A Small Embedding For Partial Directed $6k$-Cycle Systems

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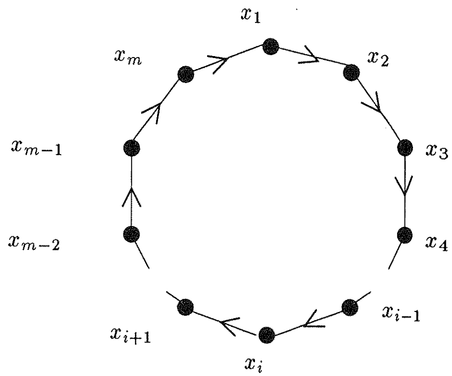
Abstract

The main result in this paper is that for $m \equiv 0 \pmod 6$ a partial directed m-cycle system of order n can be embedded in a directed m-cycle system of order less than $(mn)/2 + m^2/2 + 2m + 1$. For fixed m, this bound is asymptotic in n to $(mn)/2$ which is approximately one-half of the best known bound of $mn + (0 \text{ or } 1)$.

1 Introduction

Denote by D_n the complete directed graph on n vertices. A *directed* m-cycle of D_n is a collection of m directed edges of the edge set of D_n of the form $\{(x_1, x_2), (x_2, x_3), (x_3, x_4), \ldots, (x_{m-1}, x_m), (x_m, x_1)\}$, where $x_1, x_2, \ldots, x_m$ are m distinct vertices.



We will denote this m-cycle by any cyclic shift of $(x_1, x_2, x_3, \ldots, x_m)$.

A *directed* m-cycle system of order n is a pair (S, C), where C is a collection of directed m-cycles which partition the edge set of the complete directed graph D_n with vertex set S. The obvious necessary conditions for the existence of a directed m-cycle system of order n are:

$$\begin{cases} (1) & n \geq m, \text{ and} \\ (2) & n(n-1)/m \text{ is an integer.} \end{cases}$$

Whether or not these necessary conditions are also sufficient is an open problem. For an account of what is known the interested reader is referred to [4].

A *partial* directed m-cycle system of order n is a pair (X, P), where P is a collection of edge-disjoint directed m-cycles of the edge set of D_n with vertex set X. The difference between a partial directed m-cycle system and a directed m-cycle system is that the edge-disjoint m-cycles belonging to a partial directed m-cycle system do not necessarily include all of the edges of D_n.

Given a partial directed m-cycle system (X, P) of order n, we can ask if it is possible to decompose $E(D_n)\backslash E(P)$ (= the complement of the edge set of P in the edge set of D_n) into edge-disjoint directed m-cycles? That is, can a partial directed m-cycle system always be *completed* to a directed m-cycle system? For example, can the *partial* directed 3-cycle system (X, P) of order 5, with $X = \{1, 2, 3, 4, 5\}$ and $P = \{(1, 2, 4), (2, 3, 5), (1, 4, 3), (2, 5, 4)\}$, be *completed* to a directed 3-cycle system? The answer to this question is NO for the simple reason that a completion would produce a directed 3-cycle system of order 5 contradicting the necessary condition that the order of a directed 3-cycle system is $\equiv 0$ or 1 (mod 3). In general, it is easy to construct partial directed m-cycle systems which cannot be completed for any m.

Given the fact that a partial directed m-cycle system cannot necessarily be completed, the next question to ask is whether or not a partial directed m-cycle system can always be *embedded* in a directed m-cycle system. The partial directed m-cycle system (X, P) is said to be *embedded* in the directed m-cycle system (S, C) if and only if $X \subseteq S$ and $P \subseteq C$. For example the *partial* directed 3-cycle system (X, P) of order 5 in the above example is *embedded* in the directed 3-cycle system (S, C) of order 7 given by $S = \{1,2,3,4,5,6,7\}$ and $C = \{(\mathbf{1,\ 2,\ 4}), (\mathbf{2,\ 3,\ 5}), (\mathbf{1,\ 4,\ 3}), (\mathbf{2,\ 5,\ 4}), (3,4,6),(4,5,7),(5,6,1),(6,7,2),(7,1,3),(6,2,1),(7,3,2),(3,6,5),(4,7,6),(5,1,7)\}$.

