The Directed Almost Resolvable Hamilton-Waterloo Problem

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Abstract

In this paper, a result is proved that provides a general method of attack that can be used to solve the existence problem for partitions of the directed edges of D_n, the complete directed graph on n vertices into x and $n-x$ almost parallel classes of directed cycles of length m_1 and m_2 respectively in the case where m_1 and m_2 are even. Use of this technique is then demonstrated by essentially solving the problem when $(m_1, m_2) \in \{(4,6), (4,8)\}$.

1 Introduction

Let λK_n denote the multigraph on n vertices in which each pair of vertices is joined by λ edges, and let D_n denote the complete directed graph on n vertices.

Let M be a set of positive integers. An *M-cycle system* of G (or simply an *m-cycle system* if $M = \{m\}$) is an ordered pair $(V(G), C)$ where C is a set of cycles with lengths in M whose edges partition the edge set of G.

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A *parallel class* of C is a set of cycles in C that form a 2-factor of G. $(V(G), C)$ is said to be *resolvable* if C can be partitioned into parallel classes. The *spectrum* problem for resolvable m-cycle systems is to find the set of integers n for which there exists a resolvable m-cycle system of K_n. This is also known as the *Oberwolfach problem*, as it was first posed in Oberwolfach by Ringel in 1967 to solve the seating arrangement for n people at round tables, each table seating m people, so that after $(n-1)/2$ successive meals each person would sit beside each other person exactly once. This problem has now been completely solved [1, 9].

Clearly one necessary condition on n in any solution of the Oberwolfach problem is that $n \equiv 0 \pmod{m}$. A related problem has also been considered in the case where $n \equiv 1 \pmod{m}$. An *almost parallel class* of an m-cycle system $(V(G), C)$ of G that is *missing* the vertex v is a set of m-cycles in C that forms a 2-factor of $G - v$. The m-cycle system $(V(G), C)$ is said to be *almost resolvable* if C can be partitioned into almost parallel classes. It is easy to see that there is no almost resolvable m-cycle system of K_n for any n; the spectrum problem for almost resolvable m-cycle systems of $2K_n$ has been completely settled [4, 8]. Of course, if one is considering $2K_n$, then immediately the more difficult question of the existence of a directed analogue is raised. It has been shown that there exists an almost resolvable directed m-cycle system of D_n if and only if $n \equiv 1 \pmod{m}$ in the cases where $m = 3$ [2], $m = 4$ [3], $m = 5$ [7], m is even [5], and where $m \equiv 3 \pmod{6}$ [6].

A more general (and as yet unsolved) problem than the Oberwolfach problem is the Hamilton-Waterloo problem. In this case, a joint conference is held at two places: s days are spent at Hamilton and $(n-1)/2 - s$ days at Waterloo, where each table seats m_1 people at Hamilton and m_2 people at Waterloo. Again, a seating arrangement is required so that each of the n people sits next to each of the other people exactly once. So if s meals are held at Hamilton, then this asks for the set of integers n for which there exists a 2-factorization of K_n in which s 2-factors consist of cycles of length m_1, and $(n-1)/2 - s$ 2-factors consist of cycles of length m_2. A particularly interesting case for the Hamilton-Waterloo problem is an "extreme" situation when $(m_1, m_2) = (3, n)$, especially because it is quite different to previously considered problems in that m_1 has a fixed length whereas m_2 increases with the size of the graph.

It is also worth mentioning that Rees solved a related factorization problem: partitioning the edges of K_n into s 2-factors, each cycle in each 2-factor being a 3-cycle, and $n - 1 - 2s$ 1-factors [10].

Of course, there is also an almost resolvable directed version of the Hamilton-Waterloo problem when $n \equiv 1 \pmod{m_i}$, $i \in \{1, 2\}$. That is, we can try to find the set of integers n for which there exists an almost resolvable directed $\{m_1, m_2\}$-cycle system of D_n into x almost parallel classes of directed m_1-cycles and $n - x$ almost parallel classes of directed m_2-cycles. It is the purpose of this paper to give a general method of attack for this problem that can be used whenever m_1 and m_2 are even (see Theorem 2.1), and then to demonstrate the use of this theorem to essentially solve this spectrum problem when $(m_1, m_2) \in \{(4, 6), (4, 8)\}$ (see Corollaries 3.1 and 3.2).

