Independence and Cycles in Super Line Graphs

Jay S. Bagga

Department of Computer Science
Ball State University, Muncie, IN 47306 USA

Lowell W. Beineke

Department of Mathematical Sciences
Indiana University Purdue University, Fort Wayne, IN 46805-1499 USA

Badri N. Varma

Department of Mathematics
University of Wisconsin-Fox Valley, Menasha, WI 54952 USA

Abstract

The super line graph of index r of a graph G has the r-sets of edges as its vertices, with two being adjacent if one has an edge that is adjacent to an edge in the other. In this paper, we continue our investigation of this graph by establishing two results, the first on the independence number of super line graphs of arbitrary index and the second on pancyclicity in the index-2 case.

1 Introduction

The *super line graph of index* r, denoted by $\mathcal{L}_r(G)$ is defined for any graph G with at least r edges. Its vertices are the sets of r edges of G, and two such sets are adjacent if an edge of one is adjacent to an edge of the other.

If $r = 1$, then this is the ordinary line graph, so the super line graph is another among the line graph generalizations that have been studied. (For a discussion of many of these, see [5]). Index-r super line graphs were introduced by the authors in [1], and we have since investigated various of their properties [2, 3]. The index-2 case has been studied in greater detail [4].

In this paper, we continue our study with two further results. The first is on the independence number of super line graphs of arbitrary index, and the second on cycles in the index-2 case.

One convention that we adopt in this paper is to ignore isolated vertices in the base graph G since they have no effect on super line graphs.

2 The Independence Number

Let M be a set of independent edges in a graph G. If A and B are r-sets of M, then clearly they contain no adjacent edges, and so in $\mathcal{L}_r(G)$, A and B cannot be adjacent. However, not all pairs of nonadjacent vertices arise in this way. It is also the case that two r-sets of edges of G are nonadjacent in $\mathcal{L}_r(G)$ if they generate vertex-disjoint subgraphs. What we show is that, when one considers a set of independent vertices in $\mathcal{L}_r(G)$ of *maximum* order, then with a few exceptional families of graphs, it is produced by a maximum independent set of edges of G.

We denote the vertex-independence number and the edge-independence number of G by $\alpha(G)$ and $\alpha'(G)$ respectively, and the set of all r-sets of a set X by $\binom{X}{r}$.

Theorem 2.1 *Let G be a graph with at least r edges. Then the independence number of $\mathcal{L}_r(G)$ is*

$$\alpha(\mathcal{L}_r(G)) = \binom{\alpha'(G)}{r}.$$

Furthermore, if S is a maximum independent set of vertices in $\mathcal{L}_r(G)$, then either

(i) $S = \binom{X}{r}$ for some maximum independent set X of edges of G, or

(ii) S consists of $r+1$ disjoint stars $K_{1,r}$, or

(iii) $r = 3$ and the vertices in S are $K_{1,3}$'s or K_3's.

Proof. If X is a maximum independent set of edges of G, then clearly, r-sets of X are independent vertices in $\mathcal{L}_r(G)$. Thus,

$$\alpha(\mathcal{L}_r(G)) \geq \binom{\alpha'(G)}{r}.$$

To prove the reverse inequality, let $V_1, V_2, ..., V_k$ be r-sets of $E(G)$ which are independent vertices in $\mathcal{L}_r(G)$. Also, let

m = number of these sets which are matchings in G,
ℓ = number of these sets which are not matchings in G,
h = number of edges of G in the union of the m matchings.

Clearly, $m \leq \binom{h}{r}$. Let $U = V_1 \cup V_2 \cup ... \cup V_k$. We observe that if two edges of U are adjacent in G, they must belong to the same V_i and each such pair is in only one V_i.

Thus, we form an independent set of edges in G by taking the h edges mentioned above, and one of the nonindependent edges of each of the ℓ non-matchings. Hence, $\ell + h \leq \alpha'(G) = \alpha'$, and therefore,

$$k = \ell + m \leq \ell + \binom{h}{r} \leq \binom{\ell + h}{r} \leq \binom{\alpha'}{r} \tag{1}$$

from which we have the desired inequality.

