$ it is understood that u is not dominated by S.

In Section 2 we characterize 3-critical graphs. A characterization of disconnected 4-critical graphs is presented in Section 3. In Section 4, we show that the diameter of a connected 4-critical graph is at most 5 and that this bound is sharp. The diameter of a connected γ-critical graph, $\gamma \geq 4$, is shown in Section 5 to be at most $3\gamma - 6$.

2 3-critical graphs

The following result characterizes 3-critical graphs.

Theorem 1 *Let $K_{s,s}$ have partite sets $\mathcal{L}$ and $\mathcal{R}$. For $s \geq 2$, G is 3-critical relative to $K_{s,s}$ if and only if*

(1) *$s = 2$ and $G \cong K_2 \cup \bar{K}_2$, or*

(2) *$s \geq 3$ and there exists a partition $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}$ such that each vertex of $\mathcal{R}$ has degree $s - 1$ and is adjacent to every vertex of $\mathcal{L}_2$, and each vertex of $\mathcal{L}_1$ has degree at most $s - 2$, or*

(3) *$s \geq 3$ and there exists a partition $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}$ and $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathcal{R}$ such that*

- *Each vertex of $\mathcal{R}_1$ has degree $s-1$ and is adjacent to every vertex of $\mathcal{L}_2$;*
- *Each vertex of $\mathcal{R}_2$ has degree at most $s-2$ and dominates either $\mathcal{L}_1$ or $\mathcal{L}_2$. Furthermore, if $v \in \mathcal{R}_2$ dominates $\mathcal{L}_1$, then $deg\, v = s-2$;*
- *Each vertex of $\mathcal{L}_1$ has degree at most $s-2$ and is non-adjacent to at least one vertex of $\mathcal{R}_1$. Furthermore, if $v \in \mathcal{L}_1$ does not dominate $\mathcal{R}_2$, then $deg\, v = s-2$;*
- *Each vertex of $\mathcal{L}_2$ has degree s or $s-1$, and $\mathcal{L}_2$ contains at least one vertex of degree s.*

Proof. It is readily seen that the graphs are 3-critical. Assume that G is 3-critical. We consider two cases.

Case 1. *G has no vertex of degree s.*
For each vertex $v \in \mathcal{L}$, let $\bar{v}$ denote a vertex in $\mathcal{R}$ that is non-adjacent to v in G. Since G is 3-critical, $\gamma(G) = 3$ and $\gamma(G + v\bar{v}) = 2$. Let $\{x, y\}$ be a dominating set of $G + v\bar{v}$. Since $\{x, y\}$ is not a dominating set of G, either $x \in \{v, \bar{v}\}$ or $y \in \{v, \bar{v}\}$. Without loss of generality, we may assume that $y = v$. So $\{v, x\}$ dominates $G + v\bar{v}$. We show that $s = 2$. If $s \geq 3$, then $x \in \mathcal{R}$ and x dominates $\mathcal{L} - \{v\}$. By assumption, $deg\, x \leq s-1$ in G; consequently, $deg\, x = s-1$ and x and v are non-adjacent in G. Consider now $G + vx$. Since G is 3-critical, $\gamma(G+vx) = 2$. Furthermore, since v does not dominate $\mathcal{R} - \{x\}$ in $G + vx$, it follows that $\{x, w\}$ is a dominating set of $G + vx$ for some $w \in \mathcal{L} - \{v\}$. In particular, w dominates $\mathcal{R} - \{x\}$. However, x dominates $\mathcal{L} - \{v\}$. It follows that w dominates $\mathcal{R}$, which contradicts our assumption that G has no vertex of degree s. Hence $s = 2$. It is then readily seen that $G \cong K_2 \cup \bar{K}_2$, and so G satifies condition (1) in the statement of the theorem.

Case 2. *G has a vertex of degree s.*
Let $\Delta_{\mathcal{L}}$ ($\Delta_{\mathcal{R}}$) denote the maximum degree of a vertex of $\mathcal{L}$ ($\mathcal{R}$, respectively) in G. We may assume $\Delta_{\mathcal{L}} = s$ and that $u \in \mathcal{L}$ has degree s in G, so u dominates $\mathcal{R}$. Since $\gamma(G) = 3$, we know then that $s \geq 3$ and that $\Delta_{\mathcal{R}} \leq s-1$.

Claim 1 $\Delta_{\mathcal{R}} = s - 1$.

