On the Normality of Cayley Digraphs of Groups of Order Twice a Prime

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Abstract

We call a Cayley digraph $X = \mathrm{Cay}(G, S)$ *normal* for G if the right regular representation $R(G)$ of G is normal in the full automorphism group $\mathrm{Aut}(X)$ of X. In this paper we determine the normality of Cayley digraphs of groups of order twice a prime.

1 Introduction

Let G be a finite group and S a subset of G not containing the identity element 1. We define the *Cayley digraph* $X = \mathrm{Cay}(G, S)$ of G with respect to S by

$$\begin{aligned} V(X) &= G, \\ E(X) &= \{(g, sg) \mid g \in G, s \in S\}. \end{aligned}$$

If $S^{-1} = S$, then $\mathrm{Cay}(G, S)$ can be viewed as an undirected graph, identifying an undirected edge with two directed edges (g, h) and (h, g). This graph is called the *Cayley graph* of G with respect to S.

The following obvious facts are basic for Cayley digraphs.

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Proposition 1.1 *Let $X = \mathrm{Cay}(G, S)$ be a Cayley digraph of G with respect to S. Then*

(1) $\mathrm{Aut}(X)$ *contains the right regular representation* $R(G)$ *of* G *and so* X *is vertex-transitive.*

(2) X *is connected if and only if* $G = \langle S \rangle$.

(3) X *is undirected if and only if* $S^{-1} = S$.

Proposition 1.2 *A digraph* X *is a Cayley digraph of a group* G *if and only if* $\mathrm{Aut}(X)$ *contains a regular subgroup isomorphic to* G.

Let $X = \mathrm{Cay}(G, S)$ be a Cayley digraph of G with respect to S and

$$\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\}.$$

Obviously, $R(G)\mathrm{Aut}(G, S) \leq \mathrm{Aut}(X)$. Let $A = \mathrm{Aut}(X)$ and A_1 be the stabilizer of the identity 1 of G in A. Then we have

Proposition 1.3 (see [8])

(1) $N_A(R(G)) = R(G)\mathrm{Aut}(G, S)$;

(2) the following statements are equivalent

(2.1) $R(G) \lhd A$;

(2.2) $A = R(G)\mathrm{Aut}(G, S)$;

(2.3) $A_1 \leq \mathrm{Aut}(G, S)$.

Xu defined the so-called normal Cayley digraphs of a group in [8].

Definition 1.4 The Cayley digraph $X = \mathrm{Cay}(G, S)$ is called *normal* for G if $R(G) \lhd A$.

This concept is helpful for determining the full automorphism groups of Cayley digraphs, which is known to be very difficult in general. The reason is that if we know a Cayley digraph $\mathrm{Cay}(G, S)$ is normal for G, then by Proposition 1.3 (2.2), its full automorphism group $A = R(G)\mathrm{Aut}(G, S)$.

Recently, some results about the normal Cayley digraphs of finite groups have been obtained by several authors (see [8] for a survey.) The normality of Cayley digraphs for some special groups is known. For example, for cyclic groups of prime order p, we know that all Cayley digraphs, other than K_p or pK_1, are normal by Galois and Burnside's theorems. Unfortunately, these are the only groups for which complete information about the normality of Cayley digraphs is available. In this paper we shall study the normality of Cayley digraphs for another class of groups, namely the groups of order $2p$ where p is a prime. The result of this paper will partially solve the following problem posed by Xu. (See [8, Problem 2]).

Problem 1.5 *Determine all imprimitive nonnormal Cayley graphs of order* pq *and do the same thing for Cayley digraphs of order* pq.

Our main result is the following

Theorem 1.6 *All the Cayley digraphs of groups of order twice a prime p are normal, except for the digraphs listed in Table 1.*

