Constructions of Nested Directed BIB Designs

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Abstract

A directed BIB design $\mathrm{DB}(k, \lambda; v)$ is a BIB design $\mathrm{B}(k, 2\lambda; v)$ in which the blocks are transitively ordered k-tuples and each ordered pair of elements occurs in exactly λ blocks. A nested directed BIB design $\mathrm{NDB}(k, \lambda; v)$ of form $\Pi_{2\leq n\leq k-1}(n^{j_n}, \lambda_n)^{i_n}$ is a $\mathrm{DB}(k, \lambda; v)$ where each block contains $\sum_{2\leq n\leq k-1} i_n j_n$ mutually disjoint subblocks, $i_n j_n$ subblocks of which are partitioned into i_n mutually disjoint families of j_n subblocks of size n and the j_n subblocks of size n belong to one distinguished system which forms the collection of blocks of a $\mathrm{DB}(n, \lambda_n; v)$. In this paper we will use known and new techniques to show the existence of all $\mathrm{NDB}(k, \lambda; v)$ of the form $\Pi_{2\leq n\leq k-1}(n^{j_n}, \lambda_n)^{i_n}$ for $k = 4$ and 5.

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1. Introduction

A *balanced incomplete block* (BIB) *design* (or BIBD) $B(k, \lambda; v)$ is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a set of v elements, $\mathcal{B}$ is a collection of k-subsets, called *blocks*, of $\mathcal{V}$ such that every pair of distinct elements of $\mathcal{V}$ occurs in exactly λ blocks of $\mathcal{B}$.

Hung and Mendelsohn [8] first introduced the concept of directed BIB designs. These designs have been further studied since then, see, for example, Bennett and Mahmoodi [3], Bennett, Wei, Yin and Mahmoodi [5], Colbourn and Rosa [6], Seberry and Skillicorn [12], Street and Seberry [13], Street and Wilson [14]. A *directed BIB design* (or DBIBD) with parameters v, k and λ, denoted by $DB(k, \lambda; v)$, is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a set of v elements and $\mathcal{B}$ is a set of transitively ordered k-tuples, called blocks, of $\mathcal{V}$, such that every ordered pair of elements of $\mathcal{V}$ appears in exactly λ blocks of $\mathcal{B}$, where a transitively ordered k-tuple $(x_1, \ldots, x_k)$ is defined to be the set $\{(x_i, x_j) : 1 \leq i < j \leq k\}$ consisting of $k(k-1)/2$ ordered pairs. If we ignore the order in the blocks, a $DB(k, \lambda; v)$ becomes a $B(k, 2\lambda; v)$. In fact, a $DB(k, \lambda; v)$ is a $B(k, 2\lambda; v)$ in which the blocks are regarded as transitively ordered k-tuples and in which each ordered pair of distinct elements occurs in eactly λ blocks. A pair $\{x, y\}$ is said to occur in a block if x is written to the left of y.

A *nested BIB design* (or NBIBD) $NB(k, \lambda; v)$ of form $\Pi_{2 \leq n \leq k-1}(n^{j_n}, \lambda_n)^{i_n}$ is a $B(k, \lambda; v)$ $(\mathcal{V}, \mathcal{B})$ where each block contains $\sum_{2 \leq n \leq k-1} i_n j_n$ mutually disjoint subblocks, $i_n j_n$ subblocks of which are partitioned into i_n mutually disjoint families of j_n subblocks of size n, and the j_n subblocks of size n belong to one distinguished system $\mathcal{B}_n(\ell)$, $1 \leq \ell \leq i_n$, such that $(\mathcal{V}, \mathcal{B}_n(\ell))$ forms a $B(n, \lambda_n; v)$ for each integer n with $i_n \geq 1$.

A *nested directed BIB design* (or NDBIBD) $NDB(k, \lambda; v)$ of form $\Pi_{2 \leq n \leq k-1}$ $(n^{j_n}, \lambda_n)^{i_n}$ is a $DB(k, \lambda; v)$ $(\mathcal{V}, \mathcal{B})$ where each block contains $\sum_{2 \leq n \leq k-1} i_n j_n$ mutually disjoint subblocks, $i_n j_n$ subblocks of which are partitioned into i_n mutually disjoint families of j_n subblocks of size n and the j_n subblocks of size n belong to one distinguished system $\mathcal{B}_n(\ell)$, $1 \leq \ell \leq i_n$, such that $(\mathcal{V}, \mathcal{B}_n(\ell))$ forms a $DB(n, \lambda_n; v)$ for each integer n with $i_n \geq 1$.

An example of an NDBIBD is illustrated. As a set of 10 elements let $\mathcal{V} = Z_9 \cup \{\infty\}$ and as a collection of 4-subsets of $\mathcal{V}$ take

$$\mathcal{B} = \{(\underline{0}, \underline{\underline{1}}, \underline{3}, \underline{\underline{8}}),\ (\underline{0}, \underline{4}, \underline{\underline{1}}, \underline{\underline{3}}),\ (\underline{0}, \underline{5}, \underline{\underline{3}}, \underline{\underline{2}}),\ (\underline{\infty}, \underline{0}, \underline{\underline{4}}, \underline{\underline{5}}),\ (\underline{0}, \underline{\underline{7}}, \underline{\underline{4}}, \underline{\infty}) \mod 9\},$$

where the elements underlined "$\underline{\ }$" and "$\underline{\underline{\ }}$" within a block form two subblocks belonging to the same system $DB(2, 1; 10)$. Then $(\mathcal{V}, \mathcal{B})$ is an $NDB(4, 3; 10)$ of form $(2^2, 1)^1$.

The following necessary conditions for the existence of an $NDB(k, \lambda; v)$ of form $\Pi_{2 \leq n \leq k-1}(n^{j_n}, \lambda_n)^{i_n}$ have been established in [10]: For all integers n with $i_n \geq 1$,

$$\lambda = k(k-1)\frac{\lambda_n}{n(n-1)j_n}, \quad 2\lambda(v-1) \equiv 0 \bmod (k-1),$$
$$2\lambda v(v-1) \equiv 0 \bmod k(k-1), \quad 2\lambda_n(v-1) \equiv 0 \bmod (n-1), \tag{1.1}$$
$$2\lambda_n v(v-1) \equiv 0 \bmod n(n-1).$$

Nested directed BIB designs with parameters satisfying (1.1) are said to be *admissible.* All admissible NDBIBDs with block sizes 3 and 4 are constructed in [10] except possibly for an NDB(4, 2; 10) of form $(3,1)^1$ as the following shows.

Theorem 1.1 [10]. *The necessary conditions for the existence of an* NDB$(k, \lambda; v)$ *of any possible form are also sufficient for* $k = 3$ *and* 4 *with one possible exception: an* NDB(4, 2; 10) *of form* $(3,1)^1$.

The purpose of this paper is to show the existence of an NDB(4, 2; 10) of form $(3,1)^1$ and all admissible NDBIBDs with $k = 5$ by using known and new techniques.

In Sections 2 to 5, some constructions of NDBIBDs will be introduced. In Section 6, the existence of an NDB(4, 2; 10) of form $(3,1)^1$ will be shown.

