Large graphs with small degree and diameter: A voltage assignment approach*

Ljiljana Branković, Mirka Miller

Department of Computer Science and Software Engineering,
The University of Newcastle NSW 2308 Australia,
e-mail: {lbrankov,mirka}@cs.newcastle.edu.au

Ján Plesník

Department of Numerical and Optimization Methods,
Faculty of Mathematics and Physics,
Comenius University, 842 15 Bratislava, Slovakia,
e-mail: plesnik@fmph.uniba.sk

Joe Ryan

Department of Mathematics,
The University of Newcastle NSW 2308 Australia,
e-mail: joe@frey.newcastle.edu.au

Jozef Širáň

Department of Mathematics,
SvF Slovak Technical University
Radlinského 11, 813 68 Bratislava, Slovakia,
e-mail: siran@lux.svf.stuba.sk

Abstract

The degree/diameter problem is to determine the largest order of a graph with given degree and diameter. Although many constructions have been considered in this area, a powerful one – the covering space construction – seems to have been overlooked. Paradoxically, many examples of graphs that are known as currently largest graphs for some degrees and diameters can be obtained by the covering space construction.

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The objective of the paper is to revisit the degree/diameter problem from this new perspective. The large covering graphs (called *lifts*) of small *base graphs* are described by means of the so-called *voltage assignments* on base graphs in finite groups. We do not try to find special graphs and special voltage assignments which would provide further record examples for given degree and diameter. Instead, we are interested in the potential of this method when applied to arbitrary graphs and groups. We derive a fairly general upper bound on the diameter of a lift in terms of the properties of the base voltage graph and discuss related questions.

1 Introduction

The well known and extensively studied *degree/diameter problem* is to determine, for given numbers d and k, the largest possible order (i.e., number of vertices) $n_{d,k}$ of a graph of maximum degree d and diameter k. A straightforward general upper bound on $n_{d,k}$ is the *Moore bound* $M_{d,k}$, named after E. F. Moore who first proposed the problem (see [19]):

$$n_{d,k} \leq M_{d,k} = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$$

The equality $n_{d,k} = M_{d,k}$ holds only in the cases (cf. [19, 8, 1]) when *(i)* $k = 1$ and $d \geq 1$, or *(ii)* $k = 2$ and $d = 2,\ 3,\ 7$ (and, possibly, $d = 57$), or *(iii)* $k \geq 3$ and $d = 2$. For the remaining values of d and k the best general upper bound [2, 13] is $n_{d,k} \leq M_{d,k} - 2$; so far the only known cases when this bound is attained [12] are $(d,k) = (3,3)$, $(4,2)$ and $(5,2)$. If $d = 3$ and $k \geq 4$ the above can further be improved to $n_{d,k} \leq M_{d,k} - 4$, see [20]. The natural problem of estimating the asymptotic order of $n_{d,k}$ as $d \to \infty$ was stated in [6], and for results in this direction we refer to [3, 10].

As regards lower bounds on $n_{d,k}$, a number of researchers have contributed to computer-aided generating of large graphs with given degree and diameter; perhaps the best summarizing references are [5, 18]. An updated list of currently largest known graphs of degree d and diameter k for $d \leq 15$ and $k \leq 10$ is maintained in [11]; see also *http://www-mat.upc.es/grup_de_grafs/table_g.html* - an online table for the degree/diameter problem. Besides computer search, several techniques have been used in the past to produce large graphs of given degree and diameter (and perhaps satisfying some extra conditions). In fact, the majority of entries in the table of [11] have been obtained by applying (a combination of) these techniques. Most notably, we mention here various compounding operations [14], twisted product of graphs [4], finite geometries and polarity quotients [9, 10], and linear congruential graphs [22]; others are listed in [18] and references therein.