Table 1: Nonnormal Cayley digraphs X of groups G of order $2p$

row	digraph X	$\mathrm{Aut}(X)$	group G	p	remark
1	K_4	S_4	Z_4	2	
2	$4K_1$	S_4	Z_4	2	
3	$2pK_1$	S_{2p}	Z_{2p}, D_{2p}	$p > 2$	
4	pK_2	$Z_2 \,\mathrm{wr}\, S_p$	Z_{2p}, D_{2p}	$p > 2$	
5	$2Y$, $Y \neq pK_1$	$\mathrm{Aut}(Y) \,\mathrm{wr}\, Z_2$	Z_{2p}, D_{2p}	$p > 2$	For D_{2p}, $\mathrm{Aut}(Y) > Z_p$
6	$Y[2K_1]$, $Y \neq pK_1$	$Z_2 \,\mathrm{wr}\, \mathrm{Aut}(Y)$	Z_{2p}, D_{2p}	$p > 2$	For D_{2p}, Y undirected
7	$Y[K_2]$, $Y \neq pK_1, K_p$	$Z_2 \,\mathrm{wr}\, \mathrm{Aut}(Y)$	Z_{2p}, D_{2p}	$p > 2$	For D_{2p}, Y undirected
8	K_{2p}	S_{2p}	Z_{2p}, D_{2p}	$p > 2$	
9	$K_2[Y]$, $Y \neq K_p$	$\mathrm{Aut}(Y) \,\mathrm{wr}\, Z_2$	Z_{2p}, D_{2p}	$p > 2$	For D_{2p}, $\mathrm{Aut}(Y) > Z_p$
10	$K_{p,p} - pK_2$	$S_p \times Z_2$	Z_{2p}, D_{2p}	$p > 2$	
11	$(K_{p,p} - pK_2)^c$	$S_p \times Z_2$	Z_{2p}, D_{2p}	$p > 2$	
12	$B(H(11))$	$PGL(2, 11)$	D_{2p}	11	
13	$K_{11,11} - B(H(11))$	$PGL(2, 11)$	D_{2p}	11	
14	$(B(H(11))^c$	$PGL(2, 11)$	D_{2p}	11	
15	$(K_{11,11} - B(H(11)))^c$	$PGL(2, 11)$	D_{2p}	11	
16	$B(PG(n-1, q))$	$P\Gamma L(n, q) \rtimes Z_2$	D_{2p}	$\frac{q^n-1}{q-1}$	$n \geq 3$
17	$K_{p,p} - B(PG(n-1, q))$	$P\Gamma L(n, q) \rtimes Z_2$	D_{2p}	$\frac{q^n-1}{q-1}$	$n \geq 3$
18	$(B(PG(n-1, q)))^c$	$P\Gamma L(n, q) \rtimes Z_2$	D_{2p}	$\frac{q^n-1}{q-1}$	$n \geq 3$
19	$(K_{p,p} - B(PG(n-1, q)))^c$	$P\Gamma L(n, q) \rtimes Z_2$	D_{2p}	$\frac{q^n-1}{q-1}$	$n \geq 3$

In Table 1, "Y" denotes any transitive digraph of order p and we shall use this notation throughout this paper.

By the classification of the edge-transitive graphs of order $2p$ (see [2]), where p is a prime, we know all nonnormal edge-transitive Cayley graphs of order $2p$. However, the result of this paper does not depend on the above result.

In this paper, we use standard group- and graph-theoretic notation and terminology (see [3, 4] for example).

In the next section, we shall prove Theorem 1.6.

2 Proof of Theorem 1.6

First we assume that $p = 2$. There are two groups of order 4, namely Z_4 and Z_2^2. Since a transitive but not doubly transitive subgroup of S_4 has order 4 or 8, the only possible nonnormal Cayley digraphs of order 4 are K_4 and $4K_1$ for the group Z_4. Thus we get the first two rows in Table 1.

Now we may assume that $p > 2$. In this case G is either a cyclic group Z_{2p} or a dihedral group D_{2p}, that is,

$$G = \langle a, b \mid a^p = b^2 = 1, ab = ba \rangle,$$

or

$$G = \langle a, b \mid a^p = b^2 = 1, b^{-1}ab = a^{-1} \rangle. \tag{2.1}$$

The proof of Theorem 1.6 in this case will be divided into several lemmas.

Suppose that T is a subset of Z_p that does not contain the identity. Let $Y = \mathrm{Cay}(Z_p, T)$ and $Y \neq pK_1$. Then there is a divisor r of $p-1$ such that $\mathrm{Aut}(Y) = Z_p \rtimes H_r$ where H_r is the unique subgroup of $Z_p^* = Z_{p-1}$ of order r and T is a union of some cosets of H_r in Z_p^*. If r is even, then $-H_r = H_r$. If r is odd, then $-H_r \cup H_r = H_{2r}$. It folows that $-T = T$ (equivalently Y is undirected) if and only if r is even, i.e., $\mathrm{Aut}(Y)$ is even. In addition, $\mathrm{Aut}(Y) = Z_p$ if and only if T is not a union of some cosets of a nontrivial subgroup H_r of Z_p^*.