Proof. Let w and v be non-adjacent vertices in G with $w \in \mathcal{L}$ and $v \in \mathcal{R}$. The 3-criticality of G implies that $\gamma(G + vw) = 2$. Let D be a dominating set of $G + vw$. Either $w \in D$ or $v \in D$. Furthermore, since $s \geq 3$, one vertex of D belongs to $\mathcal{L}$ and the other to $\mathcal{R}$. If $w \in D$, then there is a vertex x of $\mathcal{R}$ that belongs to D ($x \neq v$). Since x dominates $\mathcal{L} - \{w\}$, x has degree $s-1$ in G. Hence we may assume that $v \in D$, for otherwise $\Delta_{\mathcal{R}} = s-1$. Let $\bar{v}$ denote the vertex of D that belongs to $\mathcal{L}$. Then v dominates $\mathcal{L} - \{w, \bar{v}\}$ and $\bar{v}$ dominates $\mathcal{R} - \{v\}$ in G. If $\Delta_{\mathcal{R}} < s-1$, then this shows that for each vertex $v \in \mathcal{R}$ there exists a vertex $\bar{v} \in \mathcal{L}$ that is adjacent to every vertex of $\mathcal{R}$ except for v. This would contradict the fact that $\Delta_{\mathcal{L}} = s$. Hence $\Delta_{\mathcal{R}} = s - 1$. □

We introduce the following notation. Let $\mathcal{R}_1 = \{v \in \mathcal{R} \mid deg\, v = s - 1 \text{ in } G\}$. By Claim 1, $\mathcal{R}_1 \neq \emptyset$. Let $\mathcal{L}_1$ be the set of vertices of $\mathcal{L}$ that are not adjacent to some vertex of $\mathcal{R}_1$, and let $\mathcal{L}_2 = \mathcal{L} - \mathcal{L}_1$. Since $\mathcal{R}_1 \neq \emptyset$, we know that $\mathcal{L}_1 \neq \emptyset$. Furthermore, since $u \in \mathcal{L}$ has degree s, we know that $\mathcal{L}_2 \neq \emptyset$. Moreover, every vertex of $\mathcal{L}_2$ is adjacent to every vertex of $\mathcal{R}_1$. If $\mathcal{R}_1 = \mathcal{R}$, then the graph G satisfies condition (2) in the statement of the theorem. Assume, then, that $\mathcal{R}_1 \subset \mathcal{R}$, and let $\mathcal{R}_2 = \mathcal{R} - \mathcal{R}_1$.

Claim 2 *Each $v \in \mathcal{R}_2$ dominates either $\mathcal{L}_1$ or $\mathcal{L}_2$. Furthermore, if v dominates $\mathcal{L}_1$, then $deg\, v = s - 2$.*

Proof. Let $v \in \mathcal{R}_2$ and suppose that v does not dominate $\mathcal{L}_2$. Let z be a vertex of $\mathcal{L}_2$ that is non-adjacent to v. The 3-criticality of G implies that $\gamma(G + vz) = 2$. Let $\{x, y\}$ be a dominating set of $G + vz$. We may assume that $x \in \mathcal{L}$, so $y \in \mathcal{R}$. Thus in $G + vz$, y dominates $\mathcal{L} - \{x\}$ and x dominates $\mathcal{R} - \{y\}$. Since $\{x, y\}$ is not a dominating set of G, either $x = z$ or $y = v$. If $x = z$, then y has degree $s-1$ in G and z is the only vertex of $\mathcal{L}$ that is non-adjacent to y; but then $y \in \mathcal{R}_1$ and so $z \in \mathcal{L}_1$, which produces a contradiction. Hence $y = v$ and v dominates $\mathcal{L} - \{x\}$ in $G + vz$. Thus v dominates $\mathcal{L} - \{x, z\}$ in G. This shows that $deg\, v \geq s - 2$. However since $v \in \mathcal{R}_2$, we know that $deg\, v \leq s - 2$; consequently, $deg\, v = s - 2$ and z and x are the only two vertices of $\mathcal{L}$ that are non-adjacent to v. Furthermore, since x dominates $\mathcal{R} - \{v\}$ and $v \in \mathcal{R}_2$, we know that $x \notin \mathcal{L}_1$; i.e., $x \in \mathcal{L}_2$. Thus, v dominates $\mathcal{L}_1$ in G and $deg\, v = s - 2$. □

Claim 3 *Each vertex of $\mathcal{L}_2$ has degree s or $s - 1$.*

Proof. Let $w \in \mathcal{L}_2$ and suppose that $deg\, w \leq s - 1$. Then there is a vertex v of $\mathcal{R}_2$ that is non-adjacent to w. By Claim 2, v dominates $\mathcal{L}_1$ and $deg\, v = s - 2$. Thus, v is non-adjacent to w and to exactly one other vertex z of $\mathcal{L}_2$. Since G is 3-critical, $\gamma(G + vz) = 2$. As in the proof of Claim 2, we may show that $\{v, w\}$ is a dominating set of $G + vz$. In particular, w dominates $\mathcal{R} - \{v\}$. Thus, $deg\, w = s - 1$. □

Claim 4 *Each vertex of $\mathcal{L}_1$ has degree at most $s - 2$. Furthermore, if $w \in \mathcal{L}_1$, then w dominates $\mathcal{R}_2$ or $deg\, w = s - 2$.*