The aim of this paper is to draw attention to a construction well known in topological graph theory, the *covering graph construction.* Roughly speaking, it enables to "blow up" a given *base graph* to a larger graph (called *lift*) which is a *regular covering space* of the base graph. The lift itself can be described in terms of the base graph and a mapping, called *voltage assignment,* which assigns elements of a finite group to edges of the base graph. A self-contained introduction to the topic is provided in Section 2.

We would like to justify the choice of the covering construction method as a good candidate for producing large graphs with given degree and diameter by the following three facts. First, as we shall see in Section 3, many currently largest known examples of graphs of given degree and diameter (including those found by computer search) can indeed be obtained by the voltage assignment construction. Second, as we show in another paper [7], *all* currently known best examples found as Cayley graphs of semidirect products of cyclic groups [18] can be described via voltage constructions using smaller base graphs and voltage assignments in cyclic groups. Finally, a very recent result of [21] shows that there exist *vertex-transitive* graphs of diameter two and order $\frac{8}{9}(d+\frac{1}{2})^2$ for all degrees $d = (3q-1)/2$ where $q = 4\ell + 1$ is a prime power; the construction was obtained using voltage assignments in additive groups of finite fields.

It is not our intention to try finding new special base graphs and produce sophisticated voltage assignments which would yield new largest examples of graphs with given degree and diameter (although, as documented above, the method certainly does have such a potential). Instead, we focus on the other extreme, that is, on finding tight upper bounds on diameters of lifts of *general* base graphs using voltage assignments in *arbitrary* groups (Section 4).

2 Voltage assignments and lifts

Voltage assignments on graphs have been introduced in the early 70's [16] as a dualisation of the theory of the so-called current graphs, which served as the basis for the proof of the famous Heawood Map Color Theorem. An excellent treatment of the theory of voltage graphs can be found in [17]. In order to make the paper self-contained and accessible for readers not acquainted with the theory, we sum up the basics in what follows.

Let G be an (undirected) graph, which may have loops and/or parallel edges. Each edge of G can be assigned one of the two possible orientations (directions). An edge with a preassigned direction will be called an *arc*. Let e be an arc of the (otherwise undirected) graph G. Then, e^{-1} will denote the arc arising from the same *underlying edge* but with orientation opposite to e; obviously, $(e^{-1})^{-1} = e$. The arc e^{-1} is often called the *reverse arc* of e. The set of *all possible* arcs of G will be denoted by $D(G)$. Since each (undirected) edge of G gives rise to two arcs, we have $|D(G)| = 2|E(G)|$.

Let Γ be a group and let G be a graph. A mapping $\alpha : D(G) \to \Gamma$ will be called a *voltage assignment* on G if, for each arc $e \in D(G)$, $\alpha(e^{-1}) = (\alpha(e))^{-1}$. In order to specify a voltage assignment, we usually fix in advance an orientation of the (undirected) graph G and assign voltages to the arcs thus obtained; the reverse arcs will automatically receive the corresponding inverse voltages.

Let $\alpha : D(G) \to \Gamma$ be a voltage assignment on a graph G in a group Γ. The *lift* G^α of the graph G is defined as follows. The vertex set and the arc set of the lift are $V(G^\alpha) = V(G) \times \Gamma$ and $D(G^\alpha) = D(G) \times \Gamma$. For any two vertices (u, g) and (u', g') of the lift, an arc (e, h) emanates from (u, g) and terminates at (u', g') if and only if e is an arc from u to u' in G, $h = g$, and $g' = g\alpha(e)$. Note that, according to this definition, the arc $(e^{-1}, g\alpha(e))$ of G^α emanates from (u', g') and terminates at (u, g), because $\alpha(e^{-1}) = (\alpha(e))^{-1}$. The pair of arcs $(e, g), (e^{-1}, g\alpha(e))$ constitutes an undirected *edge* of the lift G^α; for the reverse arcs in the lift we therefore have $(e, g)^{-1} = (e^{-1}, g\alpha(e))$.