Lemma 2.1 *Suppose that $X = \mathrm{Cay}(G, S)$ is a disconnected Cayley digraph of order $2p$. Then X is nonnormal for G if and only if one of the following holds:*

(1) *$G = Z_{2p}$ or D_{2p}, and $X = 2pK_1$;*

(2) *$G = Z_{2p}$ or D_{2p}, and $X = pK_2$;*

(3) *$G = Z_{2p}$ and $X = 2Y$ with $Y \neq pK_1$;*

(4) *$G = D_{2p}$, $X = 2Y$ with $Y \neq pK_1$ and $\mathrm{Aut}(Y) > Z_p$.*

The above digraphs are listed in rows 3–5 in Table 1.

Proof Suppose that X is disconnected. Then X is one of the following: $2pK_1$, pK_2 and $2Y$ where Y is a Cayley digraph of Z_p and $Y \neq pK_1$. It is clear that $2pK_1$ and pK_2 are nonnormal for both $G = Z_{2p}$ and D_{2p}. Now we deal with the case where $X = 2Y$ and $Y \neq pK_1$. In this case $A = \mathrm{Aut}(Y) \mathrm{~wr~} Z_2$. First let $G = Z_{2p}$. We claim that $R(G)$ is nonnormal in A. Assuming the contrary, the unique subgroup of order 2 of $R(G)$ would be normal in A and hence A would have 2-blocks on $V(X)$, contradicting the fact that $A = \mathrm{Aut}(Y) \mathrm{~wr~} Z_2$. Next let $G = D_{2p}$. In this case, letting H be the unique subgroup of order p of G, we have $S \subseteq H$ and $Y = \mathrm{Cay}(H, S)$. We claim that X is normal for G if and only if $\mathrm{Aut}(Y) = R(H)$. First assume that $\mathrm{Aut}(Y) = R(H)$. Then $A \cong (Z_p \times Z_p) \rtimes Z_2$. It is easy to check that A has only p involutions and has only one subgroup isomorphic to D_{2p}, which is precisely G. So G is normal in A and X is normal for $G = D_{2p}$. Conversely, assume that X is normal for $G = D_{2p}$. Then the unique subgroup of order p of $R(G)$, which consists of the right multiplication induced by H acting on G and is denoted also by $R(H)$, is normal in A and it has two p-blocks of A on $V(X) = G$, that is H and $G \setminus H$. Assume that $\mathrm{Aut}(Y) = Z_p \rtimes \langle \alpha \rangle > Z_p$, where $\langle \alpha \rangle = Z_r$. Let $\bar{\alpha}$ be the permutation on G such that $\bar{\alpha}|_H = \alpha$ and $\bar{\alpha}|_{G \setminus H} = 1$. Then $\bar{\alpha}$ is an automorphism of X and fixes pointwise one block $G \setminus H$ and has some orbits of length r on the other block H. Thus $R(H)^{\bar{\alpha}} \neq R(H)$, a contradiction. □

From now on we assume that X is connected, and we distinguish the following cases. First, suppose that A is primitive on $V(X)$. If A is doubly transitive on $V(X)$, then $X = K_{2p}$ (row 8 in Table 1), which is a nonnormal Cayley graph both for Z_{2p} and for D_{2p}. If A is simply primitive, then by [5] A must be S_5 and $|V(X)| = 10$. It follows that X is the Petersen graph or its complement, which are not Cayley graphs for any groups of order 10.

Next we assume that A is imprimitive on $V(X)$ and that B is a nontrivial block of A. Let $\Sigma = \{B_1, B_2, \ldots, B_c\}$ be a complete block system of A, where $c = 2$ or p, and that K the kernel of the action of A on Σ. We can define a block digraph of X, which we also denote by Σ, to be the digraph with vertex set Σ and edge set $\{(B_i, B_j) |$ there exist $v_i \in B_i, v_j \in B_j$ such that $(v_i, v_j) \in E(X)\}$. Since X is connected, Σ is also connected. And the group of automorphisms of Σ induced by A is also transitive on $V(\Sigma)$. Here we have to deal with two cases separately: A has only 2-blocks on $V(X)$ and A has p-blocks on $V(X)$.

