ON THE VERTEX ARBORICITY OF GRAPHS WITH PRESCRIBED SIZE

Nirmala Achuthan, N.R. Achuthan and L. Caccetta

School of Mathematics and Statistics
Curtin University of Technology
G.P.O. Box U1987
PERTH WA 6845

ABSTRACT :

Let $\mathcal{G}(n)$ denote the class of simple graphs of order n and $\mathcal{G}(n,m)$ the subclass of graphs with size m. $\overline{G}$ denotes the complement of a graph G. For a graph G, the vertex arboricity $\rho(G)$, is the minimum number of colours needed to colour the vertices of G such that every colour class is acyclic. In this paper we determine the range for the size of a graph $G \in \mathcal{G}(n)$ with prescribed arboricity. We also characterize the extremal graphs. Further, we establish sharp bounds for the sum $\rho(G) + \rho(\overline{G})$ and the product $\rho(G).\rho(\overline{G})$, where G ranges over $\mathcal{G}(n,m)$. We determine the class of graphs G for which $\rho(G).\rho(\overline{G})$ attains the minimum value.

1. INTRODUCTION AND NOTATION :

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For a graph G, V(G) denotes the vertex set, E(G) the edge set, $\nu(G)$ the number of vertices and $\varepsilon(G)$ the number of edges. The complement of a graph G is denoted by $\overline{G}$. For the most part, our notation and terminology follow that of Bondy and Murty [2].

Let $\mathcal{G}(n)$ denote the class of graphs of order n and $\mathcal{G}(n,m)$ the subclass of $\mathcal{G}(n)$ having m edges. Given a graph theoretic parameter f(G) and a positive integer n, the **Nordhaus-Gaddum (N-G)-problem** is to determine sharp bounds for the sum and the product of f(G) and $f(\overline{G})$ as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs. A further problem is to determine the set of all integer pairs (x,y) such that f(G) = x and $f(\overline{G}) = y$ for some $G \in \mathcal{G}(n)$. We refer to this latter problem as the **realizability problem**.

Australasian Journal of Combinatorics 19(1999), pp.17–36

A number of variations to the N-G problem have been considered - Dirac [3] and Plesnik [6]. Achuthan et al. [1] studied the N-G problem for the parameters chromatic number, diameter and edge-connectivity when G is restricted to the subclass $\mathcal{G}(n,m)$. In this paper we investigate N-G problem for the parameter vertex arboricity.

For a real number x, $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the largest (smallest) integer less (greater) than or equal to x. A **k-colouring** of a graph G is an assignment of k colours to its vertices so that no cycle of G has all of its vertices coloured with the same colour. The **vertex arboricity ρ(G)** of a graph G is the smallest integer k for which G has a k-colouring. A k-colouring of a graph gives rise to a partition of the vertex set of the graph into k colour classes, such that the subgraph induced on each colour class is acyclic. We denote by P_n the path on n vertices and by $\vee$ the join operation on graphs.

It is easy to verify that $\rho(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$. We now state a known result that we need for our discussion.

Theorem 1.1 : (Mitchem [5]) For $G \in \mathcal{G}(n)$, we have

$$\lceil \sqrt{n} \rceil \le \rho(G) + \rho(\overline{G}) \le \left\lfloor \frac{n+3}{2} \right\rfloor \qquad (1.1)$$

$$\left\lceil \frac{n}{4} \right\rceil \le \rho(G).\rho(\overline{G}) \le \left(\frac{n+3}{4} \right)^2 \qquad (1.2)$$

Furthermore, the upper bound in (1.1) and the lower bound in (1.2) are sharp for all n. The other two bounds are sharp for infinitely many values of n. □

Henceforth we assume without any loss of generality that m and n are integers such that $m \le \frac{1}{2}\binom{n}{2}$.

2. GRAPHS WITH PRESCRIBED VERTEX ARBORICITY :

In this section we determine the range for the number of edges of a graph G of order n and arboricity α.

Lemma 2.1 : Let $G \in \mathcal{G}(n,m)$ and $\rho(G) = \alpha$. Then

$$m \geq \binom{2\alpha - 1}{2} \tag{2.1}$$

Furthermore, if $m = \binom{2\alpha - 1}{2}$ then $G \cong K_{2\alpha-1} \cup \overline{K}_{n-2\alpha+1}$.

Proof : Consider an α-colouring of the vertices of G. This induces a partition $V_1, V_2, \ldots, V_\alpha$ of V(G) such that $G[V_i]$ is acyclic. We modify this partition of V(G) by performing the following operation i in the order $i = 2,3,\ldots,\alpha$.

Operation i : For every vertex $y \in V_i$ perform the step y.

Step y : Let j be the smallest integer $1 \leq j \leq i - 1$ such that there is no cycle in $G[V_j \cup \{y\}]$. Define a new partition of V(G) as follows :

$V_k := V_k$, $k \neq i, j$ and $1 \leq k \leq \alpha$;

$V_i := V_i - \{y\}$; and

$V_j := V_j \cup \{y\}$.

If no such j exists then the partition of V(G) remains unchanged.

Note that the above procedure yields a partition $V_1, V_2, \ldots, V_\alpha$ of V(G) with the following properties for $1 \leq i \leq \alpha$:

(i) $G[V_i]$ is acyclic;

(ii) For $y \in V_i$ and j such that $1 \leq j \leq i - 1$, $G[V_j \cup \{y\}]$ contains a cycle.