To prove the second part, let $S = \{V_1, V_2, ..., V_k\}$ be a maximum independent set in $\mathcal{L}_r(G)$ with $k = \binom{\alpha'}{r}$. It follows from (1) that

$$k = \ell + m = \ell + \binom{h}{r} = \binom{\ell + h}{r} = \binom{\alpha'}{r}.$$

If $\ell = 0$, then $k = m = \binom{h}{r} = \binom{\alpha'}{r}$ so that S satisfies (i). If $r = 1$, then again all V_i's are matchings so that $\ell = 0$. Thus we assume that $\ell > 0$ and $r > 1$. In this case $\ell + \binom{h}{r} = \binom{\ell+h}{r}$ implies that $h = 0$ and $\ell = r + 1$. Consequently, $\alpha' = r + 1$. Moreover, it follows that $m = 0$, so that no V_i is a matching. If any V_i is not a star, then it has two independent edges or it is a K_3. In the first case, one gets $r + 2$ independent edges in G, a contradiction. In the second case, it follows that $r = 3$ so that each V_i is a K_3 or a $K_{1,3}$. □

The above theorem characterizes the maximum independent sets in $\mathcal{L}_r(G)$. We observe that more can be said about the structure of G in cases (ii) and (iii), namely, that each additional edge in G must join the center of one star to some vertex in another component.

3 A Pancyclic Property

In [4], we proved that if G is connected, then $\mathcal{L}_2(G)$ is *pancyclic*, meaning that there are cycles of all lengths from 3 to the order of the graph. We now extend that result in two ways: to more graphs and to more cycles. We show that if G has no isolated edges, then every vertex of $\mathcal{L}_2(G)$ is on a cycle of each possible length, a property known as *vertex-pancyclic*. In order to prove our theorem, we require a considerable number of preliminary results. We begin with the following general result on short cycles.

Lemma 3.1 *If G has at least four edges, then every vertex of $\mathcal{L}_2(G)$ that is not an isolated vertex is on cycles of lengths 3, 4, and 5.*

Proof: Let e and f be edges of G. We consider two cases.

Case 1: e and f are adjacent in G. Let c and d be two other edges of G. Then the subgraph induced by ce, cf, de, df, and ef contains the wheel $W_{1,4}$ (with ef as its center), so ef lies on a 3-cycle, a 4-cycle, and a 5-cycle.

Case 2: e and f are not adjacent in G. Since vertex ef is not isolated in $\mathcal{L}_2(G)$, there is an edge g incident with one of them, say with f. Let d be any other edge of G. Then, as in Case 1, the subgraph induced by the five vertices df, dg, ef, eg, and fg contains $W_{1,4}$ (here fg is the center vertex), so again ef lies on a 3-cycle, a 4-cycle, and a 5-cycle. □

One family of graphs requires special treatment, and we consider that next. This is the set of graphs rP_4, consisting of r disjoint paths of length 3. Consider a pair of P_4's in rP_4. Each of them gives rise to a triangle in $\mathcal{L}_2(rP_4)$; we call these *pure triangles.* We call the subgraph induced by the nine vertices arising from edges from both P_4's a *mixed piece.* We note that a mixed piece contains a spanning wheel $W_{1,8}$, and the subgraph induced by a mixed piece and one of its associated pure triangles contains $W_{1,11}$. This is shown in Figure 1 where (a) gives a labeling of the edges of $2P_4$, and (b) and (c) give wheels in $\mathcal{L}_2(2P_4)$.