Proof. Let $w \in \mathcal{L}_1$. If w has degree $s-1$, then w together with the vertex of $\mathcal{R}_1$ that is non-adjacent to w form a dominating set of G, which produces a contradiction. Thus, $deg\, w \leq s - 2$. Suppose that w does not dominate $\mathcal{R}_2$. Let v be a vertex of $\mathcal{R}_2$ that is non-adjacent to w. By Claim 2, we know that v dominates $\mathcal{L}_2$. The 3-criticality of G implies that $\gamma(G + vw) = 2$. Let $\{x, y\}$ be a dominating set of $G + vw$. We may assume that $x \in \mathcal{L}$, so $y \in \mathcal{R}$. Thus in $G + vw$, y dominates $\mathcal{L} - \{x\}$ and x dominates $\mathcal{R} - \{y\}$. Since $\{x, y\}$ is not a dominating set of G, either $x = w$ or $y = v$. We show that $x = w$. If this is not the case, then $y = v$. Thus v dominates $\mathcal{L} - \{x\}$ and x dominates $\mathcal{R} - \{v\}$ in $G + wv$. Since v dominates $\mathcal{L}_2$, we know that $x \in \mathcal{L}_1$ and therefore x is non-adjacent to at least one vertex of $\mathcal{R}_1$. Thus x cannot dominate $\mathcal{R} - \{v\}$, which produces a contradiction. Hence $x = w$ and w dominates $\mathcal{R} - \{y\}$ in $G + wv$. Thus w dominates $\mathcal{R} - \{v, y\}$ in G. This shows that $deg\, w \geq s - 2$ in G. However, w is non-adjacent to at least one vertex of $\mathcal{R}_1$ and to at least one vertex, namely v, of $\mathcal{R}_2$, so $deg\, w \leq s - 2$. Consequently, $deg\, w = s - 2$ and w is non-adjacent to exactly one vertex of $\mathcal{R}_1$ and one vertex of $\mathcal{R}_2$. □

By Claims 2, 3, and 4 the graph G satisfies condition (3) in the statement of the theorem. This completes the proof of the theorem. □

Suppose G is a graph satisfying condition (2) or (3) in the statement of Theorem 1. Since $\mathcal{L}_2 \neq \emptyset$, and $\mathcal{L}_2$ contains at least one vertex of degree s, the graph G has a unique embedding into $K_{s,s}$. Hence if G is a graph satisfying condition (1), (2) or (3) in Theorem 1, then the graph H is unique.

We will need the following characterization of 3-critical graphs relative to $K_{3,3}$.

Corollary 1 *A graph G is 3-critical graph relative to $K_{3,3}$ if and only if G is obtained from a star $K_{1,3}$ by subdividing two edges or $G \cong K_{2,3} \cup K_1$.*

Figure 1: The two 3-critical graphs relative to $K_{3,3}$.

Using an almost identical proof to that of Theorem 1, we may establish the following result:

Theorem 2 *Let $K_{s,s-1}$ have partite sets $\mathcal{L}$ and $\mathcal{R}$ where $|\mathcal{L}| = s$ and $|\mathcal{R}| = s - 1$. For $s \geq 4$, G is 3-critical relative to $K_{s,s-1}$ if and only if*

(1) *There exists a partition $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}$ such that each vertex of $\mathcal{R}$ has degree $s - 1$ and is adjacent to every vertex of $\mathcal{L}_2$, and each vertex of $\mathcal{L}_1$ has degree at most $s - 3$, or*

(2) *There exists a partition $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}$ and $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathcal{R}$ such that*

- *Each vertex of $\mathcal{R}_1$ has degree $s - 1$ and is adjacent to every vertex of $\mathcal{L}_2$;*
- *Each vertex of $\mathcal{R}_2$ has degree at most $s - 2$ and dominates either $\mathcal{L}_1$ or $\mathcal{L}_2$. Furthermore, if $v \in \mathcal{R}_2$ dominates $\mathcal{L}_1$, then $deg\, v = s - 2$;*
- *Each vertex of $\mathcal{L}_1$ has degree at most $s - 3$ and is non-adjacent to at least one vertex of $\mathcal{R}_1$. Furthermore, if $v \in \mathcal{L}_1$ does not dominate $\mathcal{R}_2$, then $deg\, v = s - 3$;*

- *Each vertex of $\mathcal{L}_2$ has degree $s-1$ or $s-2$, and $\mathcal{L}_2$ contains at least one vertex of degree $s-1$.*

Once again, if G is a graph satisfying condition (1) or (2) in the statement of Theorem 2, then, since $\mathcal{L}_2$ contains at least one vertex adjacent to every vertex of $\mathcal{R}$, the graph G has a unique embedding into $K_{s,s}$. Hence if G is a graph satisfying condition (1) or (2) in Theorem 2, then the graph H is unique.

