$\overrightarrow{P_3}$-factorization of complete bipartite symmetric digraphs

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Abstract

In this paper, it is shown that a necessary and sufficient condition for the existence of a $\overrightarrow{P_3}$-factorization of the complete bipartite symmetric digraph $K^*_{m,n}$ is (1) $m + n \equiv 0 \pmod 3$, (2) $m \leq 2n$, (3) $n \leq 2m$, and (4) $3mn/(m + n)$ is an integer.

1. Introduction

Let $\overrightarrow{P_3}$ be the directed path on three vertices and let $K^*_{m,n}$ be the complete bipartite symmetric digraph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$. A spanning subgraph $\overrightarrow{F}$ of $K^*_{m,n}$ is called a $\overrightarrow{P_3}$-factor if each component of $\overrightarrow{F}$ is isomorphic to $\overrightarrow{P_3}$. If $K^*_{m,n}$ is expressed as an arc-disjoint sum of $\overrightarrow{P_3}$-factors, then this sum is called a $\overrightarrow{P_3}$-factorization of $K^*_{m,n}$.

The spectrum problems for P_3-factorization of the complete graph K_n, the complete bipartite graph $K_{m,n}$ and the complete multipartite graph K^m_n have been completely solved. (See [2, 4, 5, 6].) In this paper a necessary and sufficient condition for the existence of a $\overrightarrow{P_3}$-factorization of the complete symmetric digraph $K^*_{m,n}$ will be given.

Theorem 1.1 *$K^*_{m,n}$ has a $\overrightarrow{P_3}$-factorization if and only if* (1) $m+n \equiv 0 \pmod 3$, (2) $m \leq 2n$, (3) $n \leq 2m$, *and* (4) $3mn/(m + n)$ *is an integer.*

It is easy to see that a $\overrightarrow{P_3}$-factorization of $K^*_{m,n}$ gives rise to a P_3-factorization of $2K_{m,n}$. We get the following as a by-product of Theorem 1.1.

Theorem 1.2 *$2K_{m,n}$ has a P_3-factorization if and only if* (1) $m+n \equiv 0 \pmod 3$, (2) $m \leq 2n$, (3) $n \leq 2m$, *and* (4) $3mn/(m + n)$ *is an integer.*

2. Main result

From simple counting we have

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Theorem 2.1 *If $K^*_{m,n}$ has a $\overrightarrow{P_3}$-factorization then* (1) $m+n \equiv 0 \pmod 3$, (2) $m \le 2n$, (3) $n \le 2m$, *and* (4) $3mn/(m+n)$ *is an integer.*

We prove the following existence theorem, which is used later in this paper.

Theorem 2.2 *If $K^*_{m,n}$ has a $\overrightarrow{P_3}$-factorization, then $K^*_{sm,sn}$ has a $\overrightarrow{P_3}$-factorization for every positive integer s.*

Proof: Let V_1, V_2 be the independent sets of $K^*_{sm,sn}$ where $|V_1| = sm$ and $|V_2| = sn$. Divide V_1 and V_2 into s subsets of m and n vertices each, respectively. Construct a new graph G with vertex set consisting of the subsets which were just constructed. In this graph, two vertices are adjacent if and only if the subsets come from disjoint independent sets of $K^*_{sm,sn}$. Thus G is a complete bipartite graph $K_{s,s}$. Noting that the cardinality of each subset identified with a vertex set of G is m or n and that $K_{s,s}$ has a 1-factorization, we see that the desired result is obtained. (1-factorizations of $K_{s,s}$ are discussed in [1, 3].)

Now we start to prove our main result. There are three cases to consider.

Case $m = 2n$: In this case, from Theorem 2.2, $K^*_{2n,n}$ has a $\overrightarrow{P_3}$-factorization since $K^*_{2,1}$ has a $\overrightarrow{P_3}$-factorization:

$$x_1y_1x_2, \quad x_2y_1x_1.$$

Case $n = 2m$: Obviously, $K^*_{m,2m}$ has a $\overrightarrow{P_3}$-factorization.

Case $m < 2n$ and $n < 2m$: In this case, let $x = (2n-m)/3$, $y = (2m-n)/3$, $t = (m+n)/3$, and $r = 3mn/(m+n)$. Then from conditions (1)–(4), x, y, t, r are integers and $0 < x < m$ and $0 < y < n$. We have $x + 2y = m$ and $2x + y = n$. Hence $r = 2(x+y) + xy/(x+y)$. Let $z = xy/(x+y)$, which is a positive integer. And let $(x, 2y) = d$, $x = dp$, $2y = dq$, where $(p,q) = 1$. Then dq is even and $z = dpq/(2p+q)$. The following lemmas can be verified.

Lemma 2.3 *If $(p,q) = 1$, then $(pq, p+q) = 1$.*

Lemma 2.4 *If $(p,q) = 1$, then $(pq, 2p+q) = 1$ when $q \equiv 1 \pmod 2$ and $(pq, 2p+q) = 2$ when $q \equiv 0 \pmod 2$.*

Lemma 2.5 *If $(p,q) = 1$, then $(pq, 4p+q) = 1$ when $q \equiv 1 \pmod 2$, $(pq, 4p+q) = 2$ when $q \equiv 2 \pmod 4$, and $(pq, 4p+q) = 4$ when $q \equiv 0 \pmod 4$.*

Using these p, q, d, the parameters m and n satisfying conditions (1)–(4) can be expressed as follows:

Lemma 2.6 *If $(p,q) = 1$ and $dpq/(2p+q)$ is an integer, then for some positive integer s,*

(a) $m = 2(p+q)(2p+q)s$, $n = (4p+q)(2p+q)s$ *when* $q \equiv 1 \pmod 2$,
(b) $m = (p+2q')(p+q')s$, $n = (2p+q')(p+q')s$ *when* $q = 2q'$ *and* $q' \equiv 1 \pmod 2$,
(c) $m = (p+4q'')(p+2q'')s$, $n = 2(p+q'')(p+2q'')s$ *when* $q = 4q''$.