There are six possible forms for an NDB$(5, \lambda; v)$, i.e. $(4, \lambda_4)^1$, $(3, \lambda_3)^1(2, \lambda_2)^1$, $(3, \lambda_3)^1$, $(2, \lambda_2)^2$, $(2, \lambda_2)^1$, $(2^2, \lambda_2)^1$. However, since the existence of an NDB$(5, \lambda; v)$ of form $(3, \lambda_3)^1(2, \lambda_2)^1$ implies the existence of an NDB$(5, \lambda; v)$ of form $(3, \lambda_3)^1$ and the existence of an NDB$(5, \lambda; v)$ of form $(2, \lambda_2)^2$ implies the existence of an NDB$(5, \lambda; v)$ of form $(2, \lambda_2)^1$, the designs of the remaining four forms will be treated in each of Sections 7 to 10, i.e. the existence of NDB$(5, \lambda; v)$ of form $(4, \lambda_4)^1$, $(3, \lambda_3)^1(2, \lambda_2)^1$, $(2, \lambda_2)^2$ and $(2^2, \lambda_2)^1$.

The main result of this paper will be given in the last section.

2. Constructions from GDD

Let $\mathcal{V}$ be a set of v elements, $\mathcal{G}$ be a partition of $\mathcal{V}$ into subsets, called *groups*, and $\mathcal{B}$ be a collection of some subsets of $\mathcal{V}$, called *blocks*. A *group divisible design* (or GDD) (K, λ)-GDD is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ such that

(i) $|B| \in K$ for every $B \in \mathcal{B}$;

(ii) $|G \cap B| \leq 1$ for every $G \in \mathcal{G}$ and every $B \in \mathcal{B}$; and

(iii) every pair of elements $\{x, y\}$, where x and y belong to distinct groups, is contained in exactly λ blocks of $\mathcal{B}$.

The *type* of a GDD $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. An exponential notation is usually used to describe types: a type $g_1^{u_1} \cdots g_m^{u_m}$ denotes u_i occurrences of g_i, $1 \leq i \leq m$.

In order to prove that the necessary conditions (1.1) for the existence of an NDB$(k, \lambda; v)$ of any possible form are also sufficient for $k = 3$ and 4, Kageyama and Miao [10] introduced the concept of nested directed GDDs.

A *directed GDD* (or DGDD) (K,λ)-DGDD of type T is a $(K,2\lambda)$-GDD of the same type T in which the blocks are transitively ordered k-tuples and each ordered pair of elements not contained in the same group occurs in exactly λ blocks.

A *nested directed GDD* (or NDGDD) (k,λ)-NDGDD of type T and of form $\Pi_{2\le n\le k-1}(n^{j_n},\lambda_n)^{i_n}$, $(\mathcal{V},\mathcal{G},\mathcal{B})$, is a $(\{k\},\lambda)$-DGDD of type T where each block of $\mathcal{B}$ contains $\sum_{2\le n\le k-1} i_n j_n$ mutually disjoint subblocks, $i_n j_n$ subblocks of which are partitioned into i_n mutually disjoint families of j_n subblocks of size n and the j_n subblocks of size n belong to one distinguished system $\mathcal{B}_n(\ell)$, $1\le \ell\le i_n$, such that $(\mathcal{V},\mathcal{G},\mathcal{B}_n(\ell))$ forms an $(\{n\},\lambda_n)$-DGDD of type T for all integers n and ℓ with $1\le \ell\le i_n$.

Theorem 2.1 [10]. *Let $(\mathcal{V},\mathcal{G},\mathcal{B})$ be a (K,λ)-GDD. Further let $w:\mathcal{V}\longrightarrow\mathcal{N}\cup\{0\}$ be a weight function, where $\mathcal{N}$ is the set of all positive integers. For each $B\in\mathcal{B}$, suppose there exists a (k,λ')-NDGDD of type $\{w(x):x\in B\}$ and of form $\Pi_{2\le n\le k-1}(n^{j_n},\lambda_n)^{i_n}$, $(\cup_{x\in B}S(x),\ \{S(x):x\in B\},\ \mathcal{B}_B)$, where $S(x)=\{x_1,\dots,x_{w(x)}\}$ for every $x\in\mathcal{V}$ and $\mathcal{B}_B$ is the collection of blocks of this NDGDD. Then there exists a $(k,\lambda\lambda')$-NDGDD of type $\{\sum_{x\in G}w(x):G\in\mathcal{G}\}$ and of form $\Pi_{2\le n\le k-1}(n^{j_n},\lambda\lambda_n)^{i_n}$, $(\cup_{x\in\mathcal{V}}S(x),\ \{\cup_{x\in G}S(x):G\in\mathcal{G}\},\ \cup_{B\in\mathcal{B}}\mathcal{B}_B)$.*

As an immediate consequence, the following corollary can be obtained. Recall that a *pairwise balanced design* (or PBD) $\mathrm{B}(K,\lambda;v)$ can be regarded as a (K,λ)-GDD of type 1^v. A set K of positive integers is said to be *PBD-closed* if $\mathrm{B}(K)=K$, where $\mathrm{B}(K)=\{v:\text{a }\mathrm{B}(K,1;v)\text{ exists}\}$.

Corollary 2.2 [10]. *Let $\mathrm{NDB}(k,\lambda,F)=\{v:\text{an }\mathrm{NDB}(k,\lambda;v)\text{ of form }F\text{ exists}\}$. Then the $\mathrm{NDB}(k,\lambda,F)$ is a PBD-closed set.*

We also need the following construction.

Theorem 2.3 [11]. *Let $(\mathcal{V},\mathcal{G},\mathcal{B})$ be a (k,λ)-NDGDD of form F. Further let G_0 be a set of new elements, that is, $G_0\cap\mathcal{V}=\phi$, and suppose that for each group $G\in\mathcal{G}$, there exists a (k,λ)-NDGDD of form F, $(G\cup G_0,\ \mathcal{H}_G\cup\{G_0\},\ \mathcal{B}_G)$, where $\mathcal{H}_G$ is the set of groups without G_0 and $\mathcal{B}_G$ is the collection of blocks of this NDGDD. Then there exists a (k,λ)-NDGDD of form F, $(\mathcal{V}\cup G_0,\ (\cup_{G\in\mathcal{G}}\mathcal{H}_G)\cup\{G_0\},\ \mathcal{B}\cup(\cup_{G\in\mathcal{G}}\mathcal{B}_G))$.*

3. A construction from directed frames

Let $(\mathcal{V},\mathcal{G},\mathcal{B})$ be a $(\{k\},\lambda)$-DGDD. If the collection $\mathcal{B}$ of blocks can be partitioned into *partial parallel classes* each of which partitions $\mathcal{V}-G$ for some $G\in\mathcal{G}$, it is said that this DGDD is a *directed frame*, denoted by (k,λ)-directed frame. The *type* of the directed frame is the type of the underlying DGDD.

Directed frames can be used to construct NDBIBDs.

Theorem 3.1. *The existence of a (k,λ)-directed frame of type g^u implies the existence of a $(k+1,\frac{k+1}{2})$-NDGDD of type g^u and of form $(k,\frac{k-1}{2})^1$ when λ is a factor of $(k-1)/2$, or a $(k+1,\lambda+\frac{2\lambda}{k-1})$-NDGDD of type g^u and of form $(k,\lambda)^1$ when $(k-1)/2$ is a factor of λ.*

Proof. It is easy to show that for each group of a (k,λ)-directed frame of type g^u, $(\mathcal{V},\mathcal{G},\mathcal{B})$, there are $2\lambda g/(k-1)$ partial parallel classes associated with it.