Loosely speaking, in a pictorial representation of a lift, "above" each vertex u of the *base graph* G we have $|\Gamma|$ vertices (u, g), $g \in \Gamma$, in the lift G^α. Similarly, "above" each arc e from u to v in G we have $|\Gamma|$ arcs (e, g), $g \in \Gamma$; the arc (e, g) emanates from the vertex (u, g) and terminates at the vertex $(v, g\alpha(e))$. More precisely, introducing the *natural projection* $\pi : G^\alpha \to G$ by $\pi(u, g) = u$ and $\pi(e, g) = e$, the sets $\pi^{-1}(u)$ and $\pi^{-1}(e)$ are called *fibres* above the vertex u or above the arc e, respectively. (If the graphs in question are viewed as 1-dimensional CW-complexes, as is usual in algebraic topology, then π is simply a *covering projection* and G^α is a *regular covering space* of G.)

In principle, any information about the lift can be obtained in terms of walks in the base graphs. A *walk* of length m in a graph G is a sequence $W = e_1 e_2 \ldots e_m$ where e_i are *arcs* of G, such that the terminal vertex of e_{i-1} is the same as the initial vertex of e_i, $2 \le i \le m$. If the initial vertex of e_1 is u and the terminal vertex of e_m is v, we say that W is a $u-v$ walk. If $u = v$ then the walk W is said to be *closed*, or *closed at u*. If α is a voltage assignment on G, then the *net voltage* of W is simply the product $\alpha(W) = \alpha(e_1)\alpha(e_2) \ldots \alpha(e_m)$. Note that if, say, $e_j = e_i^{-1}$ for some i, j, then we are traversing the underlying edge twice (the second time in the opposite direction). Thus, if an auxiliary orientation of G has been specified, travelling along the walk W may include traversing an arc against its direction (remember that our base graph is undirected), but in computing the net voltage, when traversing an arc in the opposite direction we multiply by the inverse element. For convenience, at each vertex we also admit a *trivial* closed walk of length 0 and of unit net voltage.

The path-lifting properties known in algebraic topology translate into the language of walks as follows. For each walk W in G with initial vertex u and for each $g \in \Gamma$ there exists a *unique* walk W_g^α in the lift G^α which starts at the vertex (u, g) and such that $\pi(W_g^\alpha) = W$. Indeed, if $W = e_1 e_2 \ldots e_m$ is a walk in G emanating from u and if $\alpha(e_i) = x_i$, $1 \le i \le m$, then the walk $W_g^\alpha = (e_1, g)(e_2, gx_1) \ldots (e_m, gx_1 x_2 \ldots x_{m-1})$ emanates in the lift G^α from the vertex (u, g) and has the property that $\pi(W_g^\alpha) = W$; its uniqueness is obvious. Observe that if the walk W ends at the vertex v of G, then W_g^α terminates in G^α at the vertex $(v, g\alpha(W))$. The walk W_g^α is often called a *lift* of W; each walk in the base graph has $|\Gamma|$ different lifts.

For much more information on voltage graphs the reader is invited to consult [17]. We conclude this Section with a remark of typographical nature. In order to avoid long expressions, we will often use the subscript notation for vertices and/or arcs of the lift, and write u_g and e_g instead of (u, g) and (e, g) in what follows.

3 Examples

We illustrate the voltage graph technique on several currently known largest examples of graphs with given degree and diameter for some particular values of d and k. The ubiquitous Petersen graph is the largest graph of degree 3 and diameter 2; it can be obtained as a lift of a two-vertex "dumbell graph" with voltages in the group $\mathcal{Z}_5$, as depicted in Fig. 1. (The number appearing at an arc of the dumbell graph is the corresponding voltage; arcs without any number automatically receive zero voltage.)