3 Disconnected 4-critical graphs

If $s = 2$, then $\overline{K}_{2,2}$ is the only 4-critical graph relative to $K_{s,s}$. Hence in what follows we take $s \geq 3$. The following result characterizes disconnected 4-critical graphs.

Theorem 3 *For $s \geq 3$, G is a disconnected 4-critical graph relative to $K_{s,s}$ if and only if*

- *$s \geq 4$, G has exactly one isolated vertex v and $G - v$ is 3-critical relative to $K_{s,s-1}$, or*
- *$G \cong \bar{K}_2 \cup K_{s-1,s-1}$, or*
- *$s \geq 4$ and $G \cong \bar{K}_2 \cup K_{s,s-2}$, or*
- *$s \geq 4$ and $G \cong K_{2,2} \cup K_{s-2,s-2}$.*

Proof. It is readily seen that the graphs are disconnected 4-critical relative to $K_{s,s}$. Suppose that G is a disconnected 4-critical graph relative to $K_{s,s}$.

Claim 5 *If G has exactly one isolated vertex v, then $s \geq 4$ and $G - v$ is 3-critical relative to $K_{s,s-1}$.*

Proof. If $s = 3$, then the partite set of G containing v dominates G, which contradicts the fact that $\gamma(G) = 4$. Hence, $s \geq 4$. Furthermore, $4 = \gamma(G) = \gamma(G - v) + 1$, and so $\gamma(G - v) = 3$. If x and y are non-adjacent vertices of $G - v$ in different partite sets, then the 4-criticality of G implies that $3 = \gamma(G + xy) = \gamma((G - v) + xy) + 1$, so $\gamma((G - v) + xy) = 2$. Thus $G - v$ is 3-critical relative to $K_{s,s-1}$. □

In what follows we may assume that G has no isolated vertex or at least two isolated vertices, for otherwise the result follows from Claim 5. Let G_1 be a component of G of maximum size with partite sets $\mathcal{L}_1 \subseteq \mathcal{L}$ and $\mathcal{R}_1 \subseteq \mathcal{R}$. Further, let $\mathcal{L}_2 = \mathcal{L} - \mathcal{L}_1$ and $\mathcal{R}_2 = \mathcal{R} - \mathcal{R}_1$, and let G_2 be the subgraph of G induced by $\mathcal{L}_2 \cup \mathcal{R}_2$.

Claim 6 $|\mathcal{L}_1| \geq 2$ *and* $|\mathcal{R}_1| \geq 2$.

Proof. If $|\mathcal{L}_1| = 1$ and $|\mathcal{R}_1| = 1$, then $G_1 \cong K_2$ and every component of G is either K_1 or K_2. However, it is then readily seen that G is not 4-critical. Hence $|\mathcal{L}_1| \geq 2$ or $|\mathcal{R}_1| \geq 2$. We may assume that $|\mathcal{R}_1| = r_1 \geq 2$. We show that $|\mathcal{L}_1| \geq 2$. If this is not the case, then $\mathcal{L}_1 = \{w\}$ and $G_1 \cong K_{1,r_1}$. Let $u \in \mathcal{L}_2$ and let $v \in \mathcal{R}_1$, and consider the graph $G + uv$. Since G is 4-critical, there exists a set S such that $[u, S] \to v$ or $[v, S] \to u$. Since $r_1 \geq 2$, we may assume in both cases that $w \in S$, so v is dominated by S. Hence we must have $[v, S] \to u$. But then $S \cup \{u\}$ is a dominating set of G, which contradicts the fact that $\gamma(G) = 4$. Hence, $|\mathcal{L}_1| \geq 2$. □

Claim 7 *G_1 is a complete bipartite graph.*

Proof. Suppose that there are non-adjacent vertices $u \in \mathcal{L}_1$ and $v \in \mathcal{R}_1$. Since G is 4-critical, $3 = \gamma(G + uv) = \gamma(G_1 + uv) + \gamma(G_2)$. It follows from Claim 6 that $\gamma(G_1 + uv) \geq 2$. Hence $\gamma(G_1 + uv) = 2$ and $\gamma(G_2) = 1$. Thus G_1 is 3-critical, and $G_2 \cong K_1$ or $G_2 \cong K_{1,m}$. However, since G has no isolated vertex or at least two isolated vertices, $G_2 \cong K_{1,m}$. Without loss of generality, we may assume that $|\mathcal{L}_2| = 1$ and $|\mathcal{R}_2| = m$. Let $\mathcal{L}_2 = \{w\}$.

If $m > 1$, then let $x \in \mathcal{R}_2$, and consider the graph $G + ux$. Since G is 4-critical, there exists a set S such that $[u, S] \to x$ or $[x, S] \to u$. Since $m \geq 2$, we may assume in both cases that $w \in S$, so x is dominated by S. Hence we must have $[x, S] \to u$. But then $S \cup \{u\}$ is a dominating set of G, which contradicts the fact that $\gamma(G) = 4$. Hence, $m = 1$ and $G_2 \cong K_{1,1}$.