(I) When $\lambda \mid (k-1)/2$, let $(k-1)/2 = s\lambda$, $s \in \mathcal{N}$. Then there are g/s partial parallel classes, say, $\mathcal{P}_{G,i}$, $1 \le i \le g/s$, associated with the group G for all $G \in \mathcal{G}$. For each block $B = (b_1,\ldots,b_k)$ of a partial parallel class $\mathcal{P}_{G,i}$ with $G = \{x_1,\ldots,x_g\}$ we form s new blocks $B_1 = ((b_1,\ldots,b_k), x_{(i-1)s+1}), \ldots, B_s = ((b_1,\ldots,b_k), x_{is})$. Then $(\mathcal{V},\mathcal{G},\mathcal{B}')$ with $\mathcal{B}' = \{((b_1,\ldots,b_k), x_{(i-1)s+j}) : i = 1,\ldots,g/s\ ;\ j = 1,\ldots,s\ ;\ (b_1,\ldots,b_k) \in P_{G,i}\ ;\ G \in \mathcal{G}\}$ is a $(k+1, \frac{k+1}{2})$-NDGDD of type g^u and of form $(k, \frac{k-1}{2})^1$.

(II) When $(k-1)/2 \mid \lambda$, let $\lambda = \{(k-1)/2\}t$, $t \in \mathcal{N}$. Then there are tg partial parallel classes, say, $\mathcal{Q}_{G,i}$, $1 \le i \le tg$, associated with the group $G = \{x_1,\ldots,x_g\} \in \mathcal{G}$. For each block $B = (b_1,\ldots,b_k)$ of the t partial parallel classes $\mathcal{Q}_{G,(n-1)t+1},\ldots,\mathcal{Q}_{G,nt}$, $n = 1,\ldots,g$, a new block $((b_1,\ldots,b_k),x_n)$ is formed. Then $(\mathcal{V},\mathcal{G},\mathcal{B}'')$ with $\mathcal{B}'' = \{((b_1,\ldots,b_k),x_n) : n = 1,\ldots,g\ ;\ (b_1,\ldots,b_k) \in \cup_{j=1}^{t}\mathcal{Q}_{G,(n-1)t+j}\ ;\ G \in \mathcal{G}\}$ is a $(k+1, \lambda + \frac{2\lambda}{k-1})$-NDGDD of type g^u and of form $(k,\lambda)^1$. □

A (k,λ)-directed frame of type 1^v can be named as an *almost resolvable directed BIB design* (or ARDBIBD) ARDB$(k,\lambda;v)$. In fact, an ARDB$(k,\lambda;v)$ $(\mathcal{V},\mathcal{B})$ is a DB$(k,\lambda;v)$ in which the collection of blocks can be partitioned into partial parallel classes each of which partitions $\mathcal{V} - \{x\}$ for some $x \in \mathcal{V}$.

It is easy to show that in the ARDB$(k,\lambda;v)$, $\lambda = \{(k-1)/2\}m$ for some integer $m \in \mathcal{N}$.

Corollary 3.2. *Let $m \in \mathcal{N}$. Then the existence of an* ARDB$(k, \frac{k-1}{2}m; v)$ *implies the existence of an* NDB$(k+1, \frac{k+1}{2}m; v)$ *of form* $(k, \frac{k-1}{2}m)^1$.

Recall that an *almost resolvable BIB design* (or ARBIBD) ARB$(k,\lambda;v)$ is a BIB design B$(k,\lambda;v)$ in which the collection of blocks can be partitioned into partial parallel classes each of which partitions $\mathcal{V} - \{x\}$ for some $x \in \mathcal{V}$. It follows that the existence of an ARB$(k,\lambda;v)$ implies the existence of an ARDB$(k,\lambda;v)$. In fact, by assigning to each block of the ARB$(k,\lambda;v)$ two new blocks, one in some arbitrary but fixed order which is imposed on the elements of each block and one in the reverse order, an ARDB$(k,\lambda;v)$ is obtained.

Corollary 3.3. *Let $m \in \mathcal{N}$. Then the existence of an* ARB$(k,(k-1)m;v)$ *implies the existence of an* NDB$(k+1,(k+1)m;v)$ *of form* $(k,(k-1)m)^1$.

The existence problem of ARBIBDs has been extensively discussed in [7]. The results contained there can then be utilized to construct many such NDBIBDs.

4. A construction from idempotent MOLS

A Latin square of order v based on a set $\mathcal{V}$ of v elements is a $v \times v$ array such that each row and each column contains each element of $\mathcal{V}$ exactly once. Two Latin squares, $A = (a_{ij})$ and $B = (b_{ij})$ on $\mathcal{V}$, are said to be *orthogonal* if $\{(a_{ij}, b_{ij}) : 1 \le i, j \le v\} = \mathcal{V} \times \mathcal{V}$. Without loss of generality, we may assume $\mathcal{V} = \{1,2,\ldots,v\}$. A

Latin square on $\mathcal{V}$ is said to be *idempotent* if the (i,i)-entry is i for all i, $1 \le i \le v$. The t idempotent Latin squares $A_1, \ldots, A_t$ of order v are called t *mutually orthogonal idempotent Latin squares* if A_i and A_j are orthogonal for all i, j, $1 \le i < j \le t$, and are denoted by t idempotent MOLS(v).

The existence of t idempotent MOLS(v) has been studied extensively. For example, the following result can be found in [1].

Theorem 4.1 [1]. *For any integer $v \ge 5$, $v \ne 6$, 10, there exist 3 idempotent* MOLS(v).

This concept can be utilized to construct NDBIBDs as follows.

Theorem 4.2. *The existence of $k-2$ idempotent* MOLS(v) *implies the existence of an* NDB($k, \frac{k(k-1)}{2}; v$) *of form $\Pi_{2\le n\le k-1}(n^{j_n}, j_n\frac{n(n-1)}{2})^{i_n}$ for any possible integers n, j_n and i_n such that $\sum_{2\le n\le k-1} i_n j_n n \le k$.*

Proof. Let $\mathcal{V} = \{1, 2, \ldots, v\}$. Take $k-2$ idempotent MOLS(v) based on $\mathcal{V}$, $A_1 = (a_{ij}^{(1)}), \ldots, A_{k-2} = (a_{ij}^{(k-2)})$ for $1 \le i, j \le v$, where $1 \le a_{ij}^{(\ell)} \le v$, $1 \le \ell \le k-2$. Let $\mathcal{B} = \{(i, j, a_{ij}^{(1)}, \ldots, a_{ij}^{(k-2)}) : 1 \le i, j \le v, i \ne j\}$. Then $(\mathcal{V}, \mathcal{B})$ is a DB($k, \frac{k(k-1)}{2}; v$). Divide each block of $\mathcal{B}$ into $\sum_{2\le n\le k-1} i_n j_n$ mutually disjoint subblocks, such that $i_n j_n$ of them are partitioned into i_n mutually disjoint families of j_n subblocks of size n, the j_n subblocks of size n belong to one distinguished system $\mathcal{B}_n(\ell)$, $1 \le \ell \le i_n$, and that $(\mathcal{V}, \mathcal{B}_n(\ell))$ forms a DB($n, j_n\frac{n(n-1)}{2}; v$) for all integers n and ℓ with $1 \le \ell \le i_n$. This completes the proof. □

5. A construction from the method of differences

The method of differences is the most commonly used direct construction technique. Here we describe a construction based on this technique, which is an extension of [14].