From property (ii) it follows that every vertex of V_i is adjacent to at least two vertices of V_j, $1 \leq j \leq i - 1$. Thus each vertex of V_i is adjacent to at least 2(i-1) vertices of $\bigcup_{j=1}^{i-1} V_j$. Further, note that $G[V_i]$, $1 \leq i \leq \alpha-1$, has at least one edge, for otherwise property (ii) is violated. This in turn implies that $|V_i| \geq 2$ for $i=1,2,\ldots,\alpha-1$. Now counting the number of edges in G, we have

$$m \geq \sum_{i=2}^{\alpha} 2|V_i|(i-1)+(\alpha-1) \geq 2(\alpha-1)+4\sum_{i=2}^{\alpha-1}(i-1)+(\alpha-1) = \binom{2\alpha-1}{2}.$$

This establishes the inequality (2.1). Now if $m = \binom{2\alpha-1}{2}$, then clearly $|V_\alpha| = 1$; $|V_i| = 2$, $2 \leq i \leq \alpha-1$; and $|V_1| = n-2\alpha+3$. Using properties (i) and (ii) it is easy to show that $G \cong K_{2\alpha-1} \cup \overline{K}_{n-2\alpha+1}$. This completes the proof. □

For the rest of this section, n and α are given integers and we put $\ell = \left\lfloor \frac{n}{\alpha} \right\rfloor$ and $\ell' = n - \alpha\ell$. We define the graph $Q_{n,\alpha}$ by $Q_{n,\alpha} \cong \bigvee_{i=1}^{\alpha} T_i$, where T_i is a tree of order $\ell+1$ if $i \leq \ell'$ or of order ℓ, if $i > \ell'$.

Lemma 2.2 : Let $G \in \mathcal{G}(n,m)$ and $\rho(G) = \alpha$. Then

$$m \leq \binom{n}{2} - \ell'(\ell-1) - \alpha\binom{\ell-1}{2} \qquad (2.2)$$

with equality if and only if $G \cong Q_{n,\alpha}$.

Proof : Let $G^* \in \mathcal{G}(n)$ and $\rho(G^*) = \alpha$ such that $\varepsilon(G^*)$ is maximum. Consider an α-colouring of G^* and let $V_1, V_2, \ldots, V_\alpha$ be the induced partition of $V(G^*)$ such that $G^*[V_i]$ is acyclic for $1 \leq i \leq \alpha$. The maximality of $\varepsilon(G^*)$ implies that every vertex of V_i is adjacent to every vertex of V_j for $i \neq j$ and $G^*[V_i]$ is a tree for all i. Let $|V_i| = n_i$ for $i=1,2,\ldots,\alpha$.

Claim : n_i and n_j differ by at most 1, $\forall$ i,j.

Suppose not. Let $n_i \geq n_j + 2$ for some i and j. Let $x \in V_i$ and $y \in V_j$ such that they have degree one in $G^*[V_i]$ and $G^*[V_j]$ respectively. Such vertices always exist since $G^*[V_i]$ and $G^*[V_j]$ are trees. Let z be the neighbour of x in $G^*[V_i]$. Now we shall construct a graph G' from G^* as follows : Remove the edges of the form (x,u) where

$u \in V_j$ and $u \neq y$ and introduce the edges of the form (x,v) where $v \in V_i$ and $v \neq z$. Let G' be the resulting graph.

Consider the partition $U_1, U_2, \ldots, U_\alpha$ of the vertices of G' where $U_k = V_k$, for $k \neq i$ and j; $U_i = V_i - \{x\}$ and $U_j = V_j \cup \{x\}$. Clearly $G'[U_k]$ is acyclic for $1 \leq k \leq \alpha$ and so $\rho(G') = \alpha$. Note that $\varepsilon(G') = \varepsilon(G^*) + n_i - n_j - 1 \geq \varepsilon(G^*) + 1$, a contradiction to the maximality of $\varepsilon(G^*)$. Thus the claim is proved.

Now it is easy to see that $n_i = \ell$ or $\ell+1$, for $1 \leq i \leq \alpha$, where $\ell = \left\lfloor \frac{n}{\alpha} \right\rfloor$. Thus G^* is isomorphic to $Q_{n,\alpha}$ and simple counting establishes that

$$\varepsilon(G^*) = \binom{n-\ell}{2} + (\alpha - 1)\binom{\ell+1}{2} + (n - \alpha) = \binom{n}{2} - \ell'(\ell - 1) - \alpha\binom{\ell - 1}{2}.$$

This completes the proof of the lemma. □

Combining Lemmas 2.1 and 2.2 we have the following theorem :

Theorem 2.1 : Let $G \in \mathcal{G}(n,m)$ and $\rho(G) = \alpha$. Then

$$\binom{2\alpha - 1}{2} \leq m \leq \binom{n}{2} - \ell'(\ell - 1) - \alpha\binom{\ell - 1}{2} \qquad (2.3)$$

Furthermore, the lower bound is attained if and only if $G \cong K_{2\alpha-1} \cup \overline{K}_{n-2\alpha+1}$ and the upper bound is attained if and only if $G \cong Q_{n,\alpha}$. In addition, for every integer m satisfying (2.3), there exists a graph $G \in \mathcal{G}(n,m)$ such that $\rho(G) = \alpha$. □

3. BOUNDS FOR THE SUM $\rho(G) + \rho(\overline{G})$

In this section we will determine sharp bounds for $\rho(G) + \rho(\overline{G})$ in terms of the order n and the size m of G. From Theorem 1.1 we conclude that the sharpness of the lower bound depends on the existence of an integer β satisfying

$$\beta\left(\left\lceil \sqrt{n} \right\rceil - \beta\right) \geq \left\lceil \frac{n}{4} \right\rceil \qquad (3.1)$$

since $\rho(G).\rho(\overline{G}) \geq \left\lceil \frac{n}{4} \right\rceil$ by (1.2). When n is an odd perfect square note that

$$\left\lceil \frac{1}{2}\left\lceil \sqrt{n} \right\rceil \right\rceil \left\lfloor \frac{1}{2}\left\lceil \sqrt{n} \right\rceil \right\rfloor < \left\lceil \frac{n}{4} \right\rceil.$$