2 The General Attack

Let $(A)RD(m_1, m_2)_s$ of D denote an (almost) resolvable directed $\{m_1, m_2\}$-cycle system of the directed graph D in which s (almost)-parallel classes consist of directed m_1-cycles, the remaining (almost)-parallel classes consisting of directed m_2-cycles.

The basic tool used in the proof of Theorem 2.1 is the following lemma that was proved in [5]. Let $\mathbb{Z}_r = \{0, 1, \ldots, r-1\}$.

Lemma 2.1. *Let $F = \{\{0,1\},\{2,3\},\ldots,\{2r-2, 2r-1\}\}$, and let $r \geq 3$. There exists a 1-factorization $\{F_0, F_1, \ldots, F_{2r-1}\}$ of the multigraph $K_{2r} + F$ in which F_{2z} and F_{2z+1} each contain a copy of the edge $\{2z, 2z+1\}$ for each $z \in \mathbb{Z}_r$.*

The following result gives a method for finding the spectrum for an $ARD(m_1, m_2)_s$ of D_n. Let $D_{x,x}$ denote the complete directed bipartite graph with x vertices in each part (so each vertex has in-degree and out-degree x).

Theorem 2.1. *Let m_1, m_2 be even, let $r \geq 3$, let $\ell = lcm(m_1, m_2)$, and let $n = r\ell + 1$. Suppose $L \subseteq \mathbb{Z}_{(\ell/2)+1}$ has the property that for any $s \in \mathbb{Z}_{n+1}$ there exist $s_0, \ldots, s_{2r-1} \in L$ and $\epsilon \in \{0,1\}$ such that $s = \epsilon + \sum_{t \in \mathbb{Z}_{2r}} s_t$. Suppose there exists an $RD(m_1, m_2)_x$ of $D_{\ell/2,\ell/2}$ for each $x \in L$, and there exists an $ARD(m_1, m_2)_x$ of $D_{\ell+1}$ for each $x \in \mathbb{Z}_{\ell+1}$. Then there exists an $ARD(m_1, m_2)_s$ of D_n for each $s \in \mathbb{Z}_n$.*

Remark . Of course, if $0 < s < n$ and $\ell = lcm(m_1, m_2)$ then $n \equiv 1 \pmod{\ell}$ is a necessary condition for the existence of an $ARD(m_1, m_2)_s$ of D_n.

Proof. Let $s \in \mathbb{Z}_{n+1}$. Then by assumption there exist $s_0, \ldots, s_{2r-1} \in L$ and $\epsilon \in \{0,1\}$ such that $s = \epsilon + \sum_{t \in \mathbb{Z}_{2r}} s_t$. We produce an $ARD(m_1, m_2)_s$ of D_n $(\{\infty\} \cup (\mathbb{Z}_{2r} \times \mathbb{Z}_{\ell/2}), C)$ as follows.

Firstly, for each $z \in \mathbb{Z}_r$, let $(\{\infty\} \cup (\{2z, 2z+1\} \times \mathbb{Z}_{\ell/2}), C_z)$ be an $ARD(m_1, m_2)_{\epsilon + s_{2z} + s_{2z+1}}$ of $D_{\ell+1}$ (this exits by assumption). Name these so that:

(1) for each $i \in \mathbb{Z}_{2r}$ and each $j \in \mathbb{Z}_{\ell/2}$, $p_{i,j}$ is the almost parallel class in $C_{\lfloor i/2 \rfloor}$ that is missing vertex (i, j), and that consists of directed m_1-cycles or m_2-cycles if $0 \leq j < s_i$ or $s_i \leq j < \ell/2$ respectively; and

(2) for each $z \in \mathbb{Z}_r$, $p_{z,\infty}$, is the almost parallel class in C_z that is missing vertex ∞, and that consists of directed m_1-cycles or m_2-cycles if $\epsilon = 1$ or 0 respectively.

Secondly, let $F = \{\{0,1\},\{2,3\},\ldots,\{2r-2, 2r-1\}\}$, and let $\{F_0, F_1, \ldots, F_{2r-1}\}$ be a 1-factorization of $K_{2r} + F$ in which F_{2z} and F_{2z+1} each contain the edge $\{2z, 2z+1\}$ (see Lemma 2.1). For each $i \in \mathbb{Z}_{2r}$ and for each edge $\{x, y\} \in F_i \backslash F$, let $(\{x, y\} \times \mathbb{Z}_{\ell/2}, C_{\{x,y\}})$ be an $RD(m_1, m_2)_{s_i}$ of $D_{\ell/2,\ell/2}$ (this exists by assumption). Name these so that:

(3) for $0 \leq j < s_i$ or $s_i \leq j < \ell/2$, $p_{\{x,y\},j}$ is a parallel class in $C_{\{x,y\}}$ that consists of directed m_1-cycles or m_2-cycles respectively.