Lemma 2.2 *Suppose that A has only 2-blocks on $V(X)$, then X is nonnormal for G if and only if one of the following holds:*

(1) *$G = Z_{2p}$, and $X = Y[2K_1]$ or $X = Y[K_2]$, where $Y[Z]$ denotes the lexicographic product of Y and Z;*

(2) *$G = D_{2p}$, and $X = Y[2K_1]$ or $X = Y[K_2]$, where Y is also undirected. Moreover, to ensure the connectivity of X, Y is not pK_1. Also Y is not a complete graph for the case $Y[K_2]$. The above digraphs are listed in rows 6 and 7 in Table 1.*

Proof Let $X = Y[2K_1]$ or $Y[K_2]$, where $Y \neq pK_1$, and $Y \neq K_p$ for $Y[K_2]$. Then we have $A = \mathrm{Aut}(X) = Z_2 \mathrm{wr}\, \mathrm{Aut}(Y) = Z_2^p \rtimes \mathrm{Aut}(Y)$, and so A is imprimitive on $V(X)$, and it has only 2-blocks. It is easy to check that the subgroup Z_2^p of A has only one regular element, say γ. This element is in the center of A. Let α be a regular element of order p in A. Then α acts cyclically on $V(\Sigma)$ and $\langle \alpha, \gamma \rangle \cong Z_{2p}$ is a regular subgroup of A. It is easy to see that this subgroup is nonnormal in A, and hence both $Y[2K_1]$ and $Y[K_2]$ are nonnormal Cayley digraphs of Z_{2p}. Moreover, if Y is undirected, then $\mathrm{Aut}(Y)$ is of even order and hence $\mathrm{Aut}(Y)$ has a dihedral subgroup of order $2p$, which corresponds to an intransitive subgroup D of $\mathrm{Aut}(X)$ with the presentation: $D = \langle \alpha, \beta \mid \alpha^p = \beta^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1} \rangle$. Since $\gamma \in Z(\mathrm{Aut}(X))$, the subgroup of $\mathrm{Aut}(X)$ generated by $\gamma\beta$ and α is a regular subgroup isomorphic to D_{2p}, and it is nonnormal in A.

Now suppose that $X = \mathrm{Cay}(G, S)$, where $G = Z_{2p}$ or D_{2p}, and that $A = \mathrm{Aut}(X)$ has only 2-blocks on $V(X)$. Let $\Sigma = \{B_i \mid i \in Z_p\}$ be the complete 2-block system of A on $V(X)$ and K the kernel of the action of A on Σ. Since the B_i are also imprimitive blocks of the subgroup $R(G)$ of A, B_i can be viewed as the cosets of a subgroup of order 2 in G. Without loss of generality, we may assume that $B_i = \{a^i, a^i b\}$. So the subgraph of X induced by B_i is $2K_1$ or K_2. By [6, Lemma 3.5], one of the following holds: (a) $X = Y[2K_1]$ or $Y[K_2]$, where Y is isomorphic to the block digraph Σ; (b) $K = 1$ or Z_2.

First we deal with the case (a). Since A is imprimitive, we have that if $X = Y[K_2]$ then $Y \neq K_p$, otherwise we would have $K_p[K_2] = K_{2p}$. As proved above, we have that $\mathrm{Aut}(X) = Z_2^p \rtimes \mathrm{Aut}(Y)$ and the kernel of the action of A on the 2-blocks is $K = Z_2^p$. If $G = D_{2p}$ as is presented in (2.1), the 2-blocks will be represented as right cosets of a subgroup of order 2 in G, say $\langle b \rangle$. It follows that the right translation $R(b)$ of b will fix the 2-block $\langle b \rangle$ and interchange the other $p-1$ blocks pairly. Hence $R(b)$ is not in the kernel. So $\mathrm{Aut}(Y)$ must be of even order and hence Y is undirected.

Next we shall prove that case (b) is impossible when X is nonnormal. Set $\overline{A} = A/K$. If $\overline{A}$ were soluble, it would have a normal subgroup $\overline{H} = H/K$ of order p.

Let $P \in \mathrm{Syl}_p(H)$. Then $P \lhd H$ by Sylow's theorem and hence $P \lhd A$. So A would have p-blocks, a contradiction. Thus we have proved $\overline{A}$ is insoluble, and hence $\overline{A}$ is 2-transitive on Σ and $\Sigma = K_p$. Let $A_{\{B\}}$ be the setwise stabilizer of a 2-block B and let $v \in B$. If $K = Z_2$, we have that $A_{\{B\}} = KA_v$ and $K \cap A_v = 1$. Since $(|K|, |A : A_{\{B\}}|) = (2, p) = 1$, K has a complement M in A by Gaschütz's theorem. (See [3, Hauptsatz I.17.4] for example.) Obviously $K \in Z(A)$ and so $A = K \times M$.