Consequently in this case there does not exist an integer β satisfying (3.1) and hence there is no graph $G \in \mathcal{G}(n)$ such that $\rho(G) + \rho(\overline{G}) = \left\lceil \sqrt{n} \right\rceil$. Thus when n is an odd perfect square

$$\rho(G) + \rho(\overline{G}) \geq \left\lceil \sqrt{n} \right\rceil + 1. \qquad (3.2)$$

Combining (3.2) and (1.1) we have the following inequality for $G \in \mathcal{G}(n)$.

$$\rho(G) + \rho(\overline{G}) \geq C(n) \qquad (3.3)$$

where

$$C(n) = \begin{cases} \sqrt{n} + 1, & \text{if n is an odd perfect square,} \\ \left\lceil \sqrt{n} \right\rceil, & \text{otherwise.} \end{cases}$$

Let β be an integer such that

$$\beta(C(n) - \beta) \geq \left\lceil \frac{n}{4} \right\rceil. \qquad (3.4)$$

Define integers x_1 and x_2 such that $n = \beta x_1 + x_2$, $0 \leq x_2 \leq \beta - 1$.

In the following we describe a subclass $\mathcal{G}'$ of $\mathcal{G}(n)$ to establish the sharpness of (3.3) :

$$\mathcal{G}' = \{G_\beta : \beta \text{ satisfies } (3.4)\},$$

where G_β is defined as follows :

(i) $V(G_\beta) = \bigcup_{i=1}^{\beta} V_i$ where $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,t}\}$, $1 \leq i \leq \beta$, with

$$t = \begin{cases} x_1 + 1, & \text{if } 1 \leq i \leq x_2, \\ x_1, & \text{otherwise.} \end{cases}$$

(ii) $G_\beta[V_i]$ is isomorphic to the complement of a path, $1 \le i \le \beta$. Moreover for all i, assume that $v_{i,j}$ and $v_{i,j+1}$ are non-adjacent in $G_\beta[V_i]$ for $1 \le j \le t-1$.

(iii) G_β has no other edges.

It is easy to show that

$$\rho(\overline{G}_\beta) = \beta, \tag{3.5}$$

and

$$\rho(G_\beta) = \begin{cases} \left\lceil \frac{x_1+1}{4} \right\rceil, & \text{if} \quad x_2 \ge 1, \\ \left\lceil \frac{x_1}{4} \right\rceil, & \text{if} \quad x_2 = 0. \end{cases} \tag{3.6}$$

From (3.3) it follows that $\rho(G_\beta) \ge C(n) - \rho(\overline{G}_\beta) = C(n) - \beta$. Now we establish that $\rho(G_\beta) = C(n) - \beta$. From (3.4) we have $4\beta(C(n) - \beta) \ge n = \beta x_1 + x_2$, that is, $4(C(n) - \beta) \ge x_1 + \frac{x_2}{\beta}$. Since $4(C(n)-\beta)$ and x_1 are integers we have, $4(C(n)-\beta) \ge x_1 + 1$ or x_1 according as $x_2 \ge 1$ or $x_2 = 0$. Thus

$$C(n) - \beta \ge \begin{cases} \left\lceil \frac{x_1+1}{4} \right\rceil, & \text{if} \quad x_2 \ge 1 \\ \left\lceil \frac{x_1}{4} \right\rceil, & \text{if} \quad x_2 = 0. \end{cases}$$

Hence we have $\rho(G_\beta) = C(n) - \beta = \alpha$ (say) and $\rho(G_\beta) + \rho(\overline{G}_\beta) = C(n)$. Counting the number of edges in G_β we have

$$\varepsilon(G_\beta) = x_2\binom{x_1}{2} + (\beta - x_2)\binom{x_1-1}{2} = x_2(x_1 - 1) + \beta\binom{x_1-1}{2}. \tag{3.7}$$

In the following lemma, we prove that $\varepsilon(G_\beta)$ is a decreasing function of β.

Lemma 3.1 : Let β be an integer satisfying (3.4) and $G_\beta \in \mathcal{G}'$. Then $\varepsilon(G_\beta)$ is a decreasing function of β.

Proof : Let $\beta' < \beta$ be a positive integer such that $\beta'(C(n) - \beta') \geq \left\lceil \frac{n}{4} \right\rceil$.

We shall prove that

$$\varepsilon(G_\beta) < \varepsilon(G_{\beta'}). \tag{3.8}$$

Let y_1, y_2 be integers such that $n = y_1 \beta' + y_2$, $0 \leq y_2 \leq \beta' - 1$. Observe that :

(i) $\beta < C(n)$, for otherwise we have a contradiction to (3.4).

(ii) $\beta \leq \sqrt{n}$; this follows from (i), and the definition of C(n).

(iii) $x_1 \geq \beta$, for otherwise $n = x_1\beta + x_2 \leq \beta^2 - 1 \leq n - 1$.