Theorem 5.1. *Let S be a 5-subset of* GF(q), *and θ be a primitive element of* GF(q), *$q > 3$. If S can be arranged so that the 10 ordered differences of S contain 5 squares and 5 non-squares, then the base blocks $S, \theta^2 S, \ldots, \theta^{q-3}S$ form a* DB($5, 5; q$). *Furthermore,*

(1) *if there exists a 4-subset T_4 of the arranged S so that the 6 ordered differences of T_4 contain 3 squares and 3 non-squares, then the* DB($5,5;q$) *gives an* NDB($5,5;q$) *of form $(4,3)^1$;*

(2) *if there exist two mutually disjoint 2-subsets T_2, T_2' of the arranged S so that the 2 ordered differences of T_2 and T_2' contain 1 square and 1 non-square, then the* DB($5,5;q$) *gives an* NDB($5,5;q$) *of form $(2^2,1)^1$.*

The proof of this theorem is straightforward.

6. Construction of NDB$(4, \lambda; v)$

As pointed out in Section 1, the necessary conditions (1.1) for the existence of an NDB$(4, \lambda; v)$ of any possible form are also sufficient, except possibly for an NDB$(4, 2; 10)$ of form $(3, 1)^1$. This possible exception will be removed.

At first we need an almost resolvable directed BIB design below.

Lemma 6.1. *There exists an* ARDB$(3, 1; 10)$.

Proof. Let $\mathcal{V} = Z_5 \times Z_2$ and $\mathcal{B}$ be the development of the following base blocks modulo (5, -).

$$((1,0),(0,0),(3,0)),\quad ((2,1),(3,1),(2,0)),\quad ((4,0),(1,1),(4,1)),$$
$$((1,0),(4,1),(2,0)),\quad ((2,1),(1,1),(3,0)),\quad ((4,0),(3,1),(0,1)).$$

It is readily checked that $(\mathcal{V}, \mathcal{B})$ is an ARDB$(3, 1; 10)$, where the first three base blocks form a partition of $\mathcal{V} - \{(0,1)\}$, and the last three base blocks form a partition of $\mathcal{V} - \{(0,0)\}$. □

Theorem 6.2. *There exists an* NDB$(4, 2; 10)$ *of form* $(3, 1)^1$.

Proof. Apply Corollary 3.2 with Lemma 6.1. □

Thus we can show the entire existence of nested directed BIB designs of block size 4 as follows.

Theorem 6.3. *The necessary conditions* (1.1) *for the existence of an* NDB$(4, \lambda; v)$ *of any possible form are also sufficient.*

Proof. Take Theorems 1.1 and 6.2. □

7. Construction of NDB$(5, \lambda; v)$ of form $(4, \lambda_4)^1$

It is clear that the necessary conditions (1.1) for the existence of an NDB$(5, \lambda; v)$ of form $(4, \lambda_4)^1$ are $v \geq 5$, $\lambda_4 = 3t$, $\lambda = 5t$ and $t(v-1) \equiv 0 \bmod 2$ for some positive integer t. It will be shown that they are also sufficient.

Theorem 7.1. *The existence of an* NB$(k, \lambda; v)$ *of form* F *implies the existence of an* NDB$(k, \lambda; v)$ *of form* F.

Proof. For each block $\{x_1, \ldots, x_k\}$ of an NB$(k, \lambda; v)$ of form F, define two new blocks $(x_1, \ldots, x_k)$ and $(x_k, \ldots, x_1)$. Then these new directed blocks form the collection of blocks of an NDB$(k, \lambda; v)$ of form F. □

Corollary 7.2. *There exists an* NDB$(5, 5t; v)$ *of form* $(4, 3t)^1$ *whenever* $v \geq 5$ *and* $t(v-1) \equiv 0 \bmod 4$ *for* $t \in \mathcal{N}$.

Proof. Wang and Zhu [15] constructed all of these $\mathrm{NB}(5, 5t; v)$ of form $(4, 3t)^1$. Apply Theorem 7.1. □

Now we use DBIBDs to produce NDBIBDs.

Theorem 7.3. *The existence of a* $\mathrm{DB}(5, t; v)$ *implies the existence of an* $\mathrm{NDB}(5, 5t; v)$ *of form* $(4, 3t)^1$.

Proof. For each block (a, b, c, d, e) of an $\mathrm{DB}(5, t; v)$, define five new blocks:

$$(\underline{a}, \underline{b}, \underline{c}, \underline{d}, e), \quad (\underline{a}, \underline{b}, \underline{c}, d, \underline{e}), \quad (\underline{a}, \underline{b}, c, \underline{d}, \underline{e}), \quad (\underline{a}, b, \underline{c}, \underline{d}, \underline{e}), \quad (a, \underline{b}, \underline{c}, \underline{d}, \underline{e}),$$

where the elements underlined with "_" within a block form a subblock. Then these new blocks can form the collection of blocks of an $\mathrm{NDB}(5, 5t; v)$ of form $(4, 3t)^1$. □

Corollary 7.4. *There exists an* $\mathrm{NDB}(5, 5t; v)$ *of form* $(4, 3t)^1$ *whenever* $v \geq 5$, $(v, t) \neq (15, 1)$, $t(v-1) \equiv 0 \bmod 2$ *and* $tv(v-1) \equiv 0 \bmod 10$ *for* $t \in \mathcal{N}$.

Proof. When v and t satisfy the stated conditions, there exists a $\mathrm{DB}(5, t; v)$ (see [14]). Then apply Theorem 7.3. □

By an argument similar to those for Theorem 7.3 and Corollary 7.4, we have the following.

Theorem 7.5. *The existence of a* $(\{5\}, t)$-DGDD *of type* T *implies the existence of a* $(5, 5t)$-NDGDD *of type* T *and of form* $(4, 3t)^1$.

Corollary 7.6. *There exist* $(5, 5)$-NDGDD *of types* 2^5 *and* 2^6, *and of form* $(4, 3)^1$.

Proof. The $(\{5\}, 1)$-DGDD of types 2^5 and 2^6 can be found in [14]. □

Furthermore, we have the following.

Lemma 7.7. *There exists a* $(5, 5)$-NDGDD *of type* 2^7 *and of form* $(4, 3)^1$.

Proof. Let $\mathcal{V} = Z_2 \times Z_7$, $\mathcal{G} = \{Z_2 \times \{i\} : i \in Z_7\}$, and $\mathcal{B}$ be the development of the following base blocks modulo $(2, 7)$, where the elements underlined with "_" within a block form a subblock:

$$((\underline{0,0}), (\underline{0,1}), (\underline{0,6}), (\underline{1,3}), (1,4)),$$
$$((\underline{0,0}), (\underline{0,4}), (\underline{0,3}), (1,2), (\underline{1,5})),$$
$$((\underline{0,0}), (\underline{0,5}), (\underline{0,2}), (\underline{1,6}), (1,1)),$$
$$((\underline{1,0}), (0,1), (\underline{0,6}), (\underline{0,3}), (\underline{0,4})),$$
$$((\underline{1,0}), (0,4), (\underline{0,3}), (\underline{0,2}), (\underline{0,5})),$$
$$((\underline{1,0}), (\underline{0,5}), (0,2), (\underline{0,6}), (\underline{0,1})).$$

□

Theorem 5.1 can also be used to produce some useful NDBIBDs.