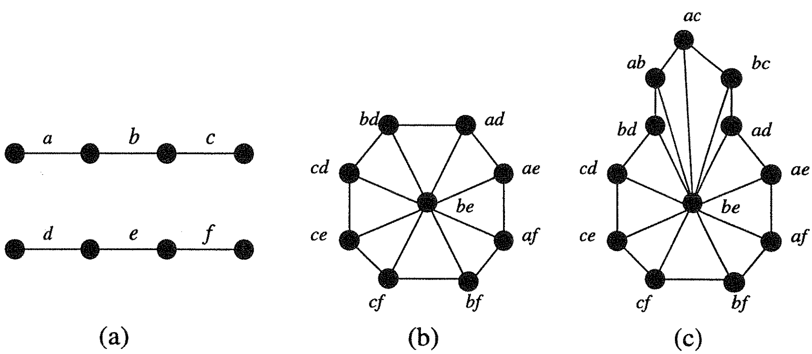

Figure 1

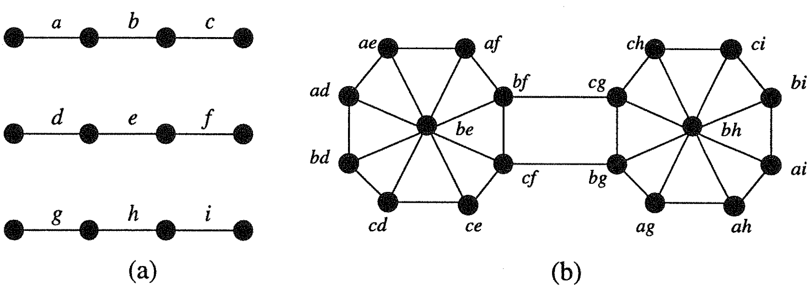

Figure 2

We now let $r > 2$ and look at some of the adjacencies between two mixed pieces that have one contributing P_4 in common. We observe that some of the rim edges from one piece lie on a 4-cycle with rim edges from the other. One such pair, the edges $[bf,\ cf]$ and $[bg,\ cg]$, is shown in Figure 2. We note that each wheel has four such edges.

Lemma 3.2 *For all r, the graph $\mathcal{L}_2(rP_4)$ is vertex-pancyclic.*

Proof: The result is clearly true for $r = 1$. It can be seen by an appropriate addition of the three vertices de, df, and ef to the graph in Figure 1(c) that $\mathcal{L}_2(2P_4)$ is spanned by the wheel $W_{1,14}$. Since every wheel is vertex-pancyclic, we have the result for $r = 2$.

The general case can be proved by induction, but for simplicity we present only the case $r = 3$ here. Assign each pure triangle to a different associated mixed piece (as in Figure 1(c). Obviously, by symmetry in the pieces, it is enough to show that each vertex in one of these augmented pieces lies on a cycle of each length $\ell = 3, 4, \cdots, 36$. For $\ell \leq 12$, this is true within the augmented piece itself. For $13 \leq \ell \leq 24$, we get the desired cycle by bridging an 11- or 12-cycle from the wheel of the first piece into the wheel of the second (the 11-cycle is needed for $\ell = 13$); and then for $25 \leq \ell \leq 36$ by bridging into the third piece. There are clearly enough bridging edges available for this to be done.

The general case follows similarly except that not all of the mixed pieces will be augmented when $r \geq 4$. □

In order to prove our main theorem of this section, we require one additional concept. A graph G of order p is *path-comprehensive* if every pair of vertices are joined by paths of all lengths $2, 3, \cdots, p-1$(but not necessarily length 1). Clearly, every wheel has this property. We observe also that every path-comprehensive graph has diameter 2, is Hamilton-connected (that is, every pair of vertices are connected by a Hamiltonian path), and is vertex-pancyclic. For reasons that will become clear, we are especially interested in complete multipartite graphs, and the next two results provide what we will need.