Then define $P_{i,j} = \bigcup_{\{x,y\} \in F_i \setminus F} p_{\{x,y\},j}$ for each $i \in \mathbb{Z}_{2r}$ and each $j \in \mathbb{Z}_{\ell/2}$.

Finally, define $\pi_\infty = \bigcup_{z \in \mathbb{Z}_r} p_{z,\infty}$, $\pi_{i,j} = p_{i,j} \cup P_{i,j}$, for each $i \in \mathbb{Z}_{2r}$ and each $j \in \mathbb{Z}_{\ell/2}$, and $C = \pi_\infty \cup (\bigcup_{\substack{i \in \mathbb{Z}_{2r} \\ j \in \mathbb{Z}_{\ell/2}}} \pi_{i,j})$. Then $(\{\infty\} \cup (\mathbb{Z}_{2r} \times \mathbb{Z}_{\ell/2}), C)$ is an $ARD(m_1, m_2)_s$ as the following indicates.

Clearly π_∞ is an almost parallel class of directed m_1-cycles or directed m_2-cycles if $\epsilon = 1$ or 0 respectively that is missing vertex ∞. For each $i \in \mathbb{Z}_{2r}$ and for $0 \leq j < s_i$, $\pi_{i,j}$ is an almost parallel class of directed m_1-cycles that is missing vertex (i,j). For each $i \in \mathbb{Z}_{2r}$ and for $s_i \leq j < \ell/2$, $\pi_{i,j}$ is an almost parallel class of directed m_2-cycles that is missing vertex (i,j). So C can be partitioned into $\epsilon + \sum_{i \in \mathbb{Z}_{2r}} s_i = s$ almost parallel classes of directed m_1-cycles and $n - s$ almost parallel classes of directed m_2-cycles, as required.

To see that a directed $\{m_1, m_2\}$-cycle system has been formed, note that: for i, $u \in \mathbb{Z}_{2r}$ and each j, $v \in \mathbb{Z}_{\ell/2}$: the directed edges $(\infty, (i,j))$ and $((i,j), \infty)$ occur in a directed cycle in $C_{\lfloor i/2 \rfloor}$; if $\lfloor i/2 \rfloor = \lfloor u/2 \rfloor$ then the directed edge $((i,j),(u,v))$ occurs in a directed cycle in $C_{\lfloor i/2 \rfloor}$; and if $\lfloor i/2 \rfloor \neq \lfloor u/2 \rfloor$ then the directed edge $((i,j),(u,v))$ occurs in a directed cycle in $C_{\{\lfloor i/2 \rfloor, \lfloor u/2 \rfloor\}}$. □

When applying Theorem 2.1, the choice of L will depend on m_1 and m_2. In any case, L could be chosen for example to be $\{0, 2, 4, \ldots, \ell/2\}$, but usually a much smaller set will suffice.

It should also be recorded that some of the ingredients needed to apply Theorem 2.1 have already been found. First, the case where $m_1 = m_2$ (or if you prefer, the cases where $s \in \{0, n\}$) has been settled.

Theorem 2.2 ([5]). *There exists a directed 2m-cycle system of D_n if and only if $n \equiv 1$ (mod m).*

Also, the existence of $RD(m_1, m_2)_x$ of $D_{\ell/2,\ell/2}$ has been considered in the case where $m_1 = m_2$.

Lemma 2.2 ([5]). *Let m_1 and m_2 be even, and let $\ell = lcm(m_1, m_2)$. There exists an $RD(m_1, m_2)_x$ of $D_{\ell/2,\ell/2}$ for each $x \in \{0, \ell/2\}$.*

3 Applications of the Main Theorem

In this section we demonstrate the use of Theorem 2.1 by applying it to the cases where $(m_1, m_2) \in \{(4, 8), (4, 6)\}$. Throughout the following we adopt the convention that $(c_1, c_2, \ldots, c_m) + i = (c_1 + i, c_2 + i, \ldots, c_m + i)$ reducing the sums modulo n (the value of n will be clear from the vertex set of the graph containing the m-cycle).