Lemma 7.8. *There exists an* NDB$(5,5;q)$ *of form* $(4,3)^1$, *where* $q \in \{7,\ 19,\ 23,\ 27,\ 43,\ 47,\ 83\}$.

Proof. Suitable orderings for S and T_4 in Theorem 5.1 are listed below:

$$\begin{array}{llll} q=7, & \theta=3, & S=(1,3,2,6,4), & T_4=(1,3,2,6); \\ q=19, & \theta=2, & S=(1,2,4,16,8), & T_4=(2,4,16,8); \\ q=23, & \theta=5, & S=(1,5,10,2,4), & T_4=(1,5,10,2); \\ q=27, & \theta^3=\theta+2, & S=(1,\theta,\theta^2,\theta+2,\theta^2+2\theta), & T_4=(1,\theta^2,\theta+2,\theta^2+2\theta); \\ q=43, & \theta=3, & S=(1,3,27,9,38), & T_4=(1,3,27,9); \\ q=47, & \theta=5, & S=(1,5,25,31,14), & T_4=(5,25,31,14); \\ q=83, & \theta=2, & S=(1,2,4,8,16), & T_4=(1,2,4,16). \end{array}$$

Then apply Theorem 5.1(1). □

A $(\{k\},\lambda)$-GDD of type g^k is called a *transversal design*, denoted by TD$(k,\lambda;g)$.

Theorem 7.9. *Let* $0 \le s,\ t \le g$. *Suppose there exists a* TD$(7,1;g)$. *If there exist* NDB$(5,5;u)$ *of form* $(4,3)^1$ *for* $u = 2g+1,\ 2s+1,\ 2t+1$, *then there exists an* NDB$(5,5;v)$ *of form* $(4,3)^1$ *with* $v = 10g+2s+2t+1$.

Proof. Delete $g-s$ elements and $g-t$ elements from two groups of the TD$(7,1;g)$ respectively. Give weight 2 to each element of the resulting $(\{5,6,7\},1)$-GDD of type $g^5s^1t^1$. Since Corollary 7.6 and Lemma 7.7 give $(5,5)$-NDGDD of types 2^5, 2^6 and 2^7, and of form $(4,3)^1$, by applying Theorem 2.1 we get a $(5,5)$-NDGDD of type $(2g)^5(2s)^1(2t)^1$ and of form $(4,3)^1$. Applying Theorem 2.3 with $|G_0|=1$, the desired NDBIBD is obtained. □

Corollary 7.10. *There exists an* NDB$(5,5;v)$ *of form* $(4,3)^1$, *where* $v \in \{99,\ 107,\ 119,\ 139,\ 143,\ 179,\ 183,\ 283\}$.

Proof. Applying Theorem 7.9 with $g = 8,\ 9,\ 11,\ 16$ and 23 (see [1] for their existence), we have the required result, since $99 = 10\cdot 8+2\cdot 5+2\cdot 4+1$, $107 = 10\cdot 8+2\cdot 8+2\cdot 5+1$, $119 = 10\cdot 9+2\cdot 9+2\cdot 5+1$, $139 = 10\cdot 11+2\cdot 9+2\cdot 5+1$, $143 = 10\cdot 11+2\cdot 8+2\cdot 8+1$, $179 = 10\cdot 16+2\cdot 5+2\cdot 4+1$, $183 = 10\cdot 16+2\cdot 6+2\cdot 5+1$ and $283 = 10\cdot 23+2\cdot 13+2\cdot 13+1$. □

Theorem 7.11. *Let* $0 \le s \le g$. *Suppose there exists a* TD$(6,1;g)$. *If there exist* NDB$(5,5;u)$ *of form* $(4,3)^1$ *for* $u = 2g+1,\ 2s+1$, *then there exists an* NDB$(5,5;v)$ *of form* $(4,3)^1$ *with* $v = 10g+2s+1$.

Proof. Delete $g-s$ elements from one group of the TD$(6,1;g)$. Give weight 2 to each element of the resulting $(\{5,6\},1)$-GDD of type g^5s^1. Since Corollary 7.6 gives $(5,5)$-NDGDD of types 2^5 and 2^6, and of form $(4,3)^1$, by applying Theorem 2.1 we get a $(5,5)$-NDGDD of type $(2g)^5(2s)^1$ and of form $(4,3)^1$. Then apply Theorem 2.3. □

Corollary 7.12. *There exists an* NDB$(5,5;v)$ *of form* $(4,3)^1$*, where* $v \in \{59, 87, 167, 243, 563\}$.

Proof. Apply Theorem 7.11 with $g = 5$, 8, 16, 23 and 55 (see [1]), where $59 = 10 \cdot 5 + 2 \cdot 4 + 1$, $87 = 10 \cdot 8 + 2 \cdot 3 + 1$, $167 = 10 \cdot 16 + 2 \cdot 3 + 1$, $243 = 10 \cdot 23 + 2 \cdot 6 + 1$ and $563 = 10 \cdot 55 + 2 \cdot 6 + 1$. □

Lemma 7.13. *There exists an* NDB$(5,5;39)$ *of form* $(4,3)^1$.

Proof. Bennett et al. [4] showed the existence of a $(\{5,7\},1)$-DGDD of type 1^{39}. For each block (a,b,c,d,e) of size 5, define five new blocks: $(\underline{a},\underline{b},\underline{c},\underline{d},e)$, $(\underline{a},\underline{b},\underline{c},d,\underline{e})$, $(\underline{a},\underline{b},c,\underline{d},\underline{e})$, $(\underline{a},b,\underline{c},\underline{d},\underline{e})$, $(a,\underline{b},\underline{c},\underline{d},\underline{e})$, where the elements underlined with "_" within a block form a subblock. For the block of size 7, we fill in with an NDB$(5,5;7)$ of form $(4,3)^1$. Then the resulting design is an NDB$(5,5;39)$ of form $(4,3)^1$. □

Lemma 7.14. *There exists an* NDB$(5,5;15)$ *of form* $(4,3)^1$.

Proof. The design is given below: $\mathcal{V} = Z_{15}$, $\mathcal{B} = \{(0,\underline{1},\underline{2},\underline{3},\underline{4})$, $(\underline{0},1,\underline{3},\underline{5},\underline{8})$, $(0,\underline{3},\underline{7},\underline{13},\underline{11})$, $(0,\underline{6},\underline{13},\underline{10},\underline{5})$, $(\underline{0},\underline{12},9,\underline{6},\underline{5})$, $(\underline{0},\underline{12},\underline{11},\underline{6},5)$, $(\underline{0},\underline{13},\underline{11},\underline{7},6) \mod 15\}$. □

Now we can state the following.

Theorem 7.15. *Let* $v \geq 5$ *be odd. Then all* NDB$(5,5t;v)$ *of form* $(4,3t)^1$ *exist for any positive integers* t.