If it is always possible to *embed* a partial directed m-cycle system in a directed m-cycle system, we would like the size of the containing system to be as small as possible.

The following table summarizes the best results to date on embedding partial directed m-cycle systems.

m	Best Embedding
ODD	$(2n+1)m,\ m > 3$ [5]
	$4n+1,\ m = 3$ [6]
EVEN	$nm,\ m \geq 8$ [6]
	$nm+1,\ m = 6$ [6]
	$\approx 2n + \sqrt{2n},\ m = 4$ [3]

The object of this paper is to reduce the bound for partial directed $6k$-cycle systems. In particular, for $m \equiv 0 \pmod 6$ we will show that a partial directed m-cycle system of order n can be embedded in a directed m-cycle system of order less than $(mn)/2 + m^2/2 + 2m + 1$. For fixed m, this is asymptotic in n to $(mn)/2$, and so far large n is roughly one-half of the best known bound of $nm + (0 \text{ or } 1)$.

2 Preliminaries

We collect together here the ingredients necessary for the construction in Section 3.

Denote by $D_{x,y}$ the complete directed bipartite graph with parts of size x and y.

Theorem 2.1 (D. Sotteau [8]) *Let $m = 2k$. The complete directed bipartite graph $D_{x,y}$ can be partitioned into directed m-cycles if and only if (i) $x \geq k, y \geq k$, and (ii) $m|2xy$.* □

Theorem 2.2 (A. Kotzig [2] and A. Rosa [7]) *There exists a directed m-cycle system of order $2m+1$ for every even m.* □

Theorem 2.3 (T. W. Tilson [9]) *There exists a directed m-cycle system of order m for all EVEN $m \notin \{4,6\}$ and one of order $m+1$ if $m \in \{4,6\}$.* □

Corollary 2.4 *Let $m \equiv 0$ (mod 6). There exists a directed m-cycle system of every order*
$n \equiv 1$ (mod m) $\geq 2m+1$.

Proof: Write $n = km+1$, $k \geq 2$. Let X be a set of size m and set $S = \{\infty\} \cup (X \times \{1,2,3,\ldots,k\})$. Further, let $(\{\infty\} \cup (X \times \{1,2\}), C(12))$ be a directed m-cycle system of order $2m+1$. There are two cases to consider: $m = 6$ and $m \geq 12$.

(a) $\mathbf{m = 6}$. For each $i = 3,4,\ldots,k$ let $(\{\infty\} \cup (X \times \{i\}), C(i))$ be a directed 6-cycle system of order 7. Define a collection C of directed 6-cycles as follows: (i)

$C(12) \subseteq C$, (ii) $C(i) \subseteq C$, (iii) for each $i \geq 3$, partition $D_{6,12}$ with parts $X \times \{i\}$ and $X \times \{1,2\}$ into directed 6-cycles and place these directed 6-cycles in C, and (iv) for each $i \neq j \geq 3$, partition $D_{6,6}$ with parts $X \times \{i\}$ and $X \times \{j\}$ into directed 6-cycles and place these directed 6-cycles in C. Then (S, C) is a directed 6-cycle system of order $n = 6k + 1$.

(b) **m $\geq$ 12.** For each $i = 3, 4, \ldots, k$ let $(X \times \{i\}, C(i))$ be a directed m-cycle system of order m. Define a collection C of directed m-cycles as follows: (i) $C(12)$, (ii) $C(i) \subseteq C$, (iii) for each $i \geq 3$, partition $D_{m,2m+1}$ with parts $X \times \{i\}$ and $\{\infty\} \cup (X \times \{1,2\})$ into directed m-cycles and place these directed m-cycles in C, and (iv) for each $i \neq j \geq 3$, partition $D_{m,m}$ with parts $X \times \{i\}$ and $X \times \{j\}$ into directed m-cycles and place these m-cycles in C. Then (S, C) is a directed m-cycle system of order $n = 6k + 1$. □

Finally, a *packing* of D_n with directed m-cycles is a triple (S, C, L), where S is the vertex set of D_n, C is a collection of edge-disjoint directed m-cycles, and L in the collection of edges *not* belonging to one of the directed m-cycles in C. L is called the *leave.*