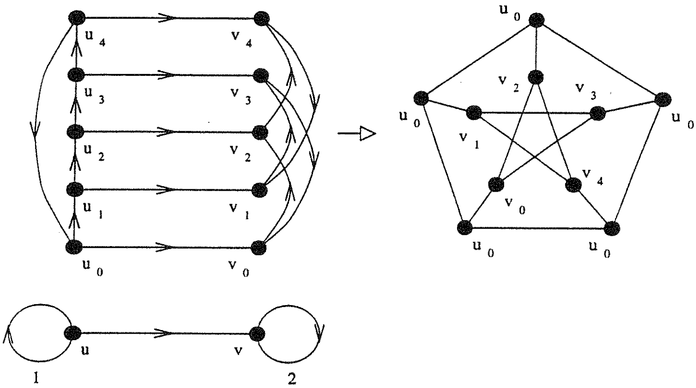

Figure 1: The Petersen Graph.

The famous Hoffman-Singleton graph is the unique largest graph of diameter 2 and degree 7; it has 50 vertices, and it can be obtained as a lift of the "inflated dumbell" (with 5 parallel edges) endowed with voltages in the group $\mathcal{Z}_5 \times \mathcal{Z}_5$ as indicated in Fig. 2.

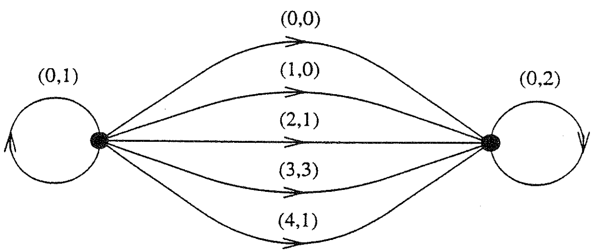

Figure 2: The base graph for the Hoffman-Singleton Graph.

As stated in the Introduction, the only three pairs (d, k) for $d \geq 4$, $d \neq 7$ and $k \geq 2$ for which the equality in $n_{d,k} \leq M_{d,k} - 2$ is known to be attainable are $(3,3)$, $(4,2)$ and $(5,2)$. We show that the corresponding largest graphs – denoted by $C_5 \star F_4$, $K_3 \star C_5$ and $K_3 \star X_8$ in the tables in [11, 18] – can all be obtained using voltage assignments. (We note that the star notation reflects the fact that the graphs can be described as twisted products, see [4].) In the next three figures, edges without direction are assumed to have zero voltage.

The unique [20] graph $C_5 \star F_4$ of degree 3 and diameter 3 on 20 vertices is a lift of the base graph in Fig 3 of order 5, with voltages in the group $\mathcal{Z}_5$.

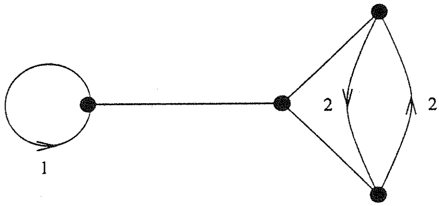

Figure 3: The base graph for $C_5 \star F_4$.

The graph $K_3 \star C_5$ of degree 4 and diameter 2 is the unique [15] graph of order 15, degree 4, and diameter 2; it can be obtained as a lift (with voltages in $\mathcal{Z}_3$) of the 5-vertex base graph in Fig 4.

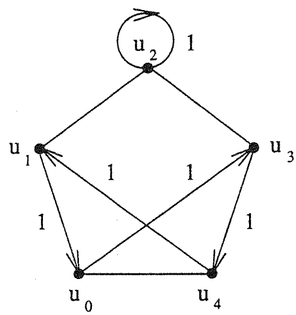

Figure 4: The base graph for $K_3 \star C_5$.

Finally, the graph $K_3 \star X_8$ of degree 5, diameter 2, and order 24 (whose uniqueness is not known, to our knowledge) can be constructed as a lift of the graph of order 8 in Fig 5; the voltage group is $\mathcal{Z}_3$.