$M \cong A/K$ is 2-transitive on Σ. By the classification of 2-transitive groups (see [1], for example), all insolvable 2-transitive groups M of degree p are almost simple, that is the socle T of M is simple and $T \leq M \leq \mathrm{Aut}T$. Moreover T is one of the following: A_p, $PSL(2, 11)$ with $p = 11$, $PSL(2, 2^{2^s})$ with $p = 1 + 2^{2^s}$ and $s > 0$, $PSL(n, q)$ with $n \geq 3$, q a prime power and $p = (q^n - 1)/(q - 1)$, M_{11} with $p = 11$ and M_{23} with $p = 23$.

Now we claim that T is transitive on $V(X)$. Since T is also normal in A, if T were intransitive on $V(X)$ then A would have p-block, as T is transitive on Σ, this contradicts our assumption. Next, since the action of T on Σ is faithful, $T_{\{B\}}$ must have a subgroup of index 2. Let us check the 2-transitive groups of degree p listed above. The groups A_p $(p \geq 5)$, $PSL(2, 11)$, M_{23} and $PSL(2, 2^{2^s})$ do not have a subgroup of index $2p$. The only possibilities are $T = PSL(n, q)$ or $T = M_{11}$. In the former case, we have that the subdegrees of T on $V(X)$ are $1, 1$ and $2(p - 1)$ by the same argument as in the proof of ([7, Lemma 4.7]). So $X = K_p[2K_1]$ or $K_p[K_2]$, this is not the case. In the latter case, $T = M_{11}$, $T_{\{B\}} = A_6 \rtimes Z_2$ and $T_v = A_6$ where $v \in B$. It is easy to check, however, that the subdegrees of T acting on $\{A_6 g \mid g \in M_{11}\}$ by the right multiplication are 1,1, 10 and 10, and two suborbits of length 10 are self-paired. This shows that X is an undirected graph. It follows from [6, Lemma 6.2] that either A has two p-blocks or $X = K_p[2K_1]$, which is a contradiction. Therefore we have proved that case (b) is impossible, completing the proof of this lemma. □

In what follows we assume that A has two p-blocks on $V(X)$, which are denoted by B_0 and B_1.

Let K be the kernel of A on the complete p-block system. Suppose that K is unfaithful on B_0 or B_1. Then the pointwise stabilizer of one block would be transitive on the other block. Hence any vertex in B_0 and any vertex in B_1 are adjacent. The digraph X is a lexicographic product of K_2 and a vertex transitive digraph Y of order p, where Y is not the complete graph. In this case the complement of X is disconnected and has edges. By Lemma 2.1 we have that X is a nonnormal Cayley digraph for Z_{2p}, and also for D_{2p} but with the additional condition that $\mathrm{Aut}(Y) > Z_p$. These digraphs are listed in row 9 in Table 1.

We assume below that K is faithful on each of B_0 and B_1, and so $K^{B_i} \cong K$ is a transitive group of degree p. Suppose that A is solvable. Then K is solvable and so $K^{B_i} \leq AGL(1, p)$. Let P be the unique subgroup of order p in A. Since $A/C_A(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P) \cong Z_{p-1}$ and $A/K \cong Z_2$, we have that $A' \leq C_A(P) \cap K = P$ and so $A' = 1$ or P. In the former case, $K = P$ and so $R(G) = A$; in the latter case, $A/P = A/A'$ is abelian and so $R(G) \lhd A$ by $R(G)/P \lhd A/P$. Therefore, in this case, X is normal for both of Z_{2p} and D_{2p}.

In what follows, we assume that A is insolvable. Hence K is insolvable by $A/K \cong$

Z_2. By Burnside's Theorem, K^{B_i} is 2-transitive.

Lemma 2.3 *Suppose that the two permutation representations of* K *on* B_0 *and* B_1 *are equivalent. Then* X *is nonnormal for* G *if and only if* X *and* G *are the graphs and the groups listed in rows 10 and 11 in Table 1.*