Since $G_2 \cong K_{1,1}$, G_1 is 3-critical relative to $K_{s-1,s-1}$. Hence, by Theorem 1, $s - 1 \geq 3$ and G_1 has a vertex z of maximum possible degree $s - 1$. Without loss of generality, we may assume $z \in \mathcal{L}_1$. Let $\mathcal{R}_2 = \{x\}$ and consider the graph $G + zx$. Since G is 4-critical, there exists a set T such that $[z, T] \to x$ or $[x, T] \to z$. If $[z, T] \to x$, then w must belong to T and x is therefore dominated by T, contrary to assumption. Hence, $[x, T] \to z$. Since z is not dominated by T, no vertex of $\mathcal{R}_1$ belongs to T. It follows that $|\mathcal{L}_1| = 3$ (so $s = 4$ and G_1 is a connected 3-critical graph relative to $K_{3,3}$) and T contains the two vertices of $\mathcal{L}_1$ different from z. However, by Corollary 1, $\mathcal{R}_1$ contains a vertex of degree exactly 1 which is adjacent only to z. Hence $T \cup \{x\}$ does not dominate $G + zx$, which produces a contradiction. We deduce, therefore, that G_1 is a complete bipartite graph. □

Claim 8 *Either G_2 is a complete bipartite graph and both partite sets have cardinality at least 2 or $G_2 \cong \bar{K}_2$.*

Proof. Since G is 4-critical, $4 = \gamma(G) = \gamma(G_1) + \gamma(G_2)$. By Claims 6 and 7, $\gamma(G_1) = 2$, so $\gamma(G_2) = 2$ and G_2 is 2-critical. By assumption G has no isolated vertex or at least two isolated vertices. Thus either $G_2 \cong \bar{K}_2$ or G_2 has no isolated vertex. If G_2 has no isolated vertex, then we show that G_2 is a complete bipartite graph. If this is not the case, then there are non-adjacent vertices $u \in \mathcal{L}_2$ and $v \in \mathcal{R}_2$. Since G_2 is 2-critical, either u or v dominates $G_2 + uv$. If u dominates $G_2 + uv$, then $\mathcal{L}_2 = \{u\}$ and v is isolated in G_2, while if v dominates $G_2 + uv$, then $\mathcal{R}_2 = \{v\}$ and u is isolated in G_2. Both cases contradict the assumption that G_2 has no isolated vertex. We deduce, therefore, that either $G_2 \cong \bar{K}_2$ or G_2 is a complete bipartite graph. Furthermore, if G_2 is a complete bipartite graph, then since $\gamma(G_2) = 2$, each partite set has cardinality at least 2. □

If $G_2 \cong \bar{K}_2$, then either $G \cong \bar{K}_2 \cup K_{s-1,s-1}$ or $G \cong \bar{K}_2 \cup K_{s,s-2}$ and $s \geq 4$. If $G_2 \ncong \bar{K}_2$, then, by Claim 8, G_2 is a complete bipartite graph and both partite sets have cardinality at least 2. If $|\mathcal{L}_2| > 2$, then the graph obtained by adding to G any edge between $\mathcal{L}_2$ and $\mathcal{R}_1$ has domination number 4, contradicting the 4-criticality of G. Consequently, $|\mathcal{L}_2| = 2$. Similarly, $|\mathcal{R}_2| = 2$. Hence if $G_2 \ncong \bar{K}_2$, then $G \cong K_{2,2} \cup K_{s-2,s-2}$ and $s \geq 4$. This completes the proof of the theorem. □

If G is a graph described in the statement of Theorem 3, then it is evident that the graph H is unique.

4 Connected 4-critical graphs

A characterization of connected 4-critical graphs seems to be difficult to obtain. In this section, we show that the diameter of a connected 4-critical graph is at most 5.

Theorem 4 *The diameter of a connected* 4*-critical graph is at most* 5.

Proof. Let $K_{s,s}$ have partite sets $\mathcal{L}$ and $\mathcal{R}$. Let G be a connected 4-critical graph relative to $K_{s,s}$ having diameter m where $m \geq 6$. Let a and b be vertices of G with $diam\, G = d(a,b) = m$. Let $a = v_0, v_1, \ldots, v_m = b$ be a shortest a-b path. For $i = 0, 1, \ldots, m$, let $V_i = \{x \,|\, d(a,x) = i\}$. Necessarily $V_0 = \{a\}$ and $v_i \in V_i$ for $i = 1, 2, \ldots, m$. Without loss of generality, we may assume that $V_i \subset \mathcal{L}$ for i odd and $V_i \subset \mathcal{R}$ for i even. Since v_2 and v_5 are non-adjacent vertices in different partite sets of G, there exists a set S such that $[v_2, S] \to v_5$ or $[v_5, S] \to v_2$. We consider the two possibilities in turn.