Proof. First we consider the case where $t = 1$. Bennett et al. [2] showed that $B(\{5,7,9\}) \supseteq (2\mathcal{N}+1) - E$, where $E = \{11, 13, 15, 17, 19, 23, 27, 29, 31, 33, 39, 43, 51, 59, 71, 75, 83, 87, 93, 95, 99, 107, 111, 113, 115, 119, 131, 135, 139, 143, 167, 173, 179, 183, 191, 195, 243, 283, 411, 563\}$. Since there exist NDB$(5,5;v)$ of form $(4,3)^1$ for $v = 5,7,9$ (see Corollary 7.2 and Lemma 7.8), by applying Corollary 2.2, we need only to construct NDB$(5,5;v)$ of form $(4,3)^1$ for $v \in E$. Corollary 7.2 settles the cases for $v = 13$, 17, 29, 33, 93, 113, 173. Corollary 7.4 covers the cases for $v = 11$, 31, 51, 71, 75, 95, 111, 115, 131, 135, 191, 195, 411. Lemma 7.8 covers the cases for $v = 19$, 23, 27, 43, 83. The remaining 15 cases are settled by Corollaries 7.10 and 7.12, and Lemmas 7.13 and 7.14. Thus all NDB$(5,5;v)$ of form $(4,3)^1$ are constructed for v odd ≥ 5.

Next take each block of an NDB$(5,5;v)$ of form $(4,3)^1$ t times. Then it follows that an NDB$(5,5t;v)$ of form $(4,3t)^1$ is obtained whenever v is odd ≥ 5. This completes the proof. □

On the other hand, when v is even, the following can be obtained.

Theorem 7.16. *Let* $v \geq 5$ *be even. Then all* NDB$(5,10s;v)$ *of form* $(4,6s)^1$ *exist for any positive integers* s.

Proof. Theorem 4.1 with Theorem 4.2 can cover all the cases except for $v = 6$ and 10, which can be removed by Corollary 7.4. □

As a summary, we have the following main result of this section.

Theorem 7.17. *The necessary conditions* (1.1) *for the existence of an* $\mathrm{NDB}(5, \lambda; v)$ *of form* $(4, \lambda_4)^1$ *are also sufficient.*

Proof. The sufficiency follows from Theorems 7.15 and 7.16. □

8. Construction of NDB$(5, \lambda; v)$ of form $(3, \lambda_3)^1(2, \lambda_2)^1$

In this case, the necessary conditions (1.1) become $v \geq 5$, $\lambda = 10\lambda_2$ and $\lambda_3 = 3\lambda_2$.

Lemma 8.1. *There exist an* $\mathrm{NDB}(5, 10; 6)$ *and an* $\mathrm{NDB}(5, 10; 10)$, *both of form* $(3, 3)^1(2, 1)^1$.

Proof. An $\mathrm{NDB}(5, 10; 6)$ of form $(3, 3)^1(2, 1)^1$ is given below:

$$\begin{aligned}
\mathcal{V} &= Z_5 \cup \{\infty\}, \\
\mathcal{B} &= \{\ (\underline{0}, \underline{1}, \underline{2}, \underline{\underline{3}}, \underline{\underline{4}}),\quad (\underline{\underline{\infty}}, \underline{3}, \underline{2}, \underline{1}, \underline{\underline{0}}),\quad (\underline{4}, \underline{2}, \underline{1}, \underline{\underline{0}}, \underline{\underline{\infty}}), \\
&\qquad (\underline{\underline{1}}, \underline{2}, \underline{\infty}, \underline{\underline{3}}, \underline{4}),\quad (\underline{\infty}, \underline{\underline{4}}, \underline{\underline{3}}, \underline{2}, \underline{0}),\quad (\underline{4}, \underline{\underline{3}}, \underline{\underline{1}}, \underline{0}, \underline{\infty}) \quad \mathrm{mod}\ 5\ \},
\end{aligned}$$

where the elements underlined with "$\underline{\ }$" and "$\underline{\underline{\ }}$" within a block form a subblock of size 3 and of size 2 respectively. An $\mathrm{NDB}(5, 10; 10)$ of form $(3, 3)^1(2, 1)^1$ is constructed below:

$$\begin{aligned}
\mathcal{V} &= \mathrm{GF}(9) \cup \{\infty\}, \\
\mathcal{B} &= \{\ (\underline{1}, \underline{2}, \underline{\underline{\infty}}, \underline{2\theta + 1}, \underline{\underline{\theta + 2}}),\quad (\underline{1}, \underline{\underline{2}}, \underline{\infty}, \underline{\underline{2\theta + 1}}, \underline{\theta + 2}),\quad (\underline{\underline{1}}, \underline{2}, \underline{\infty}, \underline{\underline{2\theta + 1}}, \underline{\theta + 2}), \\
&\qquad (\underline{\underline{1}}, \underline{2}, \underline{\infty}, \underline{2\theta + 1}, \underline{\underline{\theta + 2}}),\quad (\underline{1}, \underline{\underline{2}}, \underline{\underline{\infty}}, \underline{2\theta + 1}, \underline{\theta + 2}),\quad (\underline{\underline{0}}, \underline{\underline{2}}, \underline{1}, \underline{\underline{\theta + 2}}, \underline{2\theta + 1}), \\
&\qquad (\underline{\underline{0}}, \underline{\underline{2}}, \underline{1}, \underline{\theta + 2}, \underline{2\theta + 1}),\quad (\underline{\underline{0}}, \underline{2}, \underline{1}, \underline{\theta + 2}, \underline{\underline{2\theta + 1}}),\quad (\underline{0}, \underline{2}, \underline{1}, \underline{\underline{\theta + 2}}, \underline{\underline{2\theta + 1}}), \\
&\qquad (\underline{\underline{0}}, \underline{2}, \underline{\underline{1}}, \underline{\theta + 2}, \underline{2\theta + 1}) \quad \mathrm{mod}\ 9\ \},
\end{aligned}$$

where θ is a primitive element of $\mathrm{GF}(9)$ satisfying $\theta^2 = 2\theta + 1$. □

Theorem 8.2. *There exists an* $\mathrm{NDB}(5, 10; v)$ *of form* $(3, 3)^1(2, 1)^1$ *for* $v \geq 5$.

Proof. Theorem 4.1 with Theorem 4.2 covers all the cases except for $v = 6$ and 10, which are constructed in Lemma 8.1. □

Hence we have the following.

Theorem 8.3. *The necessary conditions* (1.1) *for the existence of an* $\mathrm{NDB}(5, \lambda; v)$ *of form* $(3, \lambda_3)^1(2, \lambda_2)^1$ *are also sufficient.*

Proof. Repeat each block (and thus each subblock) of an $\mathrm{NDB}(5, 10; v)$ of form $(3, 3)^1(2, 1)^1$ λ_2 times. □

9. Construction of NDB$(5, \lambda; v)$ of form $(2, \lambda_2)^2$

Here the necessary conditions (1.1) are that $v \geq 5$ and $\lambda = 10\lambda_2$.

Lemma 9.1. *There exist an* NDB$(5, 10; 6)$ *and an* NDB$(5, 10; 10)$*, both of form* $(2, 1)^2$.