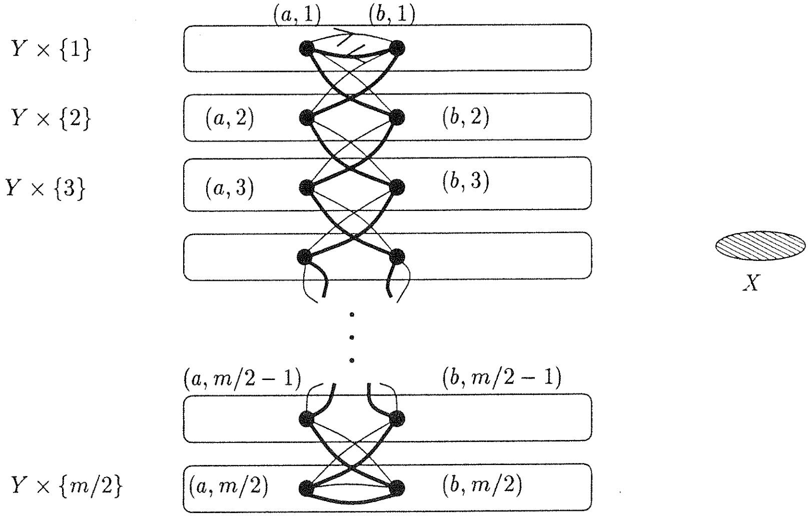
Lemma 2.5 *Let $m \equiv 0$ (mod 6). There exists a packing of D_t with directed m-cycles with leave consisting of $t/2$ vertex disjoint double edges for all $t \equiv 0$ (mod m).*

Proof: In [1] it is shown that there exists a packing of K_t (the complete undirected graph on t vertices) with m-cycles with leave a 1-factor. Replace each m-cycle with *two* directed m-cycles and each edge in the 1-factor with a double edge. □

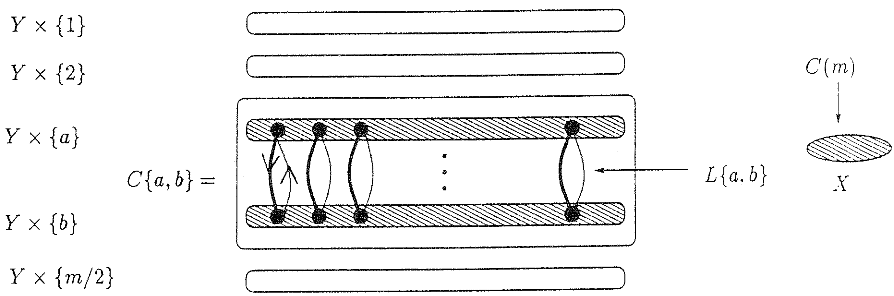
3 The $(km^2)/2 + 2m + 1$ Construction

Let $m = 6t$, Y a set of size km, and $(X, C(m))$ a directed m-cycle system of order $2m + 1$. Let $S = (Y \times \{1, 2, 3, \ldots, m/2\} \cup X$ and define a collection C of directed m-cycles of the edge set of D_s, $s = (km^2)/2 + 2m + 1$, with vertex set S as follows:

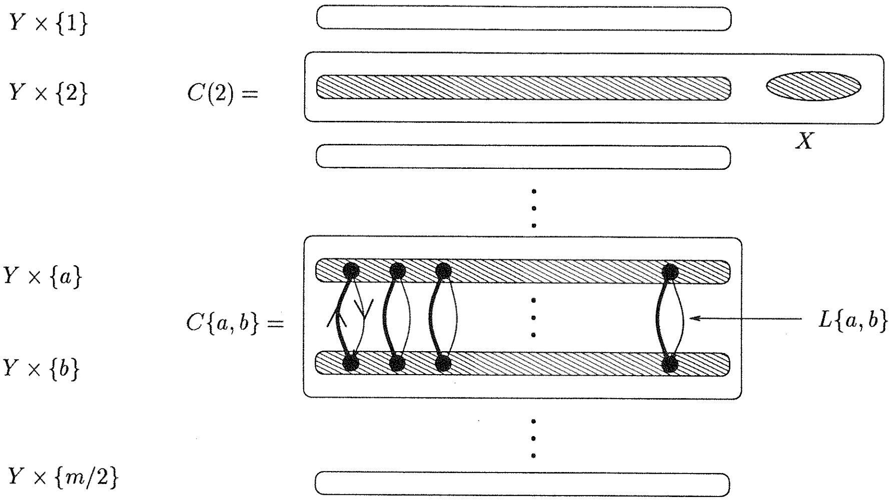
(1) For each 2-element subset $\{a, b\}$ of Y, place the two directed m-cycles $((a,1),(b,1),(a,2),(b,3),(a,4),(b,5),\ldots,(c,m/2-1),(d,m/2),(c,m/2),$ $(d,m/2-1),\ldots,(a,5),(b,4),(a,3),(b,2))$ and $((b,1),(a,1),(b,2),(a,3),(b,4),\ldots,$ $(b,5),\ldots,(c,m/2-1),(d,m/2),(c,m/2),(d,m/2-1),\ldots,(b,5),(a,4),(b,3),(a,2))$ in C, where $c = a$, $d = b$ if t is odd and $c = b$, $d = a$ if t is even.



(2a) If t is even, let π be a partition of $\{1, 2, 3, \ldots, m/2\} \backslash \{1, m/2\}$ into 2-element subsets $\{a, b\}$ such that $|a - b| \neq 1$. For each 2-element subset $\{a, b\} \in \pi$, let $(Y \times \{a, b\}, C\{a, b\}, L\{a, b\})$ be a packing of D_{2km} with vertex set $Y \times \{a, b\}$ and leave the collection of double edges $L\{a, b\} = \{((y, a), (y, b)), ((y, b), (y, a)) \mid y \in Y\}$ and place the directed m-cycles in $C\{a, b\}$ and $C(m)$ in C.

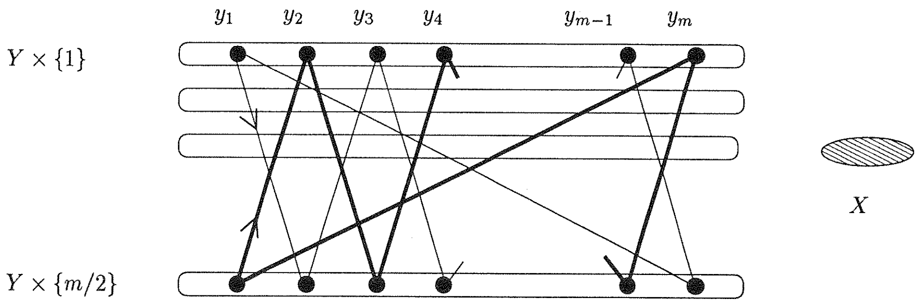


(2b) If t is odd let π be a partition of $\{1, 2, 3, \ldots, m/2\} \backslash \{1, 2, m/2\}$ into 2 element subsets $\{a, b\}$ such that $|a - b| \neq 1$. For each 2-element subset $\{a, b\} \in \pi$ let $(Y \times \{a, b\}, C\{a, b\}, L\{a, b\})$ be a packing of D_{2km} with vertex set $Y \times \{a, b\}$ and leave the collection of double edges $L\{a, b\} = \{((y, a), (y, b)), ((y, b), (y, a)) \mid y \in Y\}$. Let $((Y \times \{2\}) \cup X, C(2))$ be a directed m-cycle system of order $km + 2m + 1$. Place the directed m-cycles in $C\{a, b\}$ and $C(2)$ in C.

