Decompositions of complete tripartite graphs into k-cycles

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Abstract

We show that a complete tripartite graph with three partite sets of equal size m may be decomposed into k-cycles for any $k \geq 3$ if and only if k divides $3m^2$ and $k \leq 3m$.

1 Introduction

We begin with a few definitions. The complete tripartite graph with three partite sets of equal size m will be denoted by $K(m,m,m)$. This is sometimes known as a group divisible k-cycle system, or GDkCS, with three groups of size m. A k-circuit is a nontrivial closed trail with k edges. A k-cycle is a k-circuit with no vertices repeated. A graph G with q edges is said to be decomposable into the graph H if it can be written as the union of edge-disjoint copies of H so that every edge in G belongs to one and only one copy of H.

Although much work has gone into the decomposition of complete graphs into k-cycles (see [4] for a good survey), little attention has been paid to the same problem for arbitrary complete n-partite graphs. A significant and useful result is that of Sotteau [6], who discovered necessary and sufficient conditions for the decomposition of complete bipartite graphs into k-cycles:

> *The complete bipartite graph $K(r,s)$ can be decomposed into cycles of length k if and only if k, r and s are even, $r \geq k/2$, $s \geq k/2$, and k divides rs.*

For example, if k is even, then $K(k,k,k)$ decomposes into k-cycles, since the graph $K(k,k,k)$ may be decomposed into three copies of $K(k,k)$.

The problem becomes more difficult for complete tripartite graphs, one of the reasons being that decompositions into odd-length cycles are also permissible. For the 3-cycle case it is well known that a decomposition of $K(m,m,m)$ into triangles is equivalent to a latin square of order m. Mahmoodian and Mirzakhani [5] obtained

various results for 5-cycles. However, even when only two of the three partite sets have the same size, the problem of decomposition into 5-cycles remains unsolved.

Other known results come indirectly from more general theorems. For example, $K(k,k,k)$ decomposes into k-cycles for odd k [1]. This in fact holds for any n-partite graph where n is odd and all partite sets have size k. Another useful result is that $K(k,k,k)$ decomposes into $3k$-cycles for any k [3]. This comes from a broader result which gives all possible hamiltonian decompositions of complete n-partite graphs.

In this paper we shall give necessary and sufficient conditions for the decomposition of the complete tripartite graph $K(m,m,m)$ into k-cycles, for any $k \geq 3$. In Section 2 we give some tools for decomposition that are used in Section 3 to prove the main result.

2 A few useful tools for decomposition

The lexicographic product $G_1 \otimes G_2$ of the graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and an edge joining (u_1,u_2) to (v_1,v_2) if and only if either u_1 is adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 and v_2 are adjacent in G_2. We are only concerned with a particular kind of lexicographic product, $G \otimes \overline{K_n}$. This is a more formal way of expressing a technique often known as "blowing up" points.

Observe that $K(ml,ml,ml) = K(m,m,m) \otimes \overline{K_l}$. In fact, if $K(m,m,m)$ decomposes into copies of the graph G and $G \otimes \overline{K_l}$ decomposes into k-cycles then $K(ml,ml,ml)$ decomposes into k-cycles.

THEOREM 2.1 *If the graph $K(m,m,m)$ decomposes into k-cycles, then the graph $K(ml,ml,ml)$ decomposes into k-cycles for any positive integer l.*

Proof

From the previous observation all we need to show is that $C_k \otimes \overline{K_l}$ decomposes into copies of C_k, where C_k is just a k-cycle. Label the vertices of the original k-cycle with the integers $\{1,2,\ldots,k\}$ and the vertices of K_l with $\{1,2,\ldots,l\}$.

Take any $l \times l$ latin square and consider each element in the form (α,β,γ), where α denotes the row, β the column and γ the entry, with $1 \leq \alpha,\beta,\gamma \leq l$. From each of the l^2 elements we can construct a k-cycle. If k is even each cycle is of the form:

$$((1,\alpha),(2,\beta),(3,\alpha),\ldots,(k-1,\alpha),(k,\beta)).$$

If k is odd, each cycle is of the form:

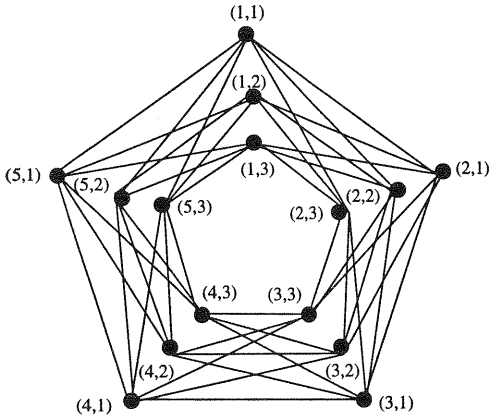
$$((1,\alpha),(2,\beta),(3,\alpha),\ldots,(k-1,\beta),(k,\gamma)).$$

Because of the properties of a latin square the decomposition is complete. ∎

The following corollary holds because in the previous proof we did not exploit the fact that $K(m,m,m)$ is tripartite.