The above claims can be checked directly by constructing the corresponding lifts. However, the advantage of voltage assignments is that, in fact, one need not actually construct the graphs. The point is that the diameter verification can be done just by doing computations on the (relatively smaller) base graphs, using the following observation:

Lemma 1 *Let α be a voltage assignment on a graph G in a group Γ. Then, diam(G^α) $\leq k$ if and only if for each ordered pair of vertices u, v (possibly, $u = v$) of G and for each $g \in \Gamma$ there exists a $u - v$ walk of length $\leq k$ of net voltage g.*

Proof. Observe that for any two distinct vertices u_g and v_h in $V(G^\alpha)$ there exists a walk $\tilde{W}$ of length at most k from u_g to v_h if and only if the projection $W = \pi(\tilde{W})$

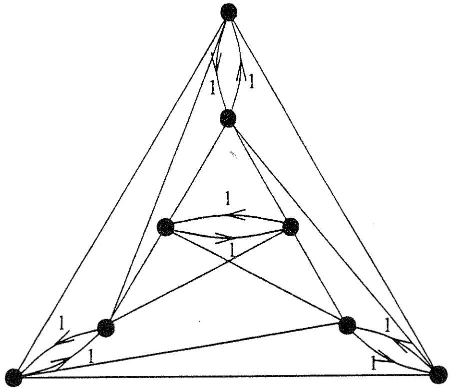

Figure 5: The base graph for $K_3 \star X_8$.

is a walk in the base graph G of length at most k from u to v with $\alpha(W) = g^{-1}h$. (The case when both $u = v$ and $g = h$ is taken care of by length 0 closed walks.) □

We emphasize that one can produce a large number of examples as above. In fact, as we show in [7], all the Cayley graphs for semidirect products of cyclic groups listed in [18] (which were found by computer search and are the currently largest known graphs for certain d and k) are lifts of smaller Cayley graphs with voltages in cyclic groups.

4 An upper bound on diameter of a lift

We start with an auxiliary result which is rather technical but important for proving the general upper bound on the diameter of a lift.

Proposition 1 *Let Γ be a group and let X be a subset of Γ such that $X^{-1} = X$ and $x^2 \neq id$ for each $x \in X$. Let G be a graph and let A be a subset of $V(G)$. Assume that each vertex in A has degree at least $|X|$ in G. Then there exists a voltage assignment $\alpha : D(G) \to \Gamma$ such that for each $v \in A$ and each $x \in X$ there exists an arc z emanating from v such that $\alpha(z) = x$.*

Proof. Our assumptions imply that $|X|$ is even, say, $|X| = 2t$. The main idea of the proof is to modify the graph G in several steps so that the final result will be a regular graph of degree $2t$. By Petersen's theorem, this new graph will have a 2-factorisation, and we will use it to induce the required voltage assignment on the original graph. Throughout, if L is a graph and u is a vertex of L then $d_L(u)$ denotes the degree of u in L.

Let $B \subset V(G)$ be the set of all odd-degree vertices of G; we know that $|B|$ is even. Take a collection of $|B|/2$ new edges (i.e., edges not in G) and add them to G in such a way that the new graph (call it H) has no vertex of odd degree. In what follows, we will refer to these added edges in the graph H (and in its subsequent modifications) as *red.*

Let $C = V(H)\setminus A$. In the next step, we split each vertex $u \in C$ into $k_u = d_H(u)/2$ new vertices $u^1, u^2, \ldots, u^{k_u}$, and we distribute the $2k_u$ edges originally incident to u in H so that each u^i will have degree 2 in the new graph. Having done this with *all* vertices in C, we obtain from H a (possibly disconnected) new graph K; note that $V(K) = A \cup_{u\in C} \{u^1, \ldots, u^{k_u}\}$ and $|E(H)| = |E(K)|$.