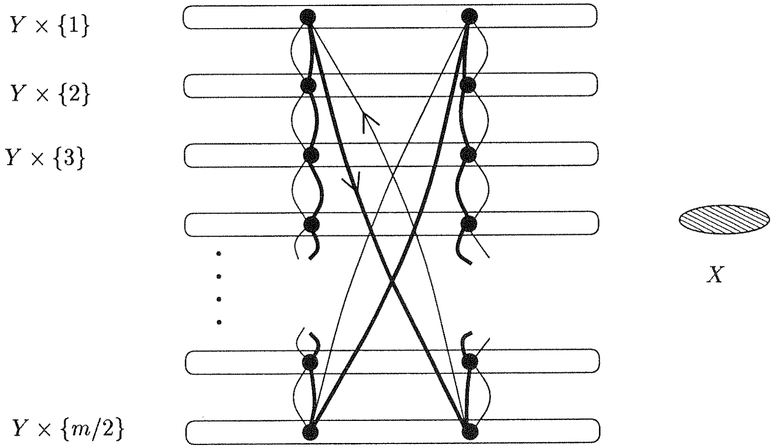


(3) For each $|i-j| \neq 1$ such that $\{i,j\} \notin \pi$ in (2a) or (2b), partition the complete directed bipartite graph with parts $Y \times \{i\}$ and $Y \times \{j\}$ into directed m-cycles and place these directed m-cycles in C. (Sotteau's Theorem 2.1.)

(4) Let (Y, P, L) be a packing of D_{km} with directed m-cycles with leave L consisting of $(km)/2$ vertex disjoint double edges. For each directed m-cycle $(y_1, y_2, y_3, \ldots, y_m) \in P$, place the TWO directed m-cycles $((y_1, 1), (y_2, m/2), (y_3, 1), (y_4, m/2), \ldots, (y_{m-1}, 1), (y_m, m/2))$ and $((y_1, m/2), (y_2, 1), (y_3, m/2), (y_4, 1), \ldots, (y_{m-1}, m/2), (y_m, 1))$ in C.



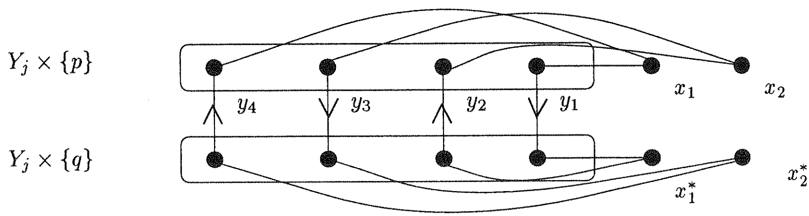
(5) For each double edge $((a,b),(b,a)) \in L$ place the TWO directed m-cycles $((a,1),(a,2),(a,3),\ldots,(a,m/2),(b,1),(b,2),(b,3),\ldots,(b,m/2))$ and $((a,1),(b,m/2),(b,m/2-1),\ldots,(b,2),(b,1),(a,m/2),(a,m/2-1),\ldots,(a,3),(a,2))$ in C.

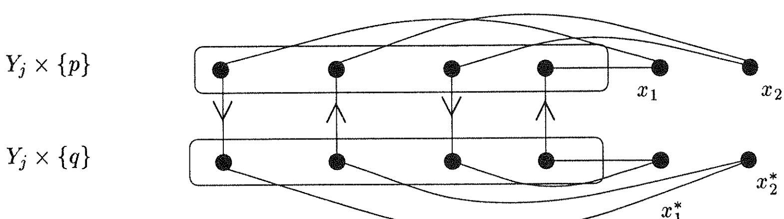


At this point the edges that have *not* been used are (i) the collection of double edges $D = \{((y,1),(y,m/2)),((y,m/2),(y,1)) \mid y \in Y\}$, (ii) the double edges in $L\{a,b\}, \{a,b\} \in \pi$ in (2a) and (2b), and (iii) the edges between X and $Y \times \{1,2,3,\ldots,m/2\}$ in (2a) and the edges between X and $Y \times (\{1,2,3,\ldots,m/2\} \backslash \{2\})$ in (2b).