Proof. An NDB$(5, 10; 6)$ of form $(2, 1)^2$ is obtained below:

$$\mathcal{V} = Z_6,$$
$$\mathcal{B} = \{(\underline{0}, \underline{1}, \underline{\underline{2}}, \underline{\underline{3}}, 4),\ (\underline{0}, \underline{\underline{1}}, \underline{2}, \underline{\underline{3}}, 4),\ (\underline{4}, \underline{3}, \underline{\underline{2}}, \underline{\underline{1}}, 0),\ (\underline{4}, \underline{\underline{3}}, \underline{2}, \underline{\underline{1}}, 0),\ (5, \underline{1}, \underline{\underline{3}}, \underline{\underline{0}}, \underline{4}) \mod 6\},$$

where the elements underlined with "$\underline{\ }$" and "$\underline{\underline{\ }}$" within a block form a subblock respectively. An NDB$(5, 10; 10)$ of form $(2, 1)^2$ is given below:

$$\mathcal{V} = \mathrm{GF}(9) \cup \{\infty\},$$
$$\begin{aligned}\mathcal{B} = \{\ & (\underline{\underline{1}}, \underline{2}, \underline{\underline{\infty}}, \underline{2\theta+1}, \theta+2),\ (\underline{\underline{1}}, \underline{2}, \underline{\infty}, \underline{\underline{2\theta+1}}, \theta+2),\ (\underline{1}, 2, \underline{\underline{\infty}}, \underline{2\theta+1}, \underline{\underline{\theta+2}}),\\ & (\underline{\underline{1}}, \underline{\underline{2}}, \underline{\infty}, \underline{2\theta+1}, \theta+2),\ (\underline{1}, \underline{\underline{2}}, \infty, \underline{\underline{2\theta+1}}, \underline{\theta+2}),\ (\underline{0}, \underline{\underline{2}}, \underline{\underline{1}}, \underline{\theta+2}, 2\theta+1),\\ & (\underline{\underline{0}}, \underline{2}, \underline{1}, \underline{\underline{\theta+2}}, 2\theta+1),\ (\underline{\underline{0}}, \underline{2}, 1, \underline{\theta+2}, \underline{\underline{2\theta+1}}),\ (\underline{0}, \underline{\underline{2}}, \underline{1}, \underline{\underline{\theta+2}}, 2\theta+1),\\ & (\underline{0}, 2, \underline{\underline{1}}, \underline{\underline{\theta+2}}, \underline{2\theta+1}) \quad \mod 9\ \},\end{aligned}$$

where θ is a primitive element of GF(9) satisfying $\theta^2 = 2\theta + 1$. □

Theorem 9.2. *There exists an* NDB$(5, 10; v)$ *of form* $(2, 1)^2$ *for* $v \geq 5$.

Proof. Theorem 4.1 with Theorem 4.2 covers all the cases except for $v = 6$ and 10, which are constructed in Lemma 9.1. □

Therefore we have the following.

Theorem 9.3. *The necessary conditions* (1.1) *for the existence of an* NDB$(5, \lambda; v)$ *of form* $(2, \lambda_2)^2$ *are also sufficient.*

Proof. Repeat each block (and thus each subblock) of an NDB$(5, 10; v)$ of form $(2, 1)^2$ λ_2 times. □

10. Construction of NDB$(5, \lambda; v)$ of form $(2^2, \lambda_2)^1$

Now the necessary conditions (1.1) are that $v \geq 5$, $\lambda = 5\lambda_2$ and $\lambda_2(v - 1) \equiv 0 \mod 2$. We first consider the case $\lambda_2 \equiv 0 \mod 2$.

Theorem 10.1. *The existence of an* NDB$(5, 10t; v)$ *of form* $(2, t)^2$ *implies the existence of an* NDB$(5, 10t; v)$ *of form* $(2^2, 2t)^1$ *for any* $t \in \mathcal{N}$.

Proof. Combine the two sub-systems of an NDB$(5, 10t; v)$ of form $(2, t)^2$ into one. □

Corollary 10.2. *There exists an* $\mathrm{NDB}(5, 5\lambda_2; v)$ *of form* $(2^2, \lambda_2)^1$ *whenever* $v \geq 5$ *and* $\lambda_2 \equiv 0 \bmod 2$.

Proof. Apply Theorem 10.1 with Theorem 9.2. Then repeat each block (and thus each subblock) $\lambda_2/2$ times. □

Next we consider the case $\lambda_2 \equiv 1 \bmod 2$. Then the necessary conditions further become that v be odd ≥ 5 and $\lambda = 5\lambda_2$.

Theorem 10.3. *There exists an* $\mathrm{NDB}(5, 5\lambda_2; v)$ *of form* $(2^2, \lambda_2)^1$ *whenever* $v \geq 5$ *and* $\lambda_2(v-1) \equiv 0 \bmod 4$ *for* $\lambda_2 \in \mathcal{N}$.

Proof. Apply Theorem 7.1, where all the $\mathrm{NB}(5, 5\lambda_2; v)$ of form $(2^2, \lambda_2)^1$ have been constructed in [9]. □

Theorem 10.4. *The existence of a* $(\{5\}, \lambda_2)$-DGDD *of type* T *implies the existence of a* $(5, 5\lambda_2)$-NDGDD *of type* T *and of form* $(2^2, \lambda_2)^1$.

Proof. For each block (a, b, c, d, e) of a $(\{5\}, \lambda_2)$-DGDD of type T, define five new blocks:

$$(\underline{a}, \underline{b}, \underline{\underline{c}}, d, \underline{\underline{e}}),\quad (\underline{a}, \underline{\underline{b}}, \underline{c}, \underline{\underline{d}}, e),\quad (\underline{a}, \underline{\underline{b}}, c, \underline{d}, \underline{\underline{e}}),\quad (\underline{a}, b, \underline{\underline{c}}, \underline{\underline{d}}, \underline{e}),\quad (a, \underline{b}, \underline{c}, \underline{\underline{d}}, \underline{\underline{e}}),$$

which can produce a $(5, 5\lambda_2)$-NDGDD of type T and of form $(2^2, \lambda_2)^1$, where the elements underlined with "$\underline{\ }$" and "$\underline{\underline{\ }}$" within a block form a subblock respectively, both of them belonging to the same system. □

Since a $\mathrm{DB}(5, \lambda; v)$ can be regarded as a $(\{5\}, \lambda)$-DGDD of type 1^v, we have the following.

Theorem 10.5. *There exists an* $\mathrm{NDB}(5, 5\lambda_2; v)$ *of form* $(2^2, \lambda_2)^1$ *whenever* $v \geq 5$, $(v, \lambda_2) \neq (15, 1)$, $\lambda_2(v-1) \equiv 0 \bmod 2$ *and* $\lambda_2 v(v-1) \equiv 0 \bmod 10$ *for* $\lambda_2 \in \mathcal{N}$.

Proof. The $\mathrm{DB}(5, \lambda_2; v)$ can be found in [14]. Then apply Theorem 10.4. □

Lemma 10.6. *There exist* $(5, 5)$-NDGDD *of types* 2^5, 2^6 *and* 2^7, *and of form* $(2^2, 1)^1$.

Proof. The first two designs can be obtained by applying Theorem 10.4, where the corresponding $(\{5\}, 1)$-DGDD of types 2^5 and 2^6 are constructed in [14]. The third design is given below:

$$\mathcal{V} = Z_2 \times Z_7, \qquad \mathcal{G} = \{Z_2 \times \{i\} : i \in Z_7\},$$

$$\mathcal{B} = \left\{ \begin{array}{l} (\underline{(0,0)}, \underline{(0,1)}, \underline{\underline{(0,6)}}, \underline{\underline{(1,3)}}, (1,4)), \\ (\underline{(0,0)}, \underline{\underline{(0,4)}}, \underline{\underline{(0,3)}}, (1,2), \underline{(1,5)}), \\ (\underline{(0,0)}, \underline{(0,5)}, \underline{\underline{(0,2)}}, (1,6), \underline{\underline{(1,1)}}), \\ (\underline{(1,0)}, \underline{(0,1)}, \underline{\underline{(0,6)}}, \underline{\underline{(0,3)}}, (0,4)), \\ (\underline{(1,0)}, (0,4), \underline{(0,3)}, \underline{\underline{(0,2)}}, \underline{\underline{(0,5)}}), \\ (\underline{(1,0)}, (0,5), \underline{(0,2)}, \underline{\underline{(0,6)}}, \underline{\underline{(0,1)}}) \quad \bmod (2,7) \end{array} \right\}.$$

□

Lemma 10.7. *There exists an* NDB$(5,5;q)$ *of form* $(2^2,1)^1$, *where* $q \in \{7,\ 19,\ 23,\ 27,\ 43,\ 47,\ 83\}$.