Corollary 3.1. *Let $0 \leq s \leq n$. There exists an $ARD(4, 8)_s$ of D_n if and only if*

$$n \equiv \begin{cases} 1 \text{ (mod 4)} & \text{if } s = n, \\ 1 \text{ (mod 8)} & \text{otherwise.} \end{cases}$$

Proof. The necessity is obvious, and the sufficiency is proven in the cases where $s \in \{0, n\}$ by Theorem 2.2. So we can assume that $1 \le s < n$. To apply Theorem 2.1, it suffices to find an $ARD(4,8)_x$ of D_9 for all $x \in \{1, \ldots, 8\}$, and an $ARD(4,8)_x$ of $D_{4,4}$ for each $x \in \{0, 2, 4\}$, since clearly $L = \{0, 2, 4\}$ suffices; the cases where $x \in \{0, 4\}$ are handled by Lemma 2.2. These ingredients are listed below with the cycles arranged into almost parallel classes. Unless otherwise stated, the vertex set is $V = \{1, \ldots, n\}$.

$\mathbf{D_9}$; x = 1: $V = \{\infty\} \cup \mathbb{Z}_8$ and $C = \{(1,4,6,5,2,3,7,\infty) + i \mid i \in \mathbb{Z}_8\} \cup \{(0,6,4,2), (1,7,5,3)\}$.

$\mathbf{D_9}$; x = 2: $C = \{(2,3,4,5), (6,7,8,9), (1,3,5,4), (6,9,8,7), (1,2,4,6,8,5,7,9), (1,5,3,8,6,2,9,7), (1,6,3,7,4,9,2,8), (1,4,8,2,7,5,9,3), (1,8,3,9,4,2,6,5), (1,9,5,6,4,7,3,2), (1,7,2,5,8,4,3,6)\}$.

$\mathbf{D_9}$; x = 3: $C = \{(2,3,4,5), (6,7,8,9), (1,3,5,4), (6,9,8,7), (1,2,6,8), (4,7,5,9), (1,5,3,7,9,2,8,6), (1,4,6,3,8,2,9,7), (1,8,3,2,7,4,9,5), (1,9,3,6,5,8,4,2), (1,7,2,5,6,4,3,9), (1,6,2,4,8,5,7,3)\}$.

$\mathbf{D_9}$; x = 4: $C = \{(2,3,4,5), (6,7,8,9), (1,3,5,4), (6,9,8,7), (1,2,6,8), (4,7,5,9), (1,5,7,9), (2,8,3,6), (1,4,2,9,7,3,8,6), (1,7,4,3,9,5,8,2), (1,8,4,9,2,5,6,3), (1,9,3,7,2,4,6,5), (1,6,4,8,5,3,2,7)\}$.

$\mathbf{D_9}$; x = 5: $C = \{(2,3,4,5), (6,7,8,9), (1,3,5,4), (6,9,8,7), (1,2,6,8), (4,7,5,9), (1,5,7,9), (2,8,3,6), (1,4,8,6), (2,7,3,9), (1,8,4,9,7,2,5,3), (1,9,3,8,5,6,4,2), (1,7,4,6,3,2,9,5), (1,6,5,8,2,4,3,7)\}$.

$\mathbf{D_9}$; x = 6: $C = \{(2,3,4,5), (6,7,8,9), (1,3,5,4), (6,9,8,7), (1,2,6,8), (4,7,5,9), (1,5,7,9), (2,8,3,6), (1,4,8,6), (2,7,3,9), (1,7,4,3), (2,9,5,8), (1,9,3,8,5,6,4,2), (1,6,5,3,2,4,9,7), (1,8,4,6,3,7,2,5)\}$.

$\mathbf{D_9}$; x = 7: $C = \{(2,3,4,5), (6,7,8,9), (1,3,5,4), (6,9,8,7), (1,2,6,8), (4,7,5,9), (1,5,7,9), (2,8,3,6), (1,7,4,6), (2,9,3,8), (1,9,7,3), (2,5,8,4), (1,8,6,5), (2,4,3,9), (1,4,9,5,6,3,2,7), (1,6,4,8,5,3,7,2)\}$.