(iv) $y_1 > x_1$, for otherwise we arrive at a contradiction to the fact that

$$y_2 \leq \beta' - 1 \leq \beta - 2.$$

Now note that

$$\varepsilon(G_\beta) = \frac{x_1 - 1}{2}(n + x_2 - 2\beta) \leq \frac{x_1 - 1}{2}(n - \beta - 1),$$

since $x_2 \leq \beta - 1$. Also

$$\varepsilon(G_{\beta'}) = \frac{y_1 - 1}{2}(n + y_2 - 2\beta') \geq \frac{y_1 - 1}{2}(n - 2\beta + 2),$$

since $\beta' \leq \beta - 1$. Now the inequality (3.8) is true if

$$\frac{x_1 - 1}{2}(n - \beta - 1) < \frac{y_1 - 1}{2}(n - 2\beta + 2). \tag{3.9}$$

Writing $y_1 = x_1 + \delta$, where δ is a positive integer, the inequality (3.9) is true if $\beta + 3x_1 - x_1\beta - 3 + \delta(n - 2\beta + 2)$ is positive. Note that this latter expression is $\geq -n + 4\beta - 3 + n - 2\beta + 2 = 2\beta - 1 > 0$. This completes the proof of the lemma. □

Given a positive integer n, we now define a function A(n) as follows :

$$A(n) = \min\{\varepsilon(G_\beta) : G_\beta \in \mathcal{G}'\}.$$

As a consequence of Lemma 3.1 we have $A(n) = \varepsilon(G_{\hat{\beta}})$, where $\hat{\beta}$ is the largest integer satisfying (3.4). In the following lemma we determine the range for the size m of $G \in \mathcal{G}(n)$ such that $\rho(G) + \rho(\overline{G}) = C(n)$.

Lemma 3.2 : For $n \geq 13$, there is a $G \in \mathcal{G}(n,m)$ with $\rho(G) + \rho(\overline{G}) = C(n)$ if and only if $m \geq A(n)$.

Proof : From Lemma 3.1 it is clear that if there is a graph $G \in \mathcal{G}(n,m)$ such that $\rho(G)+\rho(\overline{G}) = C(n)$ then $m \geq A(n)$. To complete the proof we will assume that $m \geq A(n)$ and establish the sharpness. We will construct a graph $G^* \in \mathcal{G}(n,m)$ such that $\rho(G)+\rho(\overline{G}) = C(n)$ for $n \geq 13$.

Let $\hat{\beta}$ be the largest integer satisfying (3.4) and consider the graph $G_{\hat{\beta}} \in \mathcal{G}'$. For notational convenience we shall refer to $G_{\hat{\beta}}$ as $\hat{G}$. Note that $A(n) = \varepsilon(\hat{G})$, $\rho(\overline{\hat{G}}) = \hat{\beta}$ and $\rho(\hat{G}) = C(n) - \hat{\beta} = \alpha$ (say). Firstly let $\alpha \geq 3$. Consider a partition $U_1, U_2, \ldots, U_\alpha$ of $V(\hat{G})$ defined by

$$U_k = \{v_{i,j} : 1 \leq i \leq \hat{\beta} \text{ and } 4k-3 \leq j \leq 4k\}, \quad 1 \leq k \leq \alpha - 1$$

and

$$U_\alpha = V(\hat{G}) - \bigcup_{k=1}^{\alpha-1} U_k .$$

Note that $\hat{G}[U_k]$ is acyclic for all k. Thus the partition $U_1, U_2, \ldots, U_\alpha$ gives rise to an α-colouring of $\hat{G}$. Now add edges to $\hat{G}$ such that no added edge has both its end vertices in U_k, $1 \leq k \leq \alpha$. Let G^* be the graph obtained after the addition of all possible edges. It is easy to see that $\rho(G^*) = \alpha$ and $\rho(\overline{G}^*) = \hat{\beta}$ and hence $\rho(G^*) + \rho(\overline{G}^*) = C(n)$.

It is not too difficult to show that $\varepsilon(G^*) \geq \varepsilon(\overline{G}^*)$. This can best be seen by considering the vertices in the set $V_i \cap U_j$. Observe that for $1 \leq j \leq \alpha - 1$, $G^*[V_i \cap U_j] \cong \overline{G}^*[V_i \cap U_j] \cong P_4$.

Further, $\overline{G}^*[V_i \cap U_\alpha] \cong P_t$, $t \leq 4$. Thus $\varepsilon(\overline{G}^*[U_\alpha]) \leq \varepsilon(G^*[U_\alpha]) + \hat{\beta}$. Let $u \in V_i \cap U_j = W_{ij}$. Observe that u is joined, in G^*, to every vertex of U_k, $k \neq j$, except possibly one. Since $\alpha \geq 3$, we have

$$\left|N_{G^*}(u) \cap (V(G^*) \setminus W_{ij})\right| \geq 4\hat{\beta} - 1.$$

Further, in $\overline{G}^*$, u is joined to all the vertices of $U_j \setminus W_{ij}$ and at most one vertex of $V(G^*) \setminus U_j$. Consequently

$$\left|N_{\overline{G}^*}(u) \cap (V(G^*) \setminus W_{ij})\right| \leq 4\hat{\beta} - 3.$$

Thus, for $\alpha \geq 3$

$$\varepsilon(G^*) - \varepsilon(\overline{G}^*) \geq 4(\alpha - 1)\hat{\beta} - \hat{\beta} > 0.$$

Next let $\alpha = 2$. We will now modify $\hat{G}$ as follows :

For each i, $1 \leq i \leq \hat{\beta}$, we partition V_i into two sets V_{i1} and V_{i2} such that

- $\overline{\hat{G}}[V_{i1}]$ and $\overline{\hat{G}}[V_{i2}]$ are paths.
- $|V_{i1}|$ and $|V_{i2}|$ differ by at most one.
- $\left|\bigcup_{i=1}^{\hat{\beta}} V_{i1}\right|$ and $\left|\bigcup_{i=1}^{\hat{\beta}} V_{i2}\right|$ differ by at most one.

