The α-labeling Number of Bipartite Graphs*

Chin-Lin Shiue

Department of Applied Mathematics
National Chiao Tung University
Hsinchu 30050, Taiwan, R.O.C.

Abstract

In this paper, we study the α-labeling number G_α of a bipartite graph G.

1 Introduction

Unless otherwise stated, all the graphs in this paper are finite and simple. For terms and notation used in this paper we refer to the textbook by Bondy and Murty [1]. Given a graph G with $|E(G)| = q$, an injective function $f : V(G) \longrightarrow \{0, 1, 2, ..., q\}$ is a *β-labeling* of G provided that the values $|f(u) - f(v)|$ for the q pairs of adjacent vertices u and v are distinct. A β-labeling is also known as a *graceful* labeling.

If f is a β-labeling of G and there exists an integer λ so that for each edge $uv \in E(G)$ either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$, then f is called an *α-labeling* of G. It is not difficult to see that only a bipartite graph can receive an α-labeling. But, not all bipartite graphs have an α-labeling [7]. As a matter of fact, to determine whether a bipartite graph has an α-labeling is extremely difficult. This can be seen from the following well-known conjecture.

Conjecture 1.[8] Every tree has a graceful labeling.

In [9], Snevily introduced the following notion. G is said to *eventually have an α-labeling* provided that there exists a graph H which has an α-labeling and H can be decomposed into copies of G. Such a graph H is called a *'host'* graph of G, we also say G divides H, denoted by $G|H$. If G eventually has an α-labeling, then we let $G_\alpha = \min\{t$: there exists a host graph H of G with $|E(H)| = t \cdot |E(G)|\}$ and call G_α the *α-labeling number* of G. Otherwise, we let $G_\alpha = +\infty$. As an example, the n-cube Q_n can be decomposed into 2^{n-1} copies of any tree T with n edges [3] and

*Supported in part by the National Science Council of the Repubic of China(NSC-85-1221-M-009-010)

Q_n has an α-labeling [5], thus a tree T eventually has an α-labeling and $T_\alpha \leq 2^{n-1}$. As to a general bipartite graph G, it was conjectured by Snively [9] that $G_\alpha < +\infty$. Recently, the conjecture was proved by El-Zanati *et al.*

Theorem 1.1.[2] If G is a bipartite graph, then G_α is finite.

The upper bound obtained in the proof Theorem 1.1 is huge (around $(q^2!)^2$ for a graph with q edges!). In order to obtain smaller α-labeling number of a bipartite graph other techniques are needed.

In this paper, we mainly introduce a labeling technique to directly decompose a complete bipartite graph into copies of a bipartite graph. And then we obtain smaller upper bounds of α-labeling numbers for some classes of bipartite graphs. Finally, we also give a pretty small upper bound of α-labeling number for a special class of bipartite graphs by a new method.

2 The main results

Throughout of this paper, we shall use the complete bipartite graph $K_{m,n}$ as the host graph. Therefore we need the following well-known Theorem.

Theorem 2.1.[7] *$K_{m,n}$ has an α-labeling.*

Now, if $K_{m,n}$ can be decomposed into copies of G, then we have $G_\alpha \leq \dfrac{m \cdot n}{|E(G)|}$. For example, the following result shows a class of regular bipartite graphs with small α-labeling number.

Theorem 2.2.[4] *Let G be a 3-regular bipartite graph on $4m$ vertices such that no component of G is the Heawood graph. Then $K_{6m,6m}$ can be decomposed into $6m$ copies of G. Thus $G_\alpha \leq 6m$.*

Before we consider a more general class of bipartite graphs, we need a definition. Let $G = (U, V)$ be a bipartite graph such that $U = \{u_1, u_2, ..., u_r\}$ and $\deg_G(u_i) = d_i$ for $1 \leq i \leq r$. A *partial β-labeling* f of G on V is an injection from V into N such that $\{f(v) (\text{mod } d_i) \mid v \in N_G(u_i)\} = \{0, 1, 2, ..., d_i - 1\}$ for each $1 \leq i \leq r$. Figure 2.1 is an example. For convenience, we will take q to replace the element 0 in Z_q for each positive integer q throughout this paper.

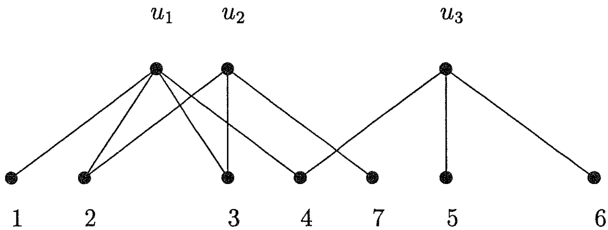

Figure 2.1

Theorem 2.3. *Let $G = (U, V)$ be a bipartite graph with n edges and let G contain a partial β-labeling f on V. Then for each positive integer m satisfying that $m \geq max\{f(v) \mid v \in V\}$ and l.c.m.$\{\deg_G(u) \mid u \in U\}|m$, $K_{n,m}$ can be decomposed into m copies of G.*

Proof. Let $U = \{u_1, u_2, ..., u_t\}$ and $\deg_G(u_i) = d_i$ for convenience. For $1 \leq i \leq t$, let $S_i = \{u_{i,1}, u_{i,2}, ..., u_{i,d_i}\}$ and $V' = \{1, 2, ..., m\}$. By letting $U' = \bigcup_{j=1}^{t} S_j$ and connecting each vertex of U' to each vertex of V', we obtain a complete bipartite graph $G' = (U', V') \cong K_{n,m}$. Now we are ready to decompose $K_{n,m}$ into m copies of G.