COROLLARY 2.2 *If the complete n-partite graph $K(a_1, a_2, \ldots, a_n)$ decomposes into k-cycles, then $K(a_1 l, a_2 l, \ldots, a_n l)$ decomposes into k-cycles.*

EXAMPLE 2.3 The graph $C_5 \otimes \overline{K_3}$ decomposes into copies of C_5.



Take the following 3×3 latin square:

1	2	3
2	3	1
3	1	2

For each entry of the latin square we construct a 5-cycle in the graph $C_5 \otimes \overline{K_3}$:

$((1,1),(2,1),(3,1),(4,1),(5,1))$, $((1,1),(2,2),(3,1),(4,2),(5,2))$,
$((1,1),(2,3),(3,1),(4,3),(5,3))$, $((1,2),(2,1),(3,2),(4,1),(5,2))$,
$((1,2),(2,2),(3,2),(4,2),(5,3))$, $((1,2),(2,3),(3,2),(4,3),(5,1))$,
$((1,3),(2,1),(3,3),(4,1),(5,3))$, $((1,3),(2,2),(3,3),(4,2),(5,1))$,
$((1,3),(2,3),(3,3),(4,3),(5,2))$.

The decomposition is complete. ∎

THEOREM 2.4 *If the graph $K(m,m,m)$ decomposes into k-cycles, then the graph $K(ml,ml,ml)$ decomposes into kl-cycles for any positive integer l.*

Proof

Note that we need only show that $C_k \otimes \overline{K_l}$ decomposes into lk-cycles. This has been done previously by Laskar [2]. We offer an alternative proof which relies on the existence of two orthogonal latin squares of order l, so the proof fails when $l = 2$ or $l = 6$. However, the construction is necessary for later, and is rather nice, so we mention it anyway.

Since we are assuming l does not equal 2 or 6, there exist at least two orthogonal latin squares of order l. Thus there exists an $l \times l$ latin square which can be partitioned into l disjoint transversals.

Label the vertices of $C_k \otimes \overline{K_l}$ as in Theorem 2.1. For each triple (row, column, entry) in the transversal, construct a k-cycle using the method from Theorem 2.1. We concatenate these l k-cycles so that the cycle corresponding to row 1 occurs first, the cycle corresponding to row 2 occurs second and so forth. We then have one lk-cycle, and from the l transversals we can form l cycles of length lk. These cycles completely decompose $C_k \otimes \overline{K_l}$. ∎

Again we have not used the fact that the graph $K(m,m,m)$ is tripartite, so the following corollary is true.

COROLLARY 2.5 *If the complete n-partite graph $K(a_1, a_2, \ldots, a_n)$ can be decomposed into k-cycles, then $K(a_1 l, a_2 l, \ldots, a_n l)$ can be decomposed into kl-cycles.*

EXAMPLE 2.6 The graph $C_5 \otimes \overline{K_3}$ decomposes into copies of C_{15}. The 3×3 latin square mentioned previously can be partitioned into 3 transversals:

1		
	3	
		2

	2	
		1
3		

		3
2		
	1	

For each (row, column, entry) triple we construct a 5-cycle (see Example 2.3). For each transversal, we concatenate the three corresponding 5-cycles to obtain a 15-cycle, ensuring that the 5-cycle obtained from the first row occurs first, the 5-cycle obtained from the second row second, and so on. This gives us three 15-cycles, which decompose $C_5 \otimes \overline{K_3}$:

$((1,1),(2,1),(3,1),(4,1),(5,1),(1,2),(2,2),(3,2),(4,2),(5,3),$
$(1,3),(2,3),(3,3),(4,3),(5,2)),$
$((1,1),(2,2),(3,1),(4,2),(5,2),(1,2),(2,3),(3,2),(4,3),(5,1),$
$(1,3),(2,1),(3,3),(4,1),(5,3)),$
$((1,1),(2,3),(3,1),(4,3),(5,3),(1,2),(2,1),(3,2),(4,1),(5,2),$
$(1,3),(2,2),(3,3),(4,2),(5,1)).$

∎

Our final tool for decomposition uses a latin square to extend a circuit decomposition of a small graph to a cyclic decomposition of a larger graph. Note that unlike the previous theorems in this section, the following construction applies only to complete tripartite graphs (although it also works if the partite sets have different size).