Now let $U_1 = \bigcup_{i=1}^{\hat{\beta}} V_{i1}$ and $U_2 = \bigcup_{i=1}^{\hat{\beta}} V_{i2}$. Since $\alpha = 2$ we find that $x_1 \leq 8$ and hence $\hat{G}[U_1]$ and $\hat{G}[U_2]$ are acyclic. Now add edges to $\hat{G}$ such that no added edge has both its end vertices in U_i, for i = 1,2. Let G^* be the graph obtained after the addition of all possible edges. Since $|U_1|$ and $|U_2|$ do not differ by more than one, it follows that $\varepsilon(G^*) \geq \varepsilon(\overline{G}^*)$. It is easy to check that $\rho(G^*) = \alpha$ and $\rho(\overline{G}^*) = \hat{\beta}$.

Next let $\alpha = 1$. In this case $\hat{\beta} = C(n)-1$. From (3.4) and the definition of C(n) it is easy to check that $n \leq 16$. Now if $13 \leq n \leq 16$, then $\left\lceil \frac{n}{4} \right\rceil = 4$ and $C(n) = 4$ and hence $\hat{\beta} = 2 = \alpha$. This completes the proof of the lemma. □

Remark 3.1 : If n = 9 then it is easy to show that the inequality (3.3) is sharp whenever $m \geq A(9) = 3$. For $n \leq 12$ and $n \neq 9$, using Lemma 2.2 it can be shown that the lower bound in (3.3) is not sharp for some values of m. These exceptional cases are listed in the following table.

Order	Range for the Size
12	$12 \leq m \leq 19$
11	$11 \leq m \leq 15$
10	$10 \leq m \leq 11$
8	$8 \leq m \leq 14$
7	$7 \leq m \leq 10$
6	$6 \leq m \leq 7$
5	$m = 5$

Table 3.1

In all other cases, the technique used in the case $\alpha = 2$, in the proof of Lemma 3.2 provides an extremal graph.

In the following Figure 3.1 we present a subclass, denoted by $\mathcal{H}_\theta$, of graphs in $\mathcal{G}(n,m)$. Here θ is an integer such that $m \geq \binom{\theta}{2}$.

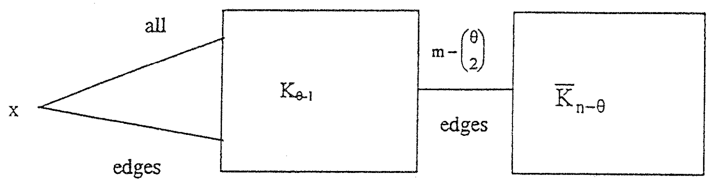

Figure 3.1 : $\mathcal{H}_\theta$, m $\geq \binom{\theta}{2}$.

This class is well defined only when $m - \binom{\theta}{2} \leq (\theta - 1)(n - \theta)$.

Let A and B denote the sets of vertices of $H_\theta \in \mathcal{H}_\theta$ which are adjacent and not adjacent respectively, to x in H_θ. Since H_θ and $\overline{H}_\theta$ contain K_θ and $K_{n-\theta+1}$ respectively, as induced subgraphs, we have

$$\rho(H_\theta) \geq \left\lfloor \frac{\theta+1}{2} \right\rfloor, \tag{3.10}$$

and

$$\rho(\overline{H}_\theta) \geq \left\lfloor \frac{n+2-\theta}{2} \right\rfloor. \tag{3.11}$$

To establish equality in (3.10) we shall colour the vertices of H_θ with $\left\lfloor \frac{\theta+1}{2} \right\rfloor$ colours. Consider an arbitrary colouring of the vertices of $A \cup \{x\}$ with $\left\lfloor \frac{\theta+1}{2} \right\rfloor$ colours such that no cycle is monocoloured. Assign the colour received by x to the vertices in B. Observe that this results in a colouring of the vertices of H_θ with $\left\lfloor \frac{\theta+1}{2} \right\rfloor$ colours such that there is no monocoloured cycle. Thus we have

$$\rho\,(H_\theta) = \left\lfloor \frac{\theta+1}{2} \right\rfloor. \tag{3.12}$$

Similarly it can be shown that

$$\rho(\overline{H}_\theta) = \left\lfloor \frac{n+2-\theta}{2} \right\rfloor. \tag{3.13}$$

Lemma 3.3 : There is a $G \in \mathcal{G}(n,m)$ with $\rho(G)+\rho(\overline{G}) = \left\lfloor \frac{n+3}{2} \right\rfloor$ except when n is odd and m = 1 or 2. In the exceptional case $\rho(G) = 1$ and $\rho(\overline{G}) = \frac{n-1}{2}$.

Proof : Let us assume that either n is odd and $m \neq 1,2$ or n is even. Let ω be an integer such that $m = \binom{\omega}{2} + t$, $0 \leq t \leq \omega - 1$. Take $G = H_\omega \in \mathcal{H}_\omega$ if n is even or both n and ω are odd; or $G \cong H_{\omega-1} \in \mathcal{H}_{\omega-1}$ if n is odd and ω is even.