For $j = 1, 2, ..., m$, let $G_j = (U_j, V_j)$ be a bipartite graph defined as follows:

(i) $U_j = \{u_{i,j \pmod{d_i}} \mid 1 \leq i \leq t\}$;

(ii) $V_j = \{(f(v) + j) (\text{mod } m) \mid v \in V\}$; and

(iii) E_j is the set of edges joining $u_{i,j \pmod{d_i}}$ and $(f(v) + j)(\text{mod } m)$ for each $v \in N_G(u_i)$, $i = 1, 2, ..., t$.

By definition, it is not difficult to see that $G_j \cong G$ for each $j = 1, 2, ..., m$. It is left to show that $G_1, G_2, ..., G_m$ form a decomposition of $K_{n,m}$, i.e., for each pair of distinct i and j, $1 \leq i, j \leq m$, $E_i \cap E_j = \emptyset$. Suppose not. Let $u_{k,x}y \in E_i \cap E_j$. Then $x \equiv i \equiv j (\text{mod } d_k)$ and there exist two distinct vertices v_1 and v_2 in $N(u_k)$ such that $f(v_1)+i \equiv f(v_2)+j \equiv y(\text{mod } m)$. This implies that $f(v_1)-f(v_2) \equiv j-i \equiv 0(\text{mod } d_k)$ since $d_k|m$. Hence, by the reason that f is a partial β-labeling on V, $v_1 = v_2$. This is a contradiction and we conclude the proof. ∎

We note here that $\{\deg_G(u) \mid u \in U\}$ (in Theorem 2.3) is a partition of n , and its l.c.m. is bounded by Landau's function $g(n)$ which is the maximum order of an element of the symmetric group S_n. The function $g(n)$ is a well-studied function in number theory. It was proved by Landau (see [6]) that $\log g(n) \sim \sqrt{n \log n}$. Hence, for a bipartite graph $G = (U, V)$ with n edges, if it contains a patial β-labeling f on V and $f(v) \leq O(g(n))$ for each vertex $v \in V$, then G_α has an upper bound $e^{O(\sqrt{n \log n})}$. For example (see Figure 2.2), if $G = (U, V)$ and each vertex of U is adjacent to consecutive vertices (ordered) in V, then G has a partial β-labeling f on V and $\max\{f(v) \mid v \in V\} = |V|$. Thus, the following result is obvious.

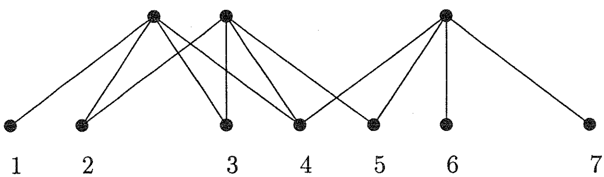

Figure 2.2

Proposition 2.4. *Let $G = (U, V)$ be a bipartite graph with n edges such that the order of the vertices of V is fixed and each vertex of U is joined to consecutive vertices of V. Then $G_\alpha \leq e^{O(\sqrt{n \log n})}$.*

In [9], Snevily pointed out that if T is a tree with n edges, then by the fact that an n-cube Q_n has an α-labeling and Q_n can be decomposed into 2^{n-1} copies of T we have $T_\alpha \leq 2^{n-1}$. But, this upper bound is too big and Snevily believes that $T_\alpha \leq n$. In what follows, we first claim that for each tree T there exists a good partial β-labeling, and then find a better upper bound for T_α.

Proposition 2.5. *Let T be a tree with n edges, then $T_\alpha \leq e^{O(\sqrt{n \log n})}$.*

Proof. Since T is a bipartite graph, let $T = (U, V)$ and $U = \{u_1, u_2, ..., u_s\}$. Furthermore, let $\deg_T(u_j) = d_j$, $j = 1, 2, ..., s$. Now, define a collection of ascending subtrees of T recursively. Let T_1 be the subtree induced by $N_T[u_1]$, where $N_T[u_1]$ is the union of $\{u_1\}$ and $N_T(u_1)$ the set of vertices which are adjacent to u_1. Let T_{i+1} be the subtree induced by $V(T_i) \cup N_T[u_{i+1}]$, $i = 1, 2, ..., s-1$. Note that $|V(T_i) \cap N_T[u_{i+1}]| = 1$ for each $i = 1, 2, ..., s-1$. Let these common vertices be denoted by $v_1, v_2, ..., v_{s-1}$.

