Induced Stars in Trees

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Abstract

We show that for $n \geq 2$, every n-vertex tree has a star forest of order exceeding $n/2$.

There are many theorems in graph theory in which a graph is required to have a particular subgraph. A stronger requirement, however, is that it be an induced subgraph: i.e. a subgraph in which two vertices must be adjacent if they are adjacent in the big graph.

A common problem of Ramsey theory, for instance, is to show that for a particular graph H, any graph G having enough vertices must have as a subgraph a copy of H or its complement H^c. Typically, the copy of H (or H^c) will contain extraneous edges, and thus will not be an induced subgraph of G.

It is trivial to partition the vertices of a tree into stars. (A *star*, $K_{1,n}$ is graph with a vertex of degree $n \geq 1$ and all its neighbors of degree one.) But this might not give an *induced* subgraph in which each component is a star.

For instance consider the *double-star* $S_{m,n}$, the graph with a vertex of degree m adjacent to a vertex of degree n, and all its other vertices of degree one. The vertices of $S_{3,3}$ can be partitioned into two disjoint copies of the star $K_{1,2}$. But this is not an induced subgraph. The largest induced subgraph consisting of disjoint stars is the single star $K_{1,3}$.

We define a *star-forest* in a graph to be an induced subgraph in which each component is a star. In this note we show that within any tree we can find a large star-forest. Extending this result to graphs other than trees seems surprisingly resistant to easy solution.

THEOREM. Every n-vertex tree has a star-forest of order strictly greater than $n/2$ (for $n \geq 2$).

The double-stars $S_{n,n}$ and $S_{n+1,n}$ show this bound is best possible.

We will refer to a vertex of degree one as an *end-vertex*, and any other vertex as an *interior vertex*.

To prove the theorem, let T be a counter-example with as few vertices as possible, say n vertices. First we claim that every interior vertex of T must have at least one neighbor that is an end-vertex.

If not, let v be an interior vertex none of whose neighbors are end-vertices. Then each component of $T - v$ has at least two vertices, so by the minimality of T the theorem is true for each component. Say component T_i has n_i vertices ($i = 1$ to p), and thus has a star forest with s_i vertices where $s_i > n_i/2$. Note that $\sum_{i=1}^{p} n_i = n-1$, and since s_i is an integer we have $s_i \geq (n_i + 1)/2$.

The union of the star-forests for the components gives a star-forest in T, as vertices in different components cannot be adjacent in T. But the number of vertices in the union is at least $\sum_{i=1}^{p} s_i \geq \sum_{i=1}^{p} (n_i + 1)/2 \geq (\sum_{i=1}^{p} n_i)/2+1 = (n-1)/2+1 > n/2$. Thus the theorem is true for T, proving the claim that each interior vertex has at least one neighbor that is an end-vertex.

Now, we can partition the vertices of T into disjoint stars by taking each interior vertex v to be the center of a star whose other vertices are the end-vertices adjacent to it.

To get a star-forest, consider a two-coloring of the vertices of T. Let T_i be the forest induced by the stars whose centers are colored i (for $i = 1$ or 2). Since T_1 and T_2 partition the vertices of T, one of them has at least $n/2$ vertices.

If each has exactly $n/2$ vertices, let v be an interior vertex whose neighbors include exactly one interior vertex. (I.e. v is an end-vertex in the subgraph of T obtained by deleting all its end-vertices.) Choose the forest T_i that does not include v, and thus does include the star centered at v', the one interior vertex adjacent to v. Then $T_i + v$ is a star-forest with v added to the star centered at v' and thus having more than $n/2$ vertices, as desired.

How does this problem generalize to graphs in general? Is there a positive constant c such that every n-vertex graph G (without isolated vertices) must have a star-forest with at least cn vertices?

If so there would be constant c' such that G has an induced subgraph H with at least $c'n$ vertices in which each vertex has odd degree. (This would be achieved by deleting one end vertex from each of the even stars.)

But this is an old conjecture [2] which has received some attention recently with quite limited success. It has been proved for trees [3] and for graphs of maximum degree three [1]; for general graphs it is known only that the constant, if it exists, is no greater than 2/7.

REFERENCES

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