Case 1. $[v_2, S] \to v_5$.
Since v_2 is adjacent to no vertex of $V_0 \cup V_4 \cup V_6$, S must contain a vertex x of $V_0 \cup V_1$ and a vertex z of V_5. Thus $m = 6$. Since S does not contain v_5, the vertices v_5 and z are distinct. Furthermore, z dominate $V_5 - \{v_5\}$, and so V_5 consists of only the two vertices v_5 and z. The vertex z dominates V_4 and V_6, while v_2 dominates V_3. Before proceeding further, we prove four claims.

Claim 9 $|V_6| = 1$.

Proof. Suppose $|V_6| \geq 2$. Consider the graph $G + v_0v_3$. There exists a set T' such that $[v_0, T'] \to v_3$ or $[v_3, T'] \to v_0$. In order to dominate $V_5 \cup V_6$, T' must consist of two vertices from $V_4 \cup V_5 \cup V_6$ since $|V_5| = 2$ and $|V_6| \geq 2$. This implies that one of v_0 or v_3 must dominate $V_1 \cup V_2$, which is impossible. □

Thus V_6 consists only of the vertex v_6 which dominates V_5.

Claim 10 $|V_2| = 1$.

Proof. Suppose $|V_2| \geq 2$. Since v_0 and v_5 are non-adjacent vertices in different partite sets of G, there exists a set T such that $[v_0, T] \rightarrow v_5$ or $[v_5, T] \rightarrow v_0$. If $[v_5, T] \rightarrow v_0$, then in order to dominate z, T must contain a vertex of $V_4 \cup V_6 \cup \{z\}$. The remaining vertex t of T must therefore dominate $V_1 \cup V_2$. Since $|V_2| \geq 2$, t must belong to V_1. But then T dominates v_0, which is impossible. On the other hand, if $[v_0, T] \rightarrow v_5$, then in order to dominate $V_6 \cup \{z\}$ we may assume without loss of generality that $z \in T$. The remaining vertex t of T must therefore dominate $V_2 \cup V_3$. Since $|V_2| \geq 2$, t must belong to V_3. Thus V_3 consists only of the vertex t (so $t = v_3$), and v_3 dominates $V_2 \cup V_3 \cup V_4$. Thus $|V_6| \geq 2$, for otherwise $\{v_0, v_3, v_6\}$ dominates G. But this contradicts Claim 9. We deduce, therefore, that $|V_2| = 1$. □

Claim 11 $|V_3| \geq 2$ *and* $|V_4| \geq 2$.

Proof. If V_3 consists only of the vertex v_3, then $\{v_0, v_3, v_6\}$ dominates G, which is impossible. Thus $|V_3| \geq 2$ and so $s = |\mathcal{L}| \geq 5$. By Claims 10 and 9, we have $s = |\mathcal{R}| = |V_4| + 3$. Hence we must have $|V_4| \geq 2$. □

Claim 12 $|V_1| = 1$.

Proof. Suppose $|V_1| \geq 2$. Consider the graph $G + v_1v_6$. There exists a set T such that $[v_1, T] \rightarrow v_6$ or $[v_6, T] \rightarrow v_1$. If $[v_1, T] \rightarrow v_6$, then in order to dominate $V_1 - \{v_1\}$ the set T must contain a vertex of $V_0 \cup V_1 \cup V_2$ different from v_1. But then the remaining element of T must dominate $V_4 \cup V_5$, which is impossible since $|V_5| = 2$ and by Claim 11, $|V_4| \geq 2$. On the other hand, if $[v_6, T] \rightarrow v_1$, then in order to dominate v_0 the set T must contain a vertex of V_1 different from v_1. But then the remaining element of T must dominate $V_3 \cup V_4$, which is impossible since by Claim 11, $|V_3| \geq 2$ and $|V_4| \geq 2$. □

We can now continue with the proof of Case 1. Consider the graph $G + v_1v_4$. There exists a set T such that $[v_1, T] \rightarrow v_4$ or $[v_4, T] \rightarrow v_1$. If $[v_4, T] \rightarrow v_1$, then in order to dominate v_0, T must contain v_0 since by Claim 12 the set V_1 consists only of v_1. But then T dominates v_1, which is impossible. On the other hand, if $[v_1, T] \rightarrow v_4$, then since T does not dominate v_4, no vertex of V_5 belongs to T. Thus in order to dominate v_6, T must contain v_6. In order to dominate V_3 and $V_4 - \{v_4\}$, T must therefore contain a vertex t of $V_4 - \{v_4\}$ since $|V_3| \geq 2$. Thus, V_4 consists of only v_4 and t, and t dominates V_3. But then $\{v_1, t, v_5\}$ dominates G, which is impossible.