Proof. Apply Theorem 5.1(2) with the suitable orderings for S and T_2, T_2' as follows.

$$
\begin{array}{llll}
q=7, & \theta=3, & S=(1,3,2,6,4), & T_2=(1,3),\ \ T_2'=(6,4); \\
q=19, & \theta=2, & S=(1,2,4,16,8), & T_2=(2,4),\ \ T_2'=(16,8); \\
q=23, & \theta=5, & S=(1,5,10,2,4), & T_2=(1,5),\ \ T_2'=(10,2); \\
q=27, & \theta^3=\theta+2, & S=(1,\theta,\theta^2,\theta+2,\theta^2+2\theta), & T_2=(1,\theta),\ \ T_2'=(\theta+2,\theta^2+2\theta); \\
q=43, & \theta=3, & S=(1,3,27,9,38), & T_2=(1,3),\ \ T_2'=(27,9); \\
q=47, & \theta=5, & S=(1,5,25,31,14), & T_2=(1,5),\ \ T_2'=(31,14); \\
q=83, & \theta=2, & S=(1,2,4,8,16), & T_2=(1,4),\ \ T_2'=(2,8).
\end{array}
$$

□

Since Lemma 10.6 gives (5, 5)-NDGDD of type 2^5, 2^6 and 2^7, and of form $(2^2,1)^1$, we have the following by arguments similar to those for Theorems 7.9 and 7.11, and Corollaries 7.10 and 7.12.

Theorem 10.8. *Let* $0 \le s,\ t \le g$. *Suppose there exists a* TD$(7,1;g)$. *If there exist* NDB$(5,5;u)$ *of form* $(2^2,1)^1$ *for* $u = 2g+1,\ 2s+1,\ 2t+1$, *then there exists an* NDB$(5,5;v)$ *of form* $(2^2,1)^1$ *with* $v = 10g+2s+2t+1$.

Theorem 10.9. *Let* $0 \le s \le g$. *Suppose there exists a* TD$(6,1;g)$. *If there exist* NDB$(5,5;u)$ *of form* $(2^2,1)^1$ *for* $u = 2g+1,\ 2s+1$, *then there exists an* NDB$(5,5;v)$ *of form* $(2^2,1)^1$ *with* $v = 10g+2s+1$.

Corollary 10.10. *There exists an* NDB$(5,5;v)$ *of form* $(2^2,1)^1$, *where* $v \in \{59,\ 87,\ 99,\ 107,\ 119,\ 139,\ 143,\ 167,\ 179,\ 183,\ 243,\ 283,\ 563\}$.

Lemma 10.11. *There exists an* NDB$(5,5;39)$ *of form* $(2^2,1)^1$.

Proof. For each block (a,b,c,d,e) of a $(\{5,7\},1)$-DGDD of type 1^{39} (see [4] for the existence), define five new blocks: $(\underline{a},\underline{b},\underline{\underline{c}},d,\underline{\underline{e}})$, $(\underline{a},\underline{\underline{b}},\underline{c},\underline{\underline{d}},e)$, $(\underline{a},\underline{\underline{b}},c,\underline{d},\underline{\underline{e}})$, $(\underline{a},b,\underline{\underline{c}},\underline{\underline{d}},\underline{e})$, $(a,\underline{b},\underline{c},\underline{\underline{d}},\underline{\underline{e}})$, and for the block of size 7 of this DGDD, fill in with an NDB$(5,5;7)$ of form $(2^2,1)^1$. □

Lemma 10.12. *There exists an* NDB$(5,5;15)$ *of form* $(2^2,1)^1$.

Proof. The design is given below: $\mathcal{V} = Z_{15}$, $\mathcal{B} = \{(\underline{14},11,\underline{\underline{4}},\underline{\underline{1}},\underline{0}),\ (\underline{\underline{8}},\underline{\underline{2}},\underline{13},7,\underline{\underline{0}}),\ (1,\underline{4},\underline{\underline{11}},\underline{14},\underline{\underline{0}}),\ (\underline{7},\underline{13},2,\underline{\underline{8}},\underline{\underline{0}}),\ (\underline{0},\underline{\underline{1}},\underline{\underline{4}},2,\underline{8}),\ (9,\underline{1},\underline{\underline{4}},\underline{\underline{6}},\underline{0}),\ (12,\underline{10},\underline{\underline{9}},\underline{\underline{3}},\underline{6}) \mod 15\}$. □

Then we have the following by an argument similar to that for Theorem 7.15.

Theorem 10.13. *Let* $v \ge 5$ *be odd. Then all* NDB$(5,5\lambda_2;v)$ *of form* $(2^2,\lambda_2)^1$ *exist for* $\lambda_2 \ge 1$.

Proof. By Theorem 10.3, Lemma 10.7 and Corollary 2.2, it suffices to show the existence of NDB$(5,5;v)$ of form $(2^2,1)^1$ for $v \in E$, where E is the same set as in the proof of Theorem 7.15. Theorem 10.3 settles the cases for $v = 13$, 17, 29, 33, 93, 113, 173. Theorem 10.5 covers the cases for $v = 11$, 31, 51, 71, 75, 95, 111, 115, 131, 135, 191, 195, 411. Lemma 10.7 covers the cases for $v = 19$, 23, 27, 43, 83. The remaining 15 cases are settled by Corollary 10.10, and Lemmas 10.11 and 10.12. Thus all NDB$(5,5;v)$ of form $(2^2,1)^1$ are constructed for $v \geq 5$. Then by taking each block and subblock of an NDB$(5,5;v)$ of form $(2^2,1)^1$ λ_2 times, an NDB$(5,5\lambda_2;v)$ of form $(2^2,\lambda_2)^1$ is obtained whenever v is odd ≥ 5. □

Combining conditions (1.1), Corollary 10.2 and Theorem 10.13, we can establish the following.

Theorem 10.14. *The necessary and sufficient conditions for the existence of an* NDB$(5,\lambda;v)$ *of form* $(2^2,\lambda_2)^1$ *are that* $v \geq 5$, $\lambda = 5\lambda_2$ *and* $\lambda_2(v-1) \equiv 0 \bmod 2$.

11. Main Result

Theorem 11.1. *The necessary conditions* (1.1) *for the existence of an* NDB$(k,\lambda;v)$ *of any possible form are also sufficient for* $k = 3$, 4 *and* 5.

Proof. The existence of an NDB$(5,\lambda;v)$ of form $(3,\lambda_3)^1(2,\lambda_2)^1$ can imply the existence of an NDB$(5,\lambda;v)$ of form $(3,\lambda_3)^1$, and the existence of an NDB$(5,\lambda;v)$ of form $(2,\lambda_2)^2$ can imply the existence of an NDB$(5,\lambda;v)$ of form $(2,\lambda_2)^1$. Hence, combining the results as in Sections 6 to 10 and as in [10], the proof is completed. □

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