We proceed with concentrating on the vertices of K which are in the distinguished set A. For each vertex $v \in A$ of degree greater than $2t$ we add another $k_v = d_K(v)/2 - t$ new vertices $v^1, \ldots, v^{k_v}$ and re-distribute the $d_K(v)$ edges originally incident in K to v in such a way that the following three requirement are fulfilled: The vertex v will have degree $2t$ in the new graph, each v^i will have degree 2 in the new graph, and if v was in K incident to a red edge, then this red edge is incident to one of the v^i in the new graph. Having done this with *all* vertices $v \in A$ of degree greater than $2t$, we obtain a graph L, with $V(L) = A \cup_{u\in C} \{u^1, \ldots, u^{k_u}\} \cup_{v\in A'} \{v^1, \ldots, v^{k_v}\}$, where $A' = \{v \in A;\ d_K(v) \geq 2t + 2\}$, and again $|E(L)| = |E(K)| = |E(H)|$.

Finally, we attach $t-1$ loops to each vertex in L of degree 2, obtaining thereby the graph L^*. Since L^* is a regular graph of degree $2t$, by Petersen's theorem it has a 2-factorisation. That is, the set $E(L^*)$ has a decomposition into t spanning subgraphs (= factors) $F_1, \ldots, F_t$ such that each F_j is a union of vertex-disjoint cycles of L^*. Now, let us choose an orientation of each cycle in each factor F_j; since each edge of L^* belongs to precisely one F_j, this will induce an orientation on each edge of L^*. We will use these temporarily introduced directed edges (= arcs) to define a suitable voltage assignment.

We may without loss of generality assume that the subset X of our group Γ has the form $X = X_0 \cup X_0^{-1}$ where $X_0 = \{x_1, x_2, \ldots, x_t\}$; we will fix this notation. Define a voltage assignment β on the arcs of L^* as follows. Let z be an arc of L^*; then there is exactly one j, $1 \leq j \leq t$ such that z belongs to F_j. Then, set $\beta(z) = x_j$ if at least one of the vertices incident to z belongs to A; otherwise let $\beta(z) = id$.

We now reassemble our original graph G from L^*: Remove all the added loops to obtain L, identify the vertices $v^1, \ldots, v^{k_v}$ with v for each $v \in A'$ (and keep the incident edges) to obtain K, and identify the vertices $u^1, \ldots, u^{k_u}$ for each $u \in C$ to obtain H. At last, remove the red edges, obtaining back the graph G together with the temporary directions on its edges. Let α be the voltage assignment on G induced by β (recall that, except of the red edges, we did not remove any other edges in the process of "shrinking" L^* back to G). Our construction guarantees that for each vertex $v \in A$, there are at least t arcs leaving v whose voltages are successively $x_1, x_2, \ldots, x_t$, and, at the same time, there are at least t arcs entering v with voltages as above. This means that, reversing the directions of the latter t arcs, they will have voltages $x_1^{-1}, x_2^{-1}, \ldots, x_t^{-1}$. It follows that for each $v \in A$ and each $x \in X$ there is an arc z leaving v such that $\alpha(z) = x$, as required. $\square$

Let us remark that, in general, Proposition 1 is no longer true if the assumption $x^2 \neq id$ is dropped. For an infinite source of such examples just consider, for $d \geq 2$, any d-regular graph G that has no perfect matching, and any group Γ with a subset X containing at least one involution and such that $|X| = d$. If $A = V(G)$ then no voltage assignment as described in Proposition 1 can exist.

We precede the statement and the proof of the main result by recalling the concept of an (undirected) Cayley graph. Let Γ be a group and let X be a generating set for Γ such that $id \notin X$, and $x^{-1} \in X$ whenever $x \in X$. The *Cayley graph* $H = Cay(\Gamma, X)$ has vertex set $V(H) = \Gamma$; two vertices g and h are adjacent in H if and only if $g^{-1}h \in X$. Observe that $g^{-1}h \in X$ is equivalent to $h^{-1}g \in X$, and hence the Cayley graph is *undirected*. It is easy to see that $Cay(\Gamma, X)$ is a connected, vertex-transitive graph of degree $|X|$.