Case 2. $[v_5, S] \rightarrow v_2$.

In order to dominate v_0, S must contain a vertex of $V_0 \cup V_1$, and in order to dominate V_3, S must also contain a vertex of $V_2 \cup V_3 \cup V_4$. Thus $m = 6$. Before proceeding further, we prove seven claims.

Claim 13 $|V_1| \geq 2$.

Proof. Suppose V_1 consists only of the vertex v_1. Then without loss of generality, we may assume that $v_1 \in S$. But then S dominates v_2, which is impossible. □

Claim 14 $|V_6| = 1$.

Proof. Suppose $|V_6| \geq 2$. Consider the graph $G + v_1v_6$. There exists a set T such that $[v_1, T] \to v_6$ or $[v_6, T] \to v_1$. If $[v_1, T] \to v_6$, then in order to dominate $V_1 - \{v_1\}$ the set T must contain a vertex of $V_0 \cup (V_1 - \{v_1\}) \cup V_2$. The remaining element of T must dominate $V_4 \cup V_5 \cup (V_6 - v_6)$. This is only possible if $|V_5| = 1$ and v_5 belongs to T. But then T dominates v_6, which is impossible. On the other hand, if $[v_6, T] \to v_1$, then in order to dominate v_0 the set T must contain a vertex of V_1 different from v_1. The remaining element of T must dominate $V_3 \cup V_4 \cup (V_6 - v_6)$, which is impossible. □

Claim 15 $|V_5| = 1$.

Proof. Suppose $|V_5| \geq 2$. Then in order to dominate $V_3 \cup (V_5 - \{v_5\})$, S must contain a vertex of V_4. Thus S contains no vertex of V_2. Hence in order to dominate $V_0 \cup V_1$, S must contain v_0 since by Claim 13, $|V_1| \geq 2$. Thus V_2 consists only of the vertex v_2. Consequently, by Claim 14 we have $s = |\mathcal{R}| = |V_4| + 3$. By Claim 13, $|V_1| \geq 2$ and by assumption, $|V_5| \geq 2$. It follows that $s = |\mathcal{L}| \geq 5$. Thus $|V_4| \geq 2$. Consider now the graph $G + v_1v_6$. There exists a set T such that $[v_1, T] \to v_6$ or $[v_6, T] \to v_1$. If $[v_1, T] \to v_6$, then in order to dominate $V_1 - \{v_1\}$ the set T must contain a vertex of $V_0 \cup (V_1 - \{v_1\}) \cup V_2$. The remaining element of T must dominate $V_4 \cup V_5$. This however is impossible since $|V_4| \geq 2$ and $|V_5| \geq 2$. On the other hand, if $[v_6, T] \to v_1$, then in order to dominate v_0 the set T must contain a vertex of $V_0 \cup (V_1 - \{v_1\})$. The remaining element t of T must dominate $V_3 \cup V_4$. Since $|V_4| \geq 2$, it follows that $t \in V_3$ and $|V_3| = 1$. Thus, since $|V_2| = 1$, v_3 dominates $V_2 \cup V_4$. Since $T \cup \{v_6\}$ dominates $G + v_1v_6$, v_6 must therefore dominate V_5. But then $\{v_0, v_3, v_6\}$ dominates G, which is impossible. □

Claim 16 $|V_1| = 2$ *and every vertex of* V_1 *is adjacent to every vertex of* V_2.

Proof. Consider the graph $G + v_1v_4$. There exists a set T such that $[v_1, T] \to v_4$ or $[v_4, T] \to v_1$. If $[v_1, T] \to v_4$, then in order to dominate v_6 the set T contains v_5 or v_6 since $|V_5| = |V_6| = 1$. We may however assume that $v_5 \in T$ (if $v_6 \in T$, then we replace v_6 by v_5). But then v_4 is dominated by T, a contradiction. Hence $[v_4, T] \to v_1$. In order to dominate v_6, we may once again take v_5 to be in T. The remaining element t of T must therefore dominate $V_0 \cup (V_1 - \{v_1\}) \cup V_2$. This is only possible if t belongs to $V_1 - \{v_1\}$ and if $|V_1| = 2$. Furthermore, t dominates V_2. If we now consider the graph $G + tv_4$, then a similar argument shows that v_1 dominates V_2. □

Claim 17 $|V_3| \geq 2$.

Proof. Suppose $|V_3| = 1$. Then, since $|V_1| = 2$ (by Claim 16) and $|V_5| = 1$ (by Claim 15), $s = |\mathcal{L}| = 4$. It follows that $|V_2| = |V_4| = 1$ (for otherwise $s = |\mathcal{R}| > 4$). But then $\gamma(G) = 3$ (for example, $\{v_0, v_2, v_5\}$ dominates G). This produces a contradiction. Hence $|V_3| \geq 2$. □

Claim 18 $|V_4| = 1$.