This completes the proof. □

From Theorems 1.1 and lemmas 3.2 and 3.3 we have :

Theorem 3.1 : Let $G \in \mathcal{G}(n,m)$. Then

$$C(n) \le \rho(G) + \rho(\overline{G}) \le D(n,m) \tag{3.14}$$

where

$$D(n,m) = \begin{cases} \left\lceil \dfrac{n+1}{2} \right\rceil, & \text{if n is odd and m} = 1 \text{ or } 2, \\ \left\lceil \dfrac{n+3}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

The upper bound in (3.14) is always sharp. The lower bound is sharp iff $m \ge A(n)$ except for the cases listed in Table 3.1. □

4. BOUNDS FOR THE PRODUCT $\rho(G) . \rho(\overline{G})$

In the following we describe a class $\mathcal{G}^*_{\alpha,\beta}$ of graphs that will be used in the later discussions. This class was motivated by the construction of Finck [4].

Consider a graph H of order $\alpha\beta$ with the following properties :

- Assume that the vertices of H are arranged into an array of α rows and β columns.
- The subgraph of H induced on vertices belonging to the same column is acyclic.
- The subgraph of H induced on vertices belonging to the same row is the complement of an acyclic graph.

Now form a new graph $G^*_{\alpha,\beta}$ of order $4\alpha\beta$ from H as follows :

- Each vertex u of H is replaced by four vertices u_1, u_2, u_3 and u_4 such that $G^*_{\alpha,\beta}[\{u_1,u_2,u_3,u_4\}]$ is isomorphic to P_4, the path on 4 vertices.
- If u and v are adjacent vertices of H belonging to the same column, then introduce in $G^*_{\alpha,\beta}$, exactly one edge between the sets $\{u_1,u_2,u_3,u_4\}$ and $\{v_1,v_2,v_3,v_4\}$.
- If u and v are non-adjacent vertices of H belonging to the same column of H then no u_i is adjacent to any v_j in $G^*_{\alpha,\beta}$.

- If u and v are adjacent vertices of H in the same row then join each u_i to each v_j in $G^*_{\alpha,\beta}$.
- If u and v are non-adjacent vertices of H belonging to the same row, then except for a specified pair $\{i',j'\} \subseteq \{1,2,3,4\}$ u_i and v_j are adjacent in $G^*_{\alpha,\beta}$.
- Let u and v be vertices of H belonging to neither the same row nor the same column. Then any vertex of $\{u_1,u_2,u_3,u_4\}$ may be joined to any vertex of $\{v_1,v_2,v_3,v_4\}$ in $G^*_{\alpha,\beta}$.

Now we define $\mathcal{G}^*_{\alpha,\beta}$ to be the class of all graphs $G^*_{\alpha,\beta}$ described above. Since each column has at least 3α edges and each row of H is missing at most β-1 edges, we have the following remark.

Remark 4.1 : Let $G \in \mathcal{G}^*_{\alpha,\beta}$. Then $\rho(G) = \beta$, $\rho(\overline{G}) = \alpha$ and

$$\binom{4\beta-1}{2}\alpha \leq \varepsilon(G) \leq \binom{4\alpha\beta}{2} - \binom{4\alpha-1}{2}\beta.$$

Observe that one can start with a graph $G \in \mathcal{G}^*_{\alpha,\beta}$ with $\varepsilon(G) = \binom{4\beta-1}{2}\alpha$ and transfer edges from $\overline{G}$ to G in such a way that $\rho(G)$ and $\rho(\overline{G})$ remain β and α, respectively. Thus we have the following remark.

Remark 4.2 : If α,β are integers such that

$$\binom{4\beta-1}{2}\alpha \leq m \leq \binom{4\alpha\beta}{2} - \binom{4\alpha-1}{2}\beta$$

then there is a graph $G \in \mathcal{G}^*_{\alpha,\beta}$ of size m.

Consider a graph $G \in \mathcal{G}^*_{\alpha,\beta}$. We obtain a new graph $G^*_{i,\alpha,\beta}$ for $1 \leq i \leq 3$ by deleting 4-i vertices from $G^*_{\alpha,\beta}$. We denote by $\mathcal{G}^*_{i,\alpha,\beta}$ the class of all graphs $G^*_{i,\alpha,\beta}$.

The following remarks are analgous to Remarks 4.1 and 4.2.

Remark 4.3 : Let $G \in \mathcal{G}^*_{i,\alpha,\beta}$ of order at least 5, for some i, $1 \le i \le 3$. Then $\rho(G) = \beta$ and $\rho(\overline{G}) = \alpha$. Moreover, if α and β are at least 4-i, then

$$\binom{4\beta-1}{2}\alpha - (4-i)(4\beta-2) \le \varepsilon(G)$$

$$\le \binom{4\alpha\beta-4+i}{2} - \binom{4\alpha-1}{2}\beta + (4-i)(4\alpha-2).$$

Further, every integer in the above range is realizable.

The cases not covered by the above remark can easily be resolved to provide the following remark.

Remark 4.4 : Let $G \in \mathcal{G}^*_{i,\alpha,\beta}$ of order at least 5, for some i = 1 or 2. Then

$$\rho(G) = \beta \quad \text{and} \quad \rho(\overline{G}) = \alpha.$$

Moreover

(i) $\binom{4\beta-5+i}{2} \le \varepsilon(G) \le \binom{4\beta-4+i}{2} - 3\beta + (4-i)2$, if $\alpha = 1$ and $\beta \ge 4$-i.

(ii) $\max\{3\alpha - (4\text{-}i)2, 1\} \le \varepsilon(G) \le 4\alpha - 5 + i$, if $\beta = 1$ and $\alpha \ge 2$.

(iii) $6 \le \varepsilon(G) \le 9$, when $i = 1$, $\alpha = 1$ and $\beta = 2$.

Further, every integer in the above range is realizable.