Proof Since K acts 2-transitively on B_0 and B_1, letting H be the stabilizer of a vertex v of B_0 in K, H has two orbits Δ_{i0} and Δ_{i1} on B_i, where $|\Delta_{i0}| = 1$ and $|\Delta_{i1}| = p - 1$, for $i = 0, 1$. Now the neighborhood $X_1(v)$ of v will be equal to one of the following seven sets: (i) $\Delta_{01} \cup \Delta_{10} \cup \Delta_{11}$; (ii) $\Delta_{10} \cup \Delta_{11}$; (iii) $\Delta_{01} \cup \Delta_{11}$; (iv) $\Delta_{01} \cup \Delta_{10}$; (v) Δ_{01}; (vi) Δ_{10}; or (vii) Δ_{11}. It is clear that the digraphs corresponding to these seven possibilities are in fact undirected, and they are (i) K_{2p}; (ii) $K_{p,p}$; (iii) $((pK_2))^c$; (iv) $((K_{p,p} - pK_2))^c$; (v) $2K_p$, (vi) pK_2, and (vii) $K_{p,p} - pK_2$. It is easy to check that the graph (i) has a primitive automorphism group, and that graphs (v) and (vi) are disconnected, and that the automorphsim group of graph (iii) is $Z_2 \operatorname{wr} S_p$ which has no p-blocks on $V(X)$. Hence we only need to consider the graphs (ii), (iv) and (vii).

For graph (ii), $A = S_p \operatorname{wr} Z_2$. So K is unfaithful on B_0 and B_1, this is not the case.

For graphs (iv) and (vii), $A = S_p \times Z_2$ and A has a regular subgroup Z_{2p} and also a regular subgroup D_{2p} which are nonnormal in A. So these two graphs are nonnormal for Z_{2p} and D_{2p}. □

Finally, we assume that K has two nonequivalent representations on B_0 and B_1. Then by [1], Soc(K) is either $PSL(n, q)$ where $n \geq 3$ and $p = (q^n - 1)/(q - 1)$, or $PSL(2, 11)$ acting on cosets of a subgroup isomorphic to A_5. In the following two lemmas we deal with these two cases separately.

Lemma 2.4 *Suppose that* Soc(K) $= PSL(n, q)$. *Then* X *is nonnormal for* G *if and only if* X *and* G *are respectively one of the graphs and the groups listed in rows 16–19 in Table 1.*

Proof In this case we may assume that the actions of $PSL(n, q)$ on B_0 and B_1 are equivalent to that on the projective points and the hyperplanes of $PG(n-1, q)$ respectively. Let H be the stabilizer of a vertex u of B_0 in K. Then it is well-known that H has two orbits Δ_{00} and Δ_{01} on B_0 with $|\Delta_{00}| = 1$ and $|\Delta_{01}| = p - 1$; and two orbits Δ_{10} and Δ_{11} on B_1 with $|\Delta_{10}| = (q^n - 1)/(q - 1)$ and $|\Delta_{01}| = q^{n-1}$. The neighborhood of u is one of the following seven sets: (i) $\Delta_{01} \cup \Delta_{10} \cup \Delta_{11}$; (ii) $\Delta_{10} \cup \Delta_{11}$; (iii) $\Delta_{01} \cup \Delta_{11}$; (iv) $\Delta_{01} \cup \Delta_{10}$; (v) Δ_{01}; (vi) Δ_{10}; or (vii) Δ_{11}. Now it is clear that the digraphs given by the above seven possibilities are in fact undirected, and they are respectively, (i) K_{2p}; (ii) $K_{p,p}$; (iii) $(B(PG(n-1, q)))^c$ (iv) $(K_{p,p} - B(PG(n-1, q)))^c$; (v) $2K_p$, (vi) $B(PG(n-1, q))$, the point-hyperplane incidence graph $B(PG(n-1, q))$ of $PG(n-1, q)$, and (vii) $K_{p,p} - B(PG(n-1, q))$. Among them, graph (i) has a primitive automorphism group, graph (v) is disconnected, and for graph (ii), $Soc(K) = A_p$, a contradiction. Hence we only need to consider graphs (iii), (iv), (vi) and (vii). For each of them, $A = Aut(PSL(n, q))$ and A has regular subgroups D_{2p} of order $2p$, which are nonnormal in A, but has no regular subgroups Z_{2p}. Hence X is a nonnormal Cayley graph of D_{2p}. □

By an argument similar to that of Lemma 2.4, we have the following

Lemma 2.5 *Suppose that* $\mathrm{Soc}(K) = PSL(2,11)$. *Then* X *is nonnormal for* G *if and only if* X *and* G *are respectively one the graphs and the groups listed in rows 12–15 in Table 1, where* $B(H(11))$ *denotes the incidence graph of the unique symmetric (11, 5, 2)-design* $H(11)$.

By combining the above lemmas, we complete the proof of Theorem 1.6.

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