Proof. Suppose $|V_4| \geq 2$. Consider now the graph $G+v_1v_6$. There exists a set T such that $[v_1, T] \to v_6$ or $[v_6, T] \to v_1$. If $[v_1, T] \to v_6$, then in order to dominate $V_1 - \{v_1\}$ the set T must contain a vertex of $V_0 \cup (V_1 - \{v_1\}) \cup V_2$. The remaining element of T must dominate $V_4 \cup V_5$. Since $|V_4| \geq 2$, T must therefore contain the vertex v_5. But then T dominates v_6, which is impossible. On the other hand, if $[v_6, T] \to v_1$, then in order to dominate v_0 the set T must contain a vertex of V_1 different from v_1. The remaining element of T must dominate $V_3 \cup V_4$. This, however, is impossible since by assumption, $|V_4| \geq 2$ and by Claim 17, $|V_3| \geq 2$. □

Claim 19 $|V_2| \geq 2$.

Proof. By Claims 14 and 18, we have $s = |\mathcal{R}| = |V_2| + 3$. By Claims 13, 15, and 17, we have $s = |\mathcal{L}| \geq 5$. Thus $|V_2| \geq 2$. □

We can now continue with the proof of Case 2. In order to dominate v_0, S must contain a vertex of $V_0 \cup V_1$. If S contains a vertex of V_1, then, by Claim 16, v_2 is dominated by S, a contradiction. Hence S must contain the vertex v_0. Thus S must contain a vertex s that dominates $V_3 \cup (V_2 - \{v_2\})$. Since $|V_3| \geq 2$, it follows that $s \in V_2 - \{v_2\}$ and $|V_2| = 2$. In particular, s dominates V_3. But then $\{v_1, s, v_5\}$ dominates G, which is impossible. This completes the proof of Case 2 and therefore of the theorem. □

That the bound given in Theorem 4 is sharp, may be seen by considering the connected 4-critical graph relative to $K_{5,5}$ with diameter 5 shown in Figure 2.

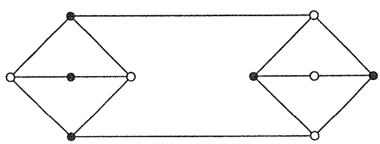

Figure 2: A connected 4-critical graph relative to $K_{5,5}$ with diameter 5.

5 Connected γ-critical graphs

We conclude with a bound on the diameter of any connected γ-critical graph having $\gamma \geq 4$.

Theorem 5 *The diameter of a connected γ-critical graph, $\gamma \geq 4$, is at most $3\gamma - 6$.*

Proof. Theorem 4 shows that the diameter of a 4-critical graph is at most $5 < 3\gamma - 6$. Hence we assume G is a connected γ-critical graph relative to $K_{s,s}$ having $\gamma \geq 5$ and diameter m where $m \geq 3\gamma - 6 \geq 9$. Let a and b be vertices of G with $diam\, G = d(a, b) = m$. Let $a = v_0, v_1, \ldots, v_m = b$ be a shortest a-b path. Without loss of generality, let $v_i \in \mathcal{L}$ for i odd and $v_i \in \mathcal{R}$ for i even. Since v_2 and v_5 are non-adjacent vertices in different partite sets of G, there exists a set S of cardinality $\gamma - 2$ such that $[v_2, S] \to v_5$ or $[v_5, S] \to v_2$. We consider the two possibilities.

Case 1. $[v_2, S] \to v_5$.

Then S must contain at least one vertex x to dominate v_0, and at least one additional vertex y to dominate v_4. Now the set $\{x, y, v_2\}$ does not dominate any of the vertices $v_7, v_8, \ldots, v_m$ on the a-b path. Hence, these $m-6$ vertices must be dominated by the remaining $\gamma - 4$ vertices in $S - \{x, y\}$. Since no vertex in S can dominate more than three consecutive vertices of the a-b path, we have $m - 6 \leq 3(\gamma - 4)$; or, equivalently, $m \leq 3\gamma - 6$.

Case 2. $[v_5, S] \to v_2$.

Then S must contain at least one vertex x to dominate v_0, and at least one additional vertex y to dominate v_3. Now the set $\{x, y, v_5\}$ does not dominate any of the vertices $v_7, v_8, \ldots, v_m$ on the a-b path. As in Case 1, the remaining $\gamma - 4$ vertices in $S - \{x, y\}$ must dominate the $m - 6$ vertices $v_7, v_8, \ldots, v_m$ on the a-b path, so $m \leq 3\gamma - 6$. □

6 Acknowledgement

We thank the referee for pointing out an error in Theorem 2 of [3]. That is, if $G \oplus H = K_{s,s}$ is a factorization of $K_{s,s}$, then the graph H is **not** necessarily unique.

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