$\mathbf{D_9}$; x = 8: $V = \{\infty\} \cup \mathbb{Z}_8$ and $C = \{(0,4,6,1) + i, (2,7,5,\infty) + i) \mid i \in \mathbb{Z}_9\} \cup \{(0,1,2,3,4,5,6,7)\}$.

$\mathbf{D_{4,4}}$; x = 2: $C = \{(1,8,2,7,3,6,4,5), (1,6,2,5,3,8,4,7), (1,7,2,8), (3,5,4,6), (1,5,2,6), (3,7,4,8)\}$. □

Since Theorem 2.1 requires $r > 2$, it only remains to find an $ARD(4,8)_s$ of D_{17} for $0 < s < n$. Since this case stands alone and is not required for the recursion of Theorem 2.1, to save space it is not included here, but can be found on the internet at http://www.dms.auburn.edu/~pikedav/publications/resolve

Corollary 3.2. *There exists an $ARD(4,6)_s$ of D_n if and only if*

$$n \equiv \begin{cases} 1 \ (mod\ 4) & \text{if } s = n, \\ 1 \ (mod\ 6) & \text{if } s = 0, \text{ and} \\ 1 \ (mod\ 12) & \text{if } 1 \le s \le n-1, \end{cases}$$

except possibly if $n = 25$ and $2 < s < 23$.

Proof. The necessity is obvious, and the sufficiency when $s \in \{0, n\}$ is settled by Theorem 2.2. By Theorem 2.1 it suffices to find an $ARD(4,6)_x$ of D_{13} for $1 \le x \le 12$

and an $ARD(4,6)_x$ of $D_{6,6}$ for each $x \in \{0,2,4,6\}$; an $ARD(4,6)_x$ of $D_{6,6}$ for each $x \in \{0,6\}$ is obtained by Lemma 2.2. These ingredients listed below with the directed cycles arranged into almost parallel classes. Unless otherwise stated, the vertex set is $V = \{1,2,\ldots,n\}$.

D_{13}; x = 1: $V = \{\infty\} \cup \mathbb{Z}_{12}$ and $C = \{(1,9,3,10,2,4)+i,\ (8,7,5,6,11,\infty)+i \mid i \in \mathbb{Z}_{12}\} \cup \{(0,3,6,9)+i \mid i \in \mathbb{Z}_3\}$.

D_{13}; x = 2: $C = \{(2,3,4,5),(6,7,8,9),(10,11,12,13),(1,3,5,4),(6,8,7,10),$ $(9,11,13,12),(1,2,4,6,5,7),(8,10,12,11,9,13),(1,5,3,2,6,9),(7,11,8,12,10,13),$ $(1,4,2,7,3,6),(8,13,11,10,9,12),(1,7,2,5,8,11),(3,12,4,13,9,10),(1,6,10,2,9,8),$ $(3,11,4,12,5,13),(1,9,2,10,4,3),(5,11,6,12,7,13),(1,11,2,8,5,10),$ $(3,13,4,7,12,6),(1,8,6,13,2,12),(3,7,5,9,4,11),(1,10,5,12,2,13),(3,9,7,6,4,8),$ $(1,13,6,2,11,5),(3,8,4,10,7,9),(1,12,3,10,8,2),(4,9,5,6,11,7)\}$.

D_{13}; x = 3: $C = \{(2,3,4,5),(6,7,8,9),(10,11,12,13),(1,3,5,4),(6,8,7,10),$ $(9,11,13,12),(1,2,4,6),(5,7,9,12),(8,10,13,11),(1,5,3,2,6,9),(7,11,10,12,8,13),$ $(1,4,2,7,3,10),(6,11,9,13,8,12),(1,7,2,5,8,11),(3,12,4,13,9,10),(1,6,2,8,3,13),$ $(4,10,9,5,12,11),(1,9,2,10,4,3),(5,11,6,12,7,13),(1,8,6,3,11,2),$ $(4,7,12,10,5,13),(1,11,3,7,4,12),(2,9,8,5,6,13),(1,13,6,5,10,7),(2,12,3,9,4,8),$ $(1,10,2,13,3,8),(4,11,5,9,7,6),(1,12,2,11,7,5),(3,6,10,8,4,9)\}$.