Lemma 3.3 *If $\ell \leq m \leq n$, the complete tripartite graph $K_{\ell,m,n}$ is path-comprehensive if and only if $\ell + m > n$.*

Proof: If $n \geq \ell + m$, then there can be no Hamiltonian path joining two vertices in the third partite set, so the inequality is clearly necessary. To prove the sufficiency we first consider the case $\ell = 1$. The condition $\ell + m > n$ then gives $m = n$. Since $K_{1,n,n}$ contains a wheel, we are done. For the case when $\ell > 1$, we proceed by induction on the total number of vertices. The result is clearly true when there are six or fewer vertices, since the only eligible graph is $K_{2,2,2}$. Assume it holds when there are p vertices ($p \geq 6$) and let $K = K_{\ell,m,n}$ be a complete tripartite graph with $\ell \leq m \leq n$, $n < \ell + m$, and $\ell + m + n = p + 1$. Let h satisfy $2 \leq h \leq p$. Let v and w be any two vertices in K, and let u be a vertex other than v and w in a partite set of maximum order. If $h \leq p-1$, then by the induction hypothesis, $K-u$, and hence K, contains a $v-w$ path of length h. Therefore, we need consider only the case $h=p$. Let P be a $v-w$ path of length $p-1$ in $K-u$. It is not difficult to show that for some edge xy in P, the vertices u, x, and y are all in different partite sets in K. Replacing the edge xy in P by the path xuy yields the requisite $v-w$ path of length p in K. The result follows. □

Our next result is a consequence of Lemma 3.3.

Corollary 3.4 *For $r \geq 3$, the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ with $n_1 \leq n_2 \leq \ldots \leq n_r$ and $p = \sum_{i=1}^{r} n_i$ is path-comprehensive if and only if $2n_r < p$.*

With those preliminaries out of the way, we turn to proving our main result, which we do first for forests. Let F be a forest without isolated edges. We consider a vertex v which is not an end vertex, and which has at most one neighbor that is not an end vertex. Let S denote the set of edges at v (as well as the star generated by those edges), and let R denote the set of remaining edges (and the subgraph generated by them). We let $\binom{R}{2}$, $\binom{S}{2}$, and RS denote the sets of pairs of edges, where both are from R, both are from S, and one is from each of R and S, respectively.

Lemma 3.5 *With F, R, and S as above, the subgraph of $\mathcal{L}_2(F)$ induced by $RS \cup \binom{S}{2}$ is path-comprehensive.*

Proof: By definition, $|\ S\ | \geq 2$. If $R = \phi$, then F is a star, and the result holds. Hence, we assume that $R \neq \phi$. Let $R = \{e_1,\ e_2,\ \cdots,\ e_r\}$ and $S = \{f_1,\ f_2,\ \cdots,\ f_s\}$. Further, for $j = 1, 2, \cdots, s$, let $V_j = \{e_i f_j : i = 1, 2, \cdots, r\}$. Then $RS = \bigcup_{j=1}^{s} V_j$, and in $\mathcal{L}_2(F)$, any two vertices in different V_j's are adjacent. Thus, the subgraph induced by RS contains the complete s-partite graph $K_{r,r,\cdots,r}$, and so the subgraph induced by $RS \cup \binom{S}{2}$ contains the join $K_{r,r,\cdots,r} * K_{\binom{S}{2}}$. Since $s \geq 2$, the property of path-comprehensiveness follows from Corollary 3.4. □

Lemma 3.6 *If F is a forest with no isolated edges, then $\mathcal{L}_2(F)$ is vertex-pancyclic.*

Proof: The proof is by induction on the number of edges in F. It is easily seen to hold for forests with two or three edges. Assume that it is true for forests with fewer than q edges, and let F be a forest with q edges, none of which is isolated. If every component of F is P_4, then $\mathcal{L}_2(F)$ is vertex-pancyclic by Lemma 3.2, and if F is just a star, the result is obvious. Therefore, we assume that F has a vertex v having at most one neighbor that is not an end vertex and for which $F-v$ has no isolated edges. As before, let S be the set of edges at v and R the set of edges of $F-v$, and let $|\ R\ | = r$ and $|\ S\ | = s$. Consider a pair of edges e and f in F. We know from Lemma 3.1 that the vertex ef lies on cycles of lengths 3, 4, and 5 in $\mathcal{L}_2(F)$. For lengths greater than 5, we consider three cases.