Also, we recall that the *eccentricity* of a vertex v in a connected graph G is the largest distance from v to a vertex in G. The *radius of* G, $rad(G)$, is the smallest eccentricity over all vertices of G. A vertex of a graph is *central* if its eccentricity is equal to the radius of the graph. If H is a graph and $u, v \in V(H)$, then the symbol $d_H(u, v)$ will stand for the distance between u and v in H.

Theorem 1 *Let $H = Cay(\Gamma, X)$ be a Cayley graph and let $x^2 \neq id$ for each $x \in X$. Let G be a connected graph of minimum degree at least $|X| + 1$. Then there exists a voltage assignment α on G in the group Γ such that*

$$diam(G^\alpha) \leq 2rad(G) + 2diam(H) - 1$$

Proof. Let w be a central vertex of G and let T be a spanning tree rooted at w such that $d_G(w, u) = d_T(w, u)$ for each $u \in V(G)$; let $r = rad(T) = rad(G)$. Let A be the set of all pendant vertices of T and let $H = G \setminus E(T)$; observe that $d_H(v) \geq |X|$ for each $v \in A$. By Proposition 1 applied to the graph H and the set A, there exists a voltage assignment $\alpha : D(G) \to \Gamma$ such that $\alpha(z) = id$ for each arc z whose underlying edge belongs to T, and such that for each $v \in A$ and each $x \in X$ there exists an arc z emanating from v with $\alpha(z) = x$. Our aim is to apply Lemma 1 after showing that for any ordered pair u, v of vertices of G and any $g \in \Gamma$ there exists a $u - v$ walk in G of length $\leq 2r + 2diam(H) - 1$, with net voltage g. The latter is obvious for $g = id$, as for any u, v there is a $u - v$ path on T of length $\leq 2r$.

Let $g \neq id$; we may w.l.o.g. assume that $d_T(u, w) \geq d_T(v, w)$. We invoke our auxiliary Cayley graph $H = Cay(\Gamma, X)$, in which there exists a path from the vertex id to the vertex g of length at most $diam(H)$. That is, there exist generators $x_1, x_2, \ldots, x_k \in X$ such that $g = x_1 x_2 \ldots x_k$ where $k \leq diam(H)$. With the help of this product we construct (by induction) a suitable $u - v$ walk W of net voltage g in the base graph G.

Let $v^0 = u$. Choose a pendant vertex u^1 in our spanning tree T of G in such a way that the unique $v^0 - u^1$ path P_1 in T does not contain the root w. By Lemma 1, there exists an arc e^1 in G but not in T, emanating from u^1 and terminating at some vertex v^1, such that $\alpha(e^1) = x_1$. We proceed by induction in much the same way: For any i, $2 \leq i \leq k$, take a pendant vertex u^i of T such that the (unique) path P_i in T from v^{i-1} to u^i avoids the root w. Again, Lemma 1 yields an arc e^i not in T

which emanates in the base graph G from u^i and terminates at some vertex v^i, with $\alpha(e^1) = x_i$. Finally, let Q be the (unique) $v^k - v$ path in T, and let W be the $u - v$ walk of the form $W = P_1 e^1 P_2 e^2 \dots P_k e^k Q$.

It is most important to note that, by the choice of our spanning tree T (which preserves distances from w) in the base graph G we have, for $1 \le i \le k$,

$$d_G(v^i, w) \ge d_G(u^i, w) - 1$$

Indeed, if this were not the case and if $d_G(v^i, w) < d_G(u^i, w) - 1$, then (using the arc $(e^i)^{-1}$ from v^i to u^i in G) we would have $d_G(w, u^i) \le d_T(w, v^i) + 1 = d_G(w, v^i) + 1 < d_G(w, u^i)$, which is absurd.