THEOREM 2.7 *If there exists a decomposition of $K(m,m,m)$ into k-circuits such that within a circuit any vertex appears at most l times (or, equivalently, any vertex has degree at most $2l$), then there exists a decomposition of $K(ml, ml, ml)$ into k-cycles.*

Proof

Let L_k be a circuit of length k in the decomposition of $K(m,m,m)$ with each vertex appearing at most l times in the circuit. Label the partite sets of $K(m,m,m)$ with a, b and c. Label the vertices of L_k with $\{1,2,\ldots,k'\}$, where k' is the number of distinct vertices in L_k, and therefore may be less than k. We then attach one of the subscripts a, b or c to each label, indicating which partite set the corresponding vertex belongs to. For example we could have

$$L_k = (1_a, 2_b, 3_a, 4_c, 5_b, 3_a, \ldots, k'_b).$$

Label the vertices of $\overline{K_l}$ with $\{1,2,\ldots,l\}$ as usual. We need to decompose $L_k \otimes \overline{K_l}$ into k-cycles.

Let $(Q,\circ)$ be a quasigroup of order l, with its elements taken from $\{1,2,\ldots,l\}$. (Q is a quasigroup if and only if its multiplication table is a latin square.) Let $i,j \in Q$. Then we can construct a k-cycle in $L_k \otimes \overline{K_l}$ from the above circuit as follows.

For each vertex of type x_a in the circuit, replace the first occurrence of x_a with (x_a,i); replace the second occurrence of x_a with $(x_a,i+1)$; and so on. In general, replace the mth occurrence of x_a with $(x_a,i+m-1)$.

For each vertex of type x_b in the circuit, replace the mth occurrence of x_b with $(x_b,j+m-1)$. Similarly, replace the mth occurrence of x_c with $(x_c,i\circ j+m-1)$. We repeat this process for each $i,j \in Q$ to obtain l^2 cycles of length k.

We now show these cycles do indeed constitute a decomposition of $L_k \otimes \overline{K_l}$ into k-cycles. Firstly, observe that since each vertex appears at most l times within L_k, from the above construction no vertex will be repeated within a cycle.

Now take an arbitrary edge $((x_a,i),(y_b,j))$ from $L_k \otimes \overline{K_l}$. The edge (x_a,y_b) must exist within L_k. Assume this is the i'th appearance of x_a and the j'th appearance of y_b. Then if we choose $i-i'+1$ and $j-j'+1$ from Q, the k-cycle constructed will include the edge $((x_a,i),(y_b,j))$. Similarly we may show that every edge of the form $((y_b,i),(z_c,j))$ and $((z_c,i),(x_a,j))$ exists for $1 \le i,j \le l$. ∎

3 Decomposition of $K(m,m,m)$ into k-cycles

THEOREM 3.1 *The graph $K(m,m,m)$ can be decomposed into k-cycles if and only if k divides $3m^2$ and $k \le 3m$.*

Proof

The number of edges in $K(m,m,m)$ is $3m^2$ so k must divide $3m^2$ for a decomposition to occur. If $k > 3m$ a vertex must be repeated within each k-cycle which is impossible, so $k \le 3m$.

Case I: 3 divides k.

Write $k = 3s^2t$, where t is square-free. From the necessary conditions, $3s^2t \mid 3m^2$, so $st \mid m$, and we can write $m = stm'$. Also, $3s^2t \le 3stm'$, so we have $m' \ge s$. Now consider $K(s,s,s)$. The degree of each vertex is even, so there exists an Eulerian circuit of length $3s^2$, with each vertex appearing exactly s times. So by Theorem 2.7,

the graph $K(sm', sm', sm')$ can be decomposed into cycles of length $3s^2$. Applying Theorem 2.4, we have that $K(sm't, sm't, sm't)$ decomposes into $3s^2t$-cycles.