Lemma 4.1 : For $G \in \mathcal{G}(n,m)$, $\rho(G) . \rho(\overline{G}) = \left\lceil \frac{n}{4} \right\rceil$ if and only if

(i) $n \equiv 0 \pmod 4$, $G \in \mathcal{G}^*_{\alpha,\beta}$ for some integers α and β such that $n = 4\alpha\beta$

or

(ii) $n = s \pmod 4$, where $1 \le s \le 3$ and $G \in \mathcal{G}^*_{s,\alpha,\beta}$ for integers α and β such that $n + 4 - s = 4\alpha\beta$.

Proof : We give only the proof of (i) as the proof of (ii) is virtually the same. The proof of the "if" part follows from Remark 4.1. To prove the "only if" part let us assume that $n = 0 \pmod 4$ and $G \in \mathcal{G}(n,m)$ with

$$\rho(G).\rho(\overline{G}) = \frac{n}{4}.$$

Let $\rho(G) = p$ and $\rho(\overline{G}) = q$. Consider a p-colouring of the vertices of G. Let $V_1, V_2, \ldots, V_p$ be the induced partition of V(G). Clearly $G[V_i]$ is acyclic for $i = 1,2,\ldots,p$.

Let $|V_1| = \max_i |V_i|$. Then $|V_1| \geq \frac{n}{p}$. Now

$$\frac{n}{4p} = q = \rho(\overline{G}) \geq \rho(\overline{G}[V_1]) \geq \frac{|V_1|}{4} \geq \frac{n}{4p}.$$

Thus $|V_1| = \frac{n}{p}$ and $\rho(\overline{G}[V_1]) = \frac{n}{4p}$. Using the fact that $n = \sum_{i=1}^{p} |V_i|$, it follows that

$$|V_i| = \frac{n}{p} \qquad \text{for } i = 1,2,\ldots,p \tag{4.2}$$

and

$$\rho(\overline{G}[V_i]) = \frac{n}{4p} \qquad \text{for } i = 1,2,\ldots,p. \tag{4.3}$$

Now consider a q-colouring of the vertices of $\overline{G}$. Let $U_1, U_2, \ldots, U_q$ be the induced partition of $V(\overline{G})$ such that $\overline{G}[U_i]$ is acyclic. Using arguments similar to the above one can verify that, for all i, $|U_i| = \frac{n}{q}$ and $\rho(G[U_i]) = \frac{n}{4q}$. Let i and j be integers such that $1 \leq i \leq p$ and $1 \leq j \leq q$. Since $\overline{G}[U_j]$ and $G[V_i]$ are acyclic it follows that $G[V_i \cap U_j]$ and $\overline{G}[V_i \cap U_j]$ are both acyclic. This implies that $|V_i \cap U_j| \leq 4$. Now, combining this with the fact that $\sum_{j=1}^{q} |V_i \cap U_j| = 4q$ we have $|V_i \cap U_j| = 4$ for $1 \leq i \leq p$ and $1 \leq j \leq q$. Now since $G[V_i \cap U_j]$ and $\overline{G}[V_i \cap U_j]$ are both acyclic it follows that they are isomorphic to P_4, the path on four vertices. Thus it is easy to see that $G \in \mathcal{G}^*_{p,q}$. This completes the proof of (i). □

Lemma 4.2 : Let $G \in \mathcal{G}(n,m)$, $n \geq 4$ and $n' = \left\lceil \frac{n}{2} \right\rceil$. Then

$$\rho(G).\rho(\overline{G}) \leq B(n,m) \tag{4.4}$$

where

$$B(n,m) = \begin{cases} \left\lfloor \frac{1}{2}\left\lfloor \frac{n+3}{2} \right\rfloor \right\rfloor \left\lceil \frac{1}{2}\left\lfloor \frac{n+3}{2} \right\rfloor \right\rceil, & \text{if } m \geq \binom{n'}{2}, \\ \left\lfloor \frac{\omega+1}{2} \right\rfloor \left(\left\lfloor \frac{n+3}{2} \right\rfloor - \left\lfloor \frac{\omega+1}{2} \right\rfloor \right), & \text{otherwise,} \end{cases}$$

and ω is an integer such that $m = \binom{\omega}{2} + t$, $0 \leq t \leq \omega - 1$. Further, this bound is sharp.

Proof : For the case of $m \geq \binom{n'}{2}$ it is routine to verify that $\rho(G) . \rho(\overline{G}) = B(n,m)$ for $G \cong H_{n}'$ if n is even or both n and n′ are odd; and for $G \cong H_{n'-1}$ if n is odd and n′ even.

Let us next assume that $m < \binom{n'}{2}$. From Lemma 2.1 it follows that $\rho(G) \leq \left\lfloor \frac{\omega+1}{2} \right\rfloor$.

Let $\rho(G) = \left\lfloor \frac{\omega+1}{2} \right\rfloor - \delta$ for $\delta \geq 0$. Now from Lemma 3.3 we have

$$\rho(\overline{G}) \leq \left(\left\lfloor \frac{n+3}{2} \right\rfloor - \rho(G) \right) = \left(\left\lfloor \frac{n+3}{2} \right\rfloor - \left\lfloor \frac{\omega+1}{2} \right\rfloor + \delta \right).$$

Therefore

$$\rho(G).\rho(\overline{G}) \leq \left(\left\lfloor \frac{\omega+1}{2} \right\rfloor - \delta \right)\left(\left\lfloor \frac{n+3}{2} \right\rfloor - \left\lfloor \frac{\omega+1}{2} \right\rfloor + \delta \right)$$

$$= \left\lfloor \frac{\omega+1}{2} \right\rfloor \left(\left\lfloor \frac{n+3}{2} \right\rfloor - \left\lfloor \frac{\omega+1}{2} \right\rfloor \right) + \delta\left(2\left\lfloor \frac{\omega+1}{2} \right\rfloor - \left\lfloor \frac{n+3}{2} \right\rfloor - \delta \right).$$

Now it is easy to verify that $2\left\lfloor \frac{\omega+1}{2} \right\rfloor - \left\lfloor \frac{n+3}{2} \right\rfloor \le 0$. For, otherwise, we arrive at a contradiction to the assumption that $m < \binom{n'}{2}$.