(6) Partition Y into $3k$ subsets $Y_1, Y_2, Y_3, \ldots, Y_{3k}$ each of size $m/3$ and let $I = \{x_1, x_2, \ldots, x_{m/6}; x_1^*, x_2^*, x_3^3, \ldots, x_{m/6}^*\}$ be any $m/3$ distinct vertices in X. (Since $|X| = 2m+1$ this is possible.) For each $Y_j = \{y_1, y_2, \ldots, y_{m/3}\}$ and each 2-element subset $\{p,q\} = \{1, m/2\}$ or $\{p,q\} = \{a,b\} \in \pi$ place the TWO directed m-cycles $((x_1,p),(y_1,p),(y_1,q),(x_1^*,q),(y_2,q),(y_2,p),(x_2,p),(y_3,p),(y_3,q),(x_2^*,q),\ldots,$ $(y_{m/3-1},p),(y_{m/3-1},q),(x_{m/6}^*,q),(y_{m/3},q),(y_{m/3},p))$ AND $((y_{m/3},p),(y_{m/3},q),$ $(x_{m/6}^*,q),(y_{m/3-1},q),(y_{m/3-1},p),\ldots,(x_2^*,q),(y_3,q),(y_3,p),(x_2,p),(y_2,p),(y_2,q),$ $(x_1^*,q),(y_1,q),(y_1,p),(x_1,p))$ in C.

For the sake of understanding we will draw a diagram for $m = 12$





Denote by $V(z_i)$, $z_i \in I$, the set of vertices in $Y \times \{1,2,3,\ldots,m/2\}$ connected to z_i by an edge in one of the m-cycles constructed in (6). A simple calculation shows that the size of $V(z_i)$ is $(6km)/4$ or $6k(m-2)/4$. In either case $|V(z_i)| \geq m$. This is *important*, because it allows us to use Sotteau's Theorem in the final part of our construction.

(7) Clearly, the sets $V(z_i)$ $z_i \in I$, partition $Y \times \{1,2,3,\ldots,m/2\}$ in (2a) and partition $Y \times (\{1,2,3,\ldots,m/2\} \backslash \{2\})$ in (2b). We now partition the complete directed bipartite graph with parts $V(z_i)$ and $X \backslash \{z_i\}$, $z_i \in I$, into directed m-cycles and place these directed m-cycles in C. (This is possible since both $|V(z_i)|$ and $2m$ are $\geq m/2$ *and* m divides twice their product.

It is now straightforward and not difficult to show that (S,C) is a directed m-cycle system of order $(km^2)/2+2m+1$. (Just count the number of directed m-cycles and show that each directed edge is in at least one of the directed m-cycles described in (1), (2), (3), (4), (5), (6), or (7).

4 The $(km^2)/2+2m+1$ embedding

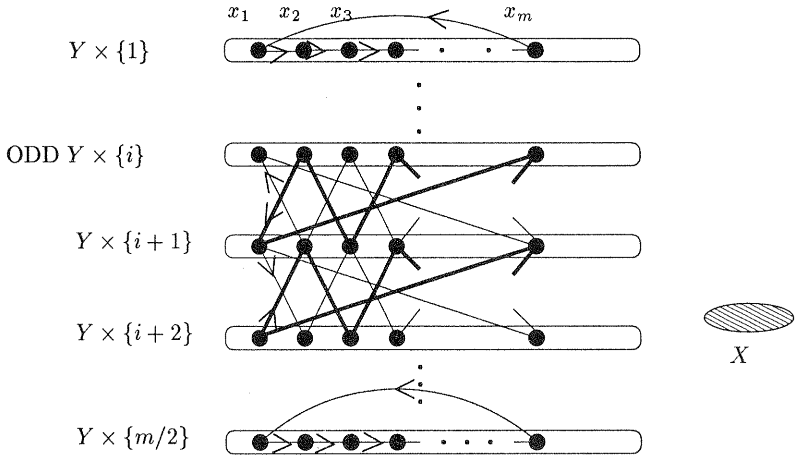
Let (Z,P) be a partial directed m-cycle system of order n, where $m = 6t$. Let km be the smallest positive integer such that $km \geq n$, Y a set of size km such that $Z \subseteq Y$, X a set of size $2m+1$, $S = (Y \times \{1,2,3,\ldots,m/2\}) \cup X$, and C the collection of directed m-cycles constructed with the $(km^2)/2+2m+1$ Construction.