Since the net voltage of Q and of each P_i is identity, we see that $\alpha(W) = x_1 x_2 \dots x_k = g$. It remains to estimate the length $\ell(W)$ of W. We first note that $\ell(P_i) = d_T(u^i, w) - d_T(v^{i-1}, w) = d_G(u^i, w) - d_G(v^{i-1}, w)$ for $1 \le i \le k$ (note that $v^0 = u$). Recalling the inequality in the preceding paragraph, we obtain $\ell(P_i) \le d_G(u^i, w) - d_G(u^{i-1}, w) + 1$, $2 \le i \le k$. Therefore,

$$\sum_{i=1}^{k} \ell(P_i) \le d_G(u^1, w) - d_G(u, w) + \sum_{i=2}^{k} (d_G(u^i, w) - d_G(u^{i-1}, w) + 1)$$

$$= d_G(u^1, w) - d_G(u, w) + d_G(u^k, w) - d_G(u^1, w) + k - 1$$

$$= d_G(u^k, w) - d_G(u, w) + k - 1$$

At last, recalling the assumption $d_G(u, w) \ge d_G(v, w)$ and using the above facts, we obtain

$$\ell(W) = \ell(Q) + \sum_{i=1}^{k} (\ell(P_i) + 1)$$

$$\le d_G(v, w) + d_G(v^k, w) + d_G(u^k, w) - d_G(u, w) + 2k - 1$$

$$\le 2rad(G) + 2diam(H) - 1$$

Thus, by Lemma 1 we have $diam(G^\alpha) \le 2rad(G) + 2diam(H) - 1$. □

Theorem 1 has the following obvious corollary.

Corollary 1 *Let $H = Cay(\Gamma, X)$ be a Cayley graph and let $x^2 \ne id$ for each $x \in X$. Let G be a connected graph of minimum degree at least $|X| + 1$. Then there exists a voltage assignment α on G in the group Γ such that*

$$diam(G^\alpha) \le 2diam(G) + 2diam(H) - 1$$ □

As pointed out earlier, involutions in the generating set cannot be in general directly treated by Proposition 1. However, an easy inspection shows that Proposition 1 is still true if the set X consists just of a single involution (and hence the group generated by X is $\mathcal{Z}_2$). In this case one can relax the degree requirement and show that for any connected graph G which is not a tree there exists a $\mathcal{Z}_2$-lift G^α of G such that $diam(G^\alpha) \le 2diam(G) + 1$.

5 Concluding remarks

Note that Theorem 1 as well as Corollary 1 are, in some sense, the best possible. To see this, let G be any graph attaining the Moore bound, that is, either a complete graph, or an odd cycle, the Petersen graph, or the Hoffman-Singleton graph. For any such G we clearly have $rad(G) = diam(G)$. Let Γ be any group of order equal to the degree of G, let $X = \Gamma \setminus \{id\}$ and let $H = Cay(\Gamma, X)$. Observe that H is a complete graph and hence $rad(H) = diam(H) = 1$. By Theorem 1 or Corollary 1, *there exists* a voltage assignment α in the group Γ such that $diam(G^\alpha) \leq 2diam(G) + 1$.

This is best possible because, as we now show, for *any* voltage assignment β on the graph G in the group Γ we have $diam(G^\beta) \geq 2diam(G) + 1$. Indeed, assume that $diam(G^\beta) \leq 2k$ where $k = diam(G)$. Then, for any two distinct vertices in the same fibre of G^β, say u_{id} and u_g, there exists a walk $\tilde{W}$ of length at most $2k$ from u_{id} to u_g. The projection $W = \pi(\tilde{W})$ is therefore a closed walk in G based at u, with net voltage $\beta(W) = g$ and length at most $2k$. However, assuming that G attains the Moore bound, the length of the shortest cycle of G is equal to $2k+1$. Consequently, the only closed walks of length at most $2k$ in G are those with trivial net voltage; hence $g = id$, a contradiction.

However, we suspect that both Theorem 1 and Corollary 1 can be greatly improved for special base graphs and voltage groups.

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