This in turn implies that

$$\rho(G).\rho(\overline{G}) \le \left\lfloor \frac{\omega+1}{2} \right\rfloor \left(\left\lfloor \frac{n+3}{2} \right\rfloor - \left\lfloor \frac{\omega+1}{2} \right\rfloor \right).$$

This proves the inequality (4.4) when $m < \binom{n'}{2}$. To establish the sharpnesss consider the graph $G \cong H_\theta$, where

$$\theta = \begin{cases} \omega - 1, & \text{if n is odd and } \omega \text{ is even,} \\ \omega, & \text{otherwise.} \end{cases}$$

Using simple algebraic manipulations one can easily verify that

$$\rho(G).\rho(\overline{G}) = \left\lfloor \frac{\omega+1}{2} \right\rfloor \left(\left\lfloor \frac{n+3}{2} \right\rfloor - \left\lfloor \frac{\omega+1}{2} \right\rfloor \right).$$

This completes the proof. □

The following definition of β is used in Theorem 4.1. Let $n \equiv i \pmod 4$, $i=1,2,3,4$. Define β as the largest integer such that 4β divides $n + 4 - i$ and

$$m \ge \binom{4\beta - 1}{2} \left(\frac{n+4-i}{4\beta} \right) - (4-i)(4\beta - 2).$$

Note that for some n and m such a β may not exist.

Theorem 4.1 : Let $n \equiv i(\text{mod } 4)$ with $i = 1,2,3,4$ and $G \in \mathcal{G}(n,m)$. Then

$$\left\lceil \frac{n}{4} \right\rceil \leq \rho(G).\rho(\overline{G}) \leq B(n,m)$$

where B(n,m) is defined as in Lemma 4.2. The upper bound is always sharp. The lower bound is sharp iff $\beta = 1$ and $\max\{3\left\lceil \frac{n}{4} \right\rceil - 2(4-i), 1\} \leq m \leq n-1$ or $\beta \geq 2$, where β is defined as above.

Proof : The upper bound and its sharpness follow from Lemma 4.2. The lower bound follows from Theorem 1.1. Now let $G \in \mathcal{G}(n,m)$ be such that $\rho(G).\rho(\overline{G}) = \frac{n}{4}$.

Case (i) i = 4. By Lemma 4.1 it follows that $G \in \mathcal{G}^{*}_{\theta,\phi}$ for some θ and ϕ such that $n = 4\theta\phi$. By Remark 4.1

$$m \geq \binom{4\phi - 1}{2}\theta = \binom{4\phi - 1}{2}\frac{n}{4\phi}.$$

From the definition of β, $\phi \leq \beta$. If $\beta \geq 2$, there is nothing to prove. Now if $\beta = 1$, then $\phi = \beta = 1$. Thus from Remark 4.1, $\frac{3n}{4} \leq m \leq n-1$. Conversely, if $\beta = 1$ and $\frac{3n}{4} \leq m \leq n-1$, then by Remark 4.2, there exists a graph $G \in \mathcal{G}(n,m)$ such that $\rho(G).\rho(\overline{G}) = \frac{n}{4}$. If $\beta \geq 2$, using the fact that

$$\binom{4\alpha\beta}{2} - \binom{4\alpha - 1}{2}\beta \geq \frac{1}{2}\binom{4\alpha\beta}{2}, \qquad m \leq \frac{1}{2}\binom{n}{2}$$

and Remark 4.2, we have a G as required.

Case (ii) i ≠ 4. Then by Lemma 4.1 it follows that $G \in \mathcal{G}^{*}_{i,\theta,\phi}$ for θ and ϕ such that $n+4-i=4\theta\phi$. By Remarks 4.3, 4.4 and the definition of β it follows that $\phi \leq \beta$. If $\beta \geq 2$, there is nothing to prove. If $\beta = 1$, then $\phi = 1$. Then from (ii) of Remark 4.4

$\max\{3\left\lceil \frac{n}{4} \right\rceil - 2(4-i), 1\} \leq m \leq n-1$. The if part can be established using the Remarks 4.3, 4.4 and the fact that $m \leq \frac{1}{2}\binom{n}{2}$. This completes the proof of the theorem. □

REFERENCES

1. N. Achuthan, N.R. Achuthan and L. Caccetta, On the Nordhaus-Gaddum Class Problems, **Australasian Journal of Combinatorics** 2 (1990), pp. 5-27.
2. J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, **The MacMillan Press,** London (1977).
3. G.A. Dirac, Graph Union and Chromatic Number, **J. London Math Soc.,** 39 (1964), pp. 451-454.
4. H.J. Finck, On the Chromatic numbers of a Graph and its Complement, in **Theory of Graphs** (P. Erdös and G. Katona Eds.), Academic Press, New York (1968), pp. 99-113.
5. J. Mitchem, On the Point-Arboricity of a Graph and its Complement, **Canadian Journal of Mathematics** 23(1971), pp. 287-292.
6. J. Plesnik, Bounds on Chromatic Numbers of Multiple Factors in a Complete Graph, **Journal of Graph of Theory** 2 (1978), pp. 9-17.

(Received 2/6/97; revised 8/12/98)