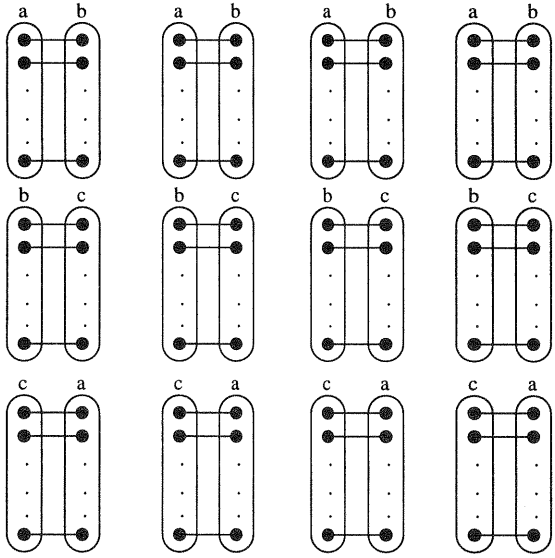
Case II: 3 does not divide k.

Let $k = s^2t$, where t is square free. From the necessary conditions, st divides m, so write $m = stm'$. Also $s^2t \leq 3stm'$, so $s \leq 3m'$ and in fact $m' \geq \lceil s/3 \rceil$.

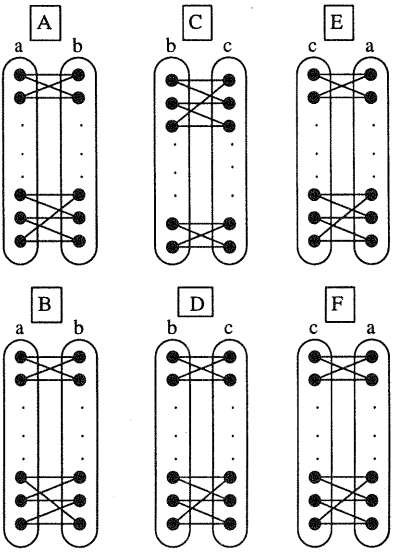
First consider when $s = 1$. Note that $K(t, t, t)$ decomposes into t-cycles either using Sotteau's result if t is even or from [1] (as mentioned in Section 1). Then, applying Theorem 2.1, $K(m't, m't, m't)$ decomposes into t-cycles. Next consider two subcases:

Case IIa: $s \equiv 1 \pmod 3$ and $s > 1$. Let $s = 3s' + 1$, so that $\lceil s/3 \rceil = s' + 1$. The graph $K(1, 1, 1)$ is a triangle, so it certainly decomposes into one 3-cycle. We next apply Theorem 2.4 to show that $K(s, s, s)$ decomposes into $3s$-cycles. We use transversals in this construction (since s does not equal 2 or 6) so that each $3s$-cycle consists of three matchings, one between each pair of partite sets.

Take any four $3s$-cycles from the decomposition of $K(s, s, s)$ to obtain twelve matchings. We label the partite sets a, b and c.



Each pair of matchings between the same two partite sets forms either a $2s$-cycle or a number of even-length cycles, the edges of which add up to $2s$. These pairs are labelled as shown.



Now, take any $s' - 1$ of the remaining $3s$-cycles together with the sets of cycles A and B to make a circuit of length $(s' - 1)3s + 2(2s) = s^2$. The remaining two circuits of length s^2 are formed in a similar fashion, using C and D, then E and F respectively. Each vertex will appear at most $s' + 1$ times, so we can apply Theorem 2.7 to obtain a decomposition of $K(sm', sm', sm')$ into s^2-cycles. Then from Theorem 2.4, we have a decomposition of $K(sm't, sm't, sm't)$ into s^2t-cycles, as required.

Case IIb: $s \equiv 2 \pmod 3$.

We let $s = 3s' + 2$, so that $\lceil s/3 \rceil = s' + 1$. This time we take only two $3s$-cycles from the decomposition of $K(s, s, s)$, and form sets of even cycles between each pair of partite sets, the edges of which add up to $2s$, as before. Note that this is not possible if $s = 2$, since no pair of 2×2 orthogonal latin squares exist, but in this case we use Sotteau's result to show that $K(2, 2, 2)$ decomposes into three 4-cycles [6].

For $s \neq 2$, simply form three circuits of length s^2 by taking s' cycles of length $3s$ and one set of even cycles with $2s$ edges, noting that $s'3s + 2s = s^2$. Each circuit is made from s cycles of length $3s$ and a set of disjoint cycles, so each vertex occurs at most $s' + 1$ times within a circuit, and we can apply Theorems 2.7 and 2.4 in the same way as before.

This completes the proof. ∎

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