By the idea of Theorem 2.3, it suffices to obtain a partial β-labeling on V, and the labeling f can be obtained recursively by the following steps:

(i) Use the elements in $\{1, 2, ..., d_1\}$ to label the vertices in $N_T(u_1)$.

(ii) For $i = 1, 2, ..., s-1$, use the elements in $\{\sum_{k=1}^{i} d_k + l \mid l = 1, 2, ..., d_{i+1}$ and $\sum_{k=1}^{i} d_k + l \neq f(v_i)(\text{mod } d_{i+1})\}$ to label the vertices in $N_T(u_{i+1})\backslash\{v_i\}$.

Now, it is not difficult to check that f is a partial β-labeling on V and the maximum label we use is at most $\sum_{i=1}^{s} d_i = |E(T)| = n$. This concludes the proof. ∎

Note that $e^{O(\sqrt{n \log n})}$ is far less than 2^{n-1} when n is sufficiently large, since $\lim_{n\to\infty} \frac{e^{c_1(\sqrt{n \log n})}}{2^{n-1}} = 0$ where c_1 is a positive constant. Actually, for some classes of trees, the upper bounds are pretty small. For example, let $T = (U, V)$ be a tree with n edges. If the degrees of vertices in U are the same, then $T_\alpha \leq O(n)$.

Finally, we will show a class of bipartite graphs with smaller α-labeling number by a new technique.

Proposition 2.6. *Let $G = (A, B)$ be a bipartite graph. Suppose A can be partitioned into r subsets $A_1, A_2, ..., A_r$ such that for $1 \leq i \leq r$, the subgraph of G induced by $G_i = (A_i, B)$ is a star forest which spans B. Then $G_\alpha \leq max\{|A_1|, |A_2|, ..., |A_r|\}$.*

Proof. Let $q=\max\{|A_1|, |A_2|, ..., |A_r|\}$. First, we claim that $K_{q,|B|}$ can be decomposed into q copies of G_i for each $1 \leq i \leq r$. Given some k, $1 \leq k \leq r$, let $A_k = \{a_1, a_2, ..., a_s\}$ and $B = \{b_1, b_2, ..., b_t\}$ where $|A_k| = s$ and $|B| = t$. We also let $\bar{G} = (U, V)$ such that $U = \{u_1, u_2, ..., u_q\}$, $V = \{v_1, v_2, ..., v_t\}$ and $u_i v_j \in E(\bar{G})$ for $1 \leq i \leq q$ and $1 \leq j \leq t$. Clearly, $\bar{G} \cong K_{q,|B|}$. Now let $\bar{G}^l = (U^l, V)$ be a subgraph of $\bar{G}$ for each $1 \leq l \leq q$ such that $|U^l| = \{u_{l+i-1 \pmod q} \mid i = 1, 2, ..., s\}$ and $u_{l+i-1 \pmod q} v_j \in E(\bar{G}^l)$ if and only if $a_i b_j \in E(G_k)$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. Therefore, $\bar{G}^l \cong G_k$. Furthermore, $K_{q,|B|}$ can be decomposed into $\bar{G}^1, \bar{G}^2, ..., \bar{G}^q$. Now, we first decompose $K_{rq,|B|}$ into $\bar{G}_1, \bar{G}_2, ..., \bar{G}_r$ such that $\bar{G}_i \cong K_{q,|B|}$ for $1 \leq i \leq r$. Then $K_{rq,|B|}$ can be decomposed into q copies of G by choosing a copy of G_i in each $\bar{G}_i$ and we have the proof. ∎

As a special case of Proposition 2.6, we prove that for the class of star forests with n edges and no isolated vertices, the upper bound of the α-labeling number is not greater than n. In fact, we can prove the upper bound is two by a simple observation. Since any two isomorphic star forests with no isolated vertices can be combined to a caterpillar which has an α-labeling.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, Elsevier North Holland, Inc. 1976.

[2] S. El-Zanati, H. L. Fu and C. L. Shiue, *A note on the α-labeling number of bipartite graphs*, submitted.

[3] J.F. Fink, *On the decomposition of n-cubes into isomorphic trees*, J. Graph Theory 14 (1990), pp 405-411.

[4] R. Häggkvist, *Decompositions of complete bipartite graphs*, in: Surveys in Combinatorics, Norwich 1989, London Math. Soc. Lecture Notes Ser., Vol. 141 (Cambridge, New York, 1989) 115-147.

[5] A. Kotzig, *Decompositions of complete graphs into isomorphic cubes*, J. Comb. Theory, Series B 31 (1981), pp 292-296.

[6] J. Nicolas, *On Landau's function $g(n)$*, The Mathematics of Paul Erdös I, R. Graham and J. Nestril (Eds.), Springer, (1995), pp 228-240.

[7] A. Rosa, *On certain valuations of the vertices of a graph*, in: Théorie des graphes, journées internationals d'études, Rome 1966 (Dunod, Paris, 1967) 349-355.

[8] G. Ringel, *Theory of Graphs and its Applications*, Academic Press, New York (1963), 162.

[9] H. S. Snevily, *New families of graphs that have α-labelings* , Discrete Mathematics 170 (1997), pp 185-194.

(Received 23/2/98)