Case 1. $e,\ f \in R$. By the induction hypothesis, $\mathcal{L}_2(F-v)$ is vertex-pancyclic, so ef lies on cycles of length 3, 4, …, $\binom{r}{2}$ within this subgraph. Let C be such a cycle, and let gh be one of the neighbors of ef on C. Without loss of generality, we assume that f and g are adjacent in F. Let d be any edge in S, the star at v. Then df and dg are in RS, and so in $\mathcal{L}_2(F)$ we have $ef \sim dg \sim df \sim gh$. By Lemma 3.5, there is within $\langle RS \cup \binom{S}{2}\rangle$ a path of each possible length between df and dg. Thus, if the length of C is ℓ, this procedure generates cycles of lengths $\ell+2$, $\ell+3$, $\cdots$, $\ell+rs+\binom{s}{2}$. Since ℓ can equal 3, 4, $\cdots$, $\binom{r}{2}$, and since ef is on a cycle of length 4, this case is complete.

Case 2. $e \in R$, $f \in S$ (say). The argument is quite similar to Case 1. We know that ef is on cycles of lengths 3, 4, $\cdots$, $rs+\binom{s}{2}$ in $\langle RS \cup \binom{S}{2}\rangle$. To get a cycle of length $\ell > rs+\binom{s}{2}$, we choose numbers i and j with $i+j=\ell$, $3 \le i \le \binom{r}{2}$ and $3 \le j \le rs+\binom{s}{2}$. Let d be another edge of R, let C be a cycle of length i containing de, and let bc be a neighbor of de on C. Let a denote one of d or e that is adjacent to b or c, and let g be another edge of S. By Lemma 3.5, there is a path P of length $j-1$ in $\langle RS \cup \binom{S}{2}\rangle$ joining ag and ef. The union of this path, the edge joining ef and de, the path of length $i-1$ joining de and bc (from C), and the edge joining bc and ag gives the desired cycle.

Case 3. $e,\ f \in S$. Let $v = ef$ in $\mathcal{L}_2(F)$. The argument just given must apply to $\mathcal{L}_2(F)-v$ since $\langle RS \cup \binom{S}{2} - \{v\}\rangle$ still has the path-comprehensive property. Hence, $\mathcal{L}_2(F)-v$ is vertex-pancyclic, and since v is adjacent to all of the other vertices of $RS \cup \binom{S}{2}$, v will be on cycles of all possible lengths. This completes the proof. □

The proof of the extension of this result to graphs in general follows one given in [4]; we include it here for completeness.

Theorem 3.7 *If G is a graph with no isolated edges, then the index-2 super line graph $\mathcal{L}_2(G)$ is vertex-pancyclic.*

Proof: The proof is by induction on the number of cycles. By Lemma 3.6, the result holds for graphs with no cycles. Assume that it holds for graphs with fewer than n cycles and let G be a graph with n cycles and no isolated edges. Replace an edge uv on a cycle of G by an edge uw, where w is a new vertex. The resulting graph G' has fewer cycles than G (and no isolated edges), so by the induction hypothesis $\mathcal{L}_2(G')$ is vertex-pancyclic. Since $\mathcal{L}_2(G')$ is isomorphic to a subgraph of $\mathcal{L}_2(G)$, the latter is also vertex-pancyclic. The theorem follows by the Principle of Mathematical Induction. □

While it may be possible to extend this theorem to some graphs having isolated edges, it cannot be done for all graphs, even to their nontrivial components. For example, let G be the disjoint union $P_4 \cup 2K_2$.Then the nontrivial component of

$\mathcal{L}_2(G)$ is the complete multipartite graph $K_{1,1,2,5}$, which clearly is not Hamiltonian, and thus is not pancyclic. However, we believe that, for a graph G with at most one isolated edge, $\mathcal{L}_2(G)$ is pancyclic, and that it may even be vertex-pancyclic.

Further problems along these lines suggest themselves: Find conditions on G under which $\mathcal{L}_2(G)$ has isolated vertices but the nontrivial component is Hamiltonian, pancyclic, or vertex-pancyclic. Another open problem is to study $\mathcal{L}_2(G)$ in relation to Hamiltonian-connectedness and panconnectedness.

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