For each directed m-cycle $p = (x_1, x_2, \ldots, x_m) \in P$ let mp be the collection of m directed m-cycles given by:

(1) $((x_1,1),(x_2,1),(x_3,1),\ldots,(x_m,1))$ and $((x_1,m/2),(x_2,m/2),(x_3,m/2),\ldots,(x_m,m/2))$; and

(2) for each $(i,i+1)$, where $i \in \{1,2,\ldots,m/2-1\}$ is EVEN, the two directed m-cycles $((x_1,i),(x_2,i+1),(x_3,i),(x_4,i+1),\ldots,(x_{m-1},i),(x_m,i+1))$ and $((x_1,i+1),(x_2,i),(x_3,i+1),(x_2,i),\ldots,(x_{m-1},i+1),(x_m,i))$.

(3) for each $(i,i+1)$, where $i \in \{1,2,\ldots,m/2-1\}$ is ODD, the two directed m-cycles $((x_m,i+1)),(x_{m-1},i),\ldots,(x_4,i+1),(x_3,i),(x_2,i+1),(x_1,i))$ and $((x_m,i),(x_{m-1},i+1),\ldots,(x_2,i),(x_3,i+1),(x_2,i),(x_1,i+1))$.



For each $a \neq b \in Y$, denote by $v(a,b)$ the cycle of C containing the edge $((a,1),(b,1))$ in part (1) of the $(km^2)/2 + 2m + 1$ Construction. For each $p = (x_1, x_2, x_3, \ldots, x_m) \in P$ let $vp = \{v(x_i, x_{i+1}) \mid (x_i, x_{i+1}) \text{ is an edge of } p\}$.

Then mp and vp are *mutually balanced*; i.e., they contain *exactly* the same edges. Furthermore, if $p_1 \neq p_2$, the edge sets of vp_1 and vp_2 are disjoint. Now set $C^* = (C \backslash \{vp \mid p \in P\}) \cup \{mp \mid p \in P\}$. Then (S, C^*) is a directed m-cycle system of order $(km^2)/2 + 2m + 1$ which contains (at least) two disjoint copies of the partial directed m-cycle system (Z, P); namely, the directed m-cycles of type (1) in each collection mp.

Theorem 4.1 *Let $m \equiv 0$ (mod 6). A partial directed m-cycle system of order n can be embedded in a directed m-cycle system of order $(km^2)/2 + 2m + 1$, where k is the smallest positive integer such that $km \geq n$.* □

5 Concluding remarks

If k is the smallest positive integer such that $km \geq n$, then $(km^2)/2 + 2m + 1 \leq (nm)/2 + m^2/2 + 2m + 1$. For fixed m, this is asymptotic in n to $(nm)/2$ and so for large n, as advertised in Section (1), is *roughly one-half* of the best known bound of $nm+$ (0 or 1).

References

[1] P. Horák and C. C. Lindner, *A small embedding for partial even-cycle systems*, (to appear).

[2] A. Kotzig, *On the decomposition of complete graphs into 4k-gons*, Mat.-Fyz. Časopis Sloven. Akad. Vied., 15 (1965), 229-233.

[3] C. C. Lindner and C. A. Rodger, *Small embeddings for partial directed cycle systems*, Bull. Aust. Math. Soc., 46 (1992), 213-224.

[4] C. C. Lindner and C. A. Rodger, *Decomposition into cycles II: Cycle Systems*, in Contemporary Design Theory: A collection of surveys, (John Wiley and Sons), eds. J. H. Dinitz and D. R. Stinson, (1992), 325-369.

[5] C. C. Lindner and C. A. Rodger, *Embedding directed and undirected partial cycle systems of index* $\lambda > 1$, J. Combinatorial Designs, 2 (1993), 113-123.

[6] C. A. Rodger, *Embedding partial Mendelsohn triple systems*, Discrete Math. 65 (1987), 187-196.

[7] A. Rosa, *On cyclic decompositions of the complete graph into* $(4m+2)$*-gons*, Mat.-Fyz. Časopis Sloven. Akad. Vied., 16 (1966), 349-353.

[8] D. Sotteau, *Decompositions of* $K_{m,n}(K^*_{m,n})$ *into cycles (circuits) of length* $2k$, J. Combin. Th. (B), 30 (1981), 75-81.

[9] T. W. Tilson, *A Hamilton decomposition of* K^*_{2m}, $2m > 8$, J. Combinatorial Theory, Ser. B., 30 (1980), 68-74.

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