D_{13}; x = 4: $C = \{(2,3,4,5),(6,7,8,9),(10,11,12,13),(1,3,5,4),(6,8,7,10),$ $(9,11,13,12),(1,2,4,6),(5,7,9,12),(8,10,13,11),(1,5,3,2),(6,9,7,13),$ $(8,11,10,12),(1,4,2,6,3,10),(7,11,9,13,8,12),(1,7,2,5,8,13),(3,9,10,4,12,11),$ $(1,6,2,8,3,11),(4,13,5,12,10,9),(1,9,2,7,3,12),(4,10,5,11,6,13),(1,10,3,7,4,8),$ $(2,12,6,11,5,13),(1,8,2,11,4,7),(3,13,9,5,6,12),(1,12,2,13,7,5),(3,6,10,8,4,9),$ $(1,13,3,8,5,9),(2,10,7,6,4,11),(1,11,7,12,4,3),(2,9,8,6,5,10)\}$.

D_{13}; x = 5: $C = \{(2,3,4,5),(6,7,8,9),(10,11,12,13),(1,3,5,4),(6,8,7,10),$ $(9,11,13,12),(1,2,4,6),(5,7,9,12),(8,10,13,11),(1,5,3,2),(6,9,7,13),$ $(8,11,10,12),(1,4,2,7),(3,6,12,11),(8,13,9,10),(1,7,2,5,8,12),(3,10,4,11,9,13),$ $(1,6,2,8,3,11),(4,12,10,9,5,13),(1,9,2,6,3,13),(4,10,5,11,7,12),$ $(1,8,2,13,5,10),(3,12,7,11,6,4),(1,13,8,5,12,3),(2,9,4,7,6,11),(1,10,2,12,6,5),$ $(3,9,8,4,13,7),(1,11,4,9,3,8),(2,10,7,5,6,13),(1,12,2,11,5,9),(3,7,4,8,6,10)\}$.

D_{13}; x = 6: $V = \{\infty\} \cup (\mathbb{Z}_6 \times \mathbb{Z}_2)$ and $C = \{((0,0),(1,0),(4,1),(2,1)) + (i,0),((3,0),(3,1),(0,1),(1,1))+(i,0),((5,0),(2,0),(4,0),\infty)+(i,0),$ $((1,0),(0,1),(2,1),(1,1),(4,0),(3,0))+(i,0),((5,1),(5,0),(3,1),(2,0),(4,1),\infty)+(i,0) \mid i \in \mathbb{Z}_6\} \cup \{((0,0),(1,1),(2,0),(3,1),(4,0),(5,1))+(i,0) \mid i \in \mathbb{Z}_2\}$.

D_{13}; x = 7: $V = \{\infty\} \cup (\mathbb{Z}_6 \times \mathbb{Z}_2)$ and $C = \{((0,0),(3,1),(1,0),(5,1))+(i,0),$ $((3,0),(2,1),(0,1),(1,1))+(i,0),((5,0),(2,0),(4,0),\infty)+(i,0) \mid i \in \mathbb{Z}_6\}$ $\cup\{((1,0),(2,1),(4,1),(3,0),(4,0),(2,0))+(i,0),((5,1),(5,0),(1,1),(0,1),(3,1),\infty)+(i,0) \mid i \in \mathbb{Z}_6\} \cup \{((0,0),(0,1),(3,0),(3,1))+(i,0) \mid i \in \mathbb{Z}_3\}$.

D_{13}; x = 8: $C = \{(2,3,4,5),(6,7,8,9),(10,11,12,13),(1,3,5,4),(6,8,7,10),$ $(9,11,13,12),(1,2,4,6),(5,7,9,12),(8,10,13,11),(1,5,3,2),(6,9,7,13),$ $(8,11,10,12),(1,4,2,7),(3,6,12,11),(8,13,9,10),(1,7,2,5),(3,8,12,10),$ $(4,11,9,13),(1,6,2,11),(3,10,9,8),(4,13,5,12),(1,9,2,10),(3,7,5,13),(4,12,6,11),$ $(1,10,5,6,4,8),(2,13,7,12,3,11),(1,8,6,13,2,12),(3,9,5,11,7,4),(1,12,7,6,3,13),$ $(2,8,5,9,4,10),(1,13,8,2,9,3),(4,7,11,6,5,10),(1,11,5,8,4,9),(2,6,10,7,3,12)\}$.

$\mathbf{D_{13}}$; x = 9: $C = \{(2,3,4,5), (6,7,8,9), (10,11,12,13), (1,3,5,4), (6,8,7,10), (9,11,13,12), (1,2,4,6), (5,7,9,12), (8,10,13,11), (1,5,3,2), (6,9,7,13), (8,11,10,12), (1,4,2,7), (3,6,12,11), (8,13,9,10), (1,7,2,5), (3,8,12,10), (4,11,9,13), (1,6,2,11), (3,10,9,8), (4,13,5,12), (1,9,2,10), (3,11,4,12), (5,6,13,7), (1,11,7,12), (2,6,3,13), (4,10,5,8), (1,12,6,11,5,13), (2,9,3,7,4,8), (1,13,3,12,2,8), (4,9,5,10,7,6), (1,10,4,7,3,9), (2,13,8,6,5,11), (1,8,5,9,4,3), (2,12,7,11,6,10)\}$.

$\mathbf{D_{13}}$; x = 10: $C = \{(2,3,4,5), (6,7,8,9), (10,11,12,13), (1,3,5,4), (6,8,7,10), (9,11,13,12), (1,2,4,6), (5,7,9,12), (8,10,13,11), (1,5,3,2), (6,9,7,13), (8,11,10,12), (1,4,2,7), (3,6,12,11), (8,13,9,10), (1,7,2,5), (3,8,12,10), (4,11,9,13), (1,6,2,11), (3,10,9,8), (4,13,5,12), (1,9,2,12), (3,11,5,13), (4,10,7,6), (1,10,4,8), (2,6,5,11), (3,13,7,12), (1,8,2,13), (3,7,5,9), (4,12,6,11), (1,13,8,5,6,10), (2,9,4,7,3,12), (1,11,7,4,3,9), (2,10,5,8,6,13), (1,12,7,11,6,3), (2,8,4,9,5,10)\}$.

$\mathbf{D_{13}}$; x = 11: $C = \{(2,3,4,5), (6,7,8,9), (10,11,12,13), (1,3,5,4), (6,8,7,10), (9,11,13,12), (1,2,4,6), (5,7,9,12), (8,10,13,11), (1,5,3,2), (6,9,7,13), (8,11,10,12), (1,4,2,7), (3,6,12,11), (8,13,9,10), (1,7,2,5), (3,8,12,10), (4,11,9,13), (1,6,2,11), (3,10,9,8), (4,13,5,12), (1,9,2,12), (3,13,7,4), (5,11,6,10), (1,10,4,8), (2,6,5,13), (3,11,7,12), (1,8,2,13), (3,7,5,9), (4,12,6,11), (1,12,7,3), (2,9,4,10), (5,6,13,8), (1,13,3,9,5,10), (2,8,6,4,7,11), (1,11,5,8,4,9), (2,10,7,6,3,12)\}$.

$\mathbf{D_{13}}$; x = 12: $V = \{\infty\} \cup \mathbb{Z}_{12}$ and $C = \{(1,8,7,5)+i, (4,9,6,10)+i, (11,2,3,\infty)+i \mid i \in \mathbb{Z}_{12}\} \cup \{(0,2,4,6,8,10)+i \mid i \in \mathbb{Z}_2\}$.

$\mathbf{D_{6,6}}$; x = 2: $C = \{(1,7,2,8), (3,9,4,10), (5,11,6,12), (1,8,2,7), (3,10,4,9), (5,12,6,11), (1,9,2,11,3,12), (4,7,5,10,6,8), (1,10,2,12,3,11), (4,8,5,9,6,7), (1,11,4,12,2,9), (3,7,6,10,5,8), (1,12,4,11,2,10), (3,8,6,9,5,7)\}$.

$\mathbf{D_{6,6}}$; x = 4: $C = \{(1,7,2,8), (3,9,4,10), (5,11,6,12), (1,8,2,7), (3,10,4,9), (5,12,6,11), (1,9,2,10), (3,11,4,12), (5,7,6,8), (1,11,2,12), (3,7,4,8), (5,10,6,9), (1,10,5,8,4,11), (2,9,6,7,3,12), (1,12,4,7,5,9), (2,11,3,8,6,10)\}$.

As in the case of the previous corollary, it remains to find an $ARD(4,6)_x$ of $D_{2\ell+1} = D_{25}$ for each $x \in \{0,1,24,25\}$; these can be found on the internet at http://www.dms.auburn.edu/~pikedav/publications/resolve for $x \in \{1,24\}$, and by using Theorem 2.2 otherwise. □

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