We use the following notation for sequences. Let A and B be two sequences of the same length:

$$A: \; a_1, a_2, \ldots, a_u \qquad B: \; b_1, b_2, \ldots, b_u.$$

If $b_i = a_i + c$ $(1 \le i \le u)$, then we write $B = A + c$. If $b_i = a_i + c \pmod{w}$ $(1 \le i \le u)$, then we write $B = A + c \pmod{w}$, where the residues $a_i + c \pmod{w}$ are integers in the set $\{1, 2, \cdots, w\}$.

For the parameters m and n in (a)–(c) when $s = 1$, we can construct a $\overrightarrow{P_3}$-factorization of $K^*_{m,n}$.

It is easy to see that the existence of a P_3-factorization of $K_{m,n}$ implies the existence of a $\overrightarrow{P_3}$-factorization of $K^*_{m,n}$. The following two lemmas come from [5, Lemma 4 and Lemma 6].

Lemma 2.7 *If* $(p, q) = 1$, $q \equiv 1 \pmod{2}$, *and* $m = 2(p+q)(2p+q)$, $n = (4p+q)(2p+q)$, *then* $K^*_{m,n}$ *has a* $\overrightarrow{P_3}$*-factorization.*

Lemma 2.8 *If* $(p, q) = 1$, $q = 4q''$, *and* $m = (p+4q'')(p+2q'')$, $n = 2(p+q'')(p+2q'')$, *then* $K^*_{m,n}$ *has a* $\overrightarrow{P_3}$*-factorization.*

For our main result we need only to prove the following lemma.

Lemma 2.9 *If* $(p, q) = 1$, $q = 2q'$, $q' \equiv 1 \pmod{2}$, *and* $m = (p+2q')(p+q')$, $n = (2p+q')(p+q')$, *then* $K^*_{m,n}$ *has a* $\overrightarrow{P_3}$*-factorization.*

Proof: Let $x = (2n-m)/3$, $y = (2m-n)/3$, $t = (m+n)/3$, and $r = 3mn/(m+n)$. Then we have $x = p(p+q')$, $y = q'(p+q')$, $t = (p+q')^2$, and $r = (p+2q')(2p+q')$. Let $r_1 = p+2q'$, $r_2 = 2p+q'$, $m_0 = m/r_1 = (p+q')$, and $n_0 = n/r_2 = (p+q')$. Consider the two sequences R and C both of length $2(p+q')$

$$R:\ R', R'' \qquad C:\ C', C''$$

in which

$$\begin{aligned}
R' &:\ 1, 1, 2, 2, \cdots, \tfrac{1}{2}(p+q'), \tfrac{1}{2}(p+q') \\
R'' &:\ \tfrac{1}{2}(p+q')+1, \tfrac{1}{2}(p+q')+1, \cdots, (p+q'), (p+q') \\
C' &:\ 1, 2, 3, 4, \cdots, (p+q')-1, (p+q') \\
C'' &:\ (p+q')+1, (p+q')+2, \cdots, 2(p+q')-1, 2(p+q').
\end{aligned}$$

Construct p sequences R_i where $R_i = R + (i-1)(p+q')$ $(1 \le i \le p)$. Construct p sequences C_i where $C_i = C + (i-1) \pmod{2(p+q')} + 2(i-1)(p+q')$ $(1 \le i \le p)$. Construct two sequences S and T both of length $2(p+q')$

$$S:\ S', S'' \qquad T:\ T', T''$$

in which

$$\begin{aligned}
S' &:\ 1, 2, \cdots, (p+q')-1, (p+q') \\
S'' &:\ (p+q')+1, (p+q')+2, \cdots, 2(p+q')-1, 2(p+q') \\
T' &:\ 1, 3, \cdots, (p+q')-1, 1, 3, \cdots, (p+q')-1 \\
T'' &:\ 2, 4, \cdots, (p+q'), 2, 4, \cdots (p+q').
\end{aligned}$$

Construct q' sequences S_i where $S_i = S + 2(i-1)(p+q') + p(p+q')$ $(1 \le i \le q')$. Construct q' sequences T_i where $T_i = T + (i-1) + p \pmod{(p+q')} + (i-1)(p+q') + 2p(p+q')$ $(1 \le i \le q')$. Consider the two sequences I and J both of the same length

$$I:\ I', I'' \qquad J:\ J', J''$$

in which

$$I':\ R_1, R_2, \cdots, R_p \qquad I'':\ S_1, S_2, \cdots, S_{q'}$$
$$J':\ C_1, C_2, \cdots, C_p \qquad J'':\ T_1, T_2, \cdots, T_{q'}.$$

Then the length of I and J is $2t$. Divide R_i into two subsequences R_i' and R_i'' of equal lengths ($i = 1, 2, \ldots, p$). And divide T_i into two subsequences T_i' and T_i'' of equal lengths ($i = 1, 2, \ldots, q'$). Thus we have $R_i:\ R_i', R_i''$ and $T_i:\ T_i', T_i''$. Let h_k, j_k be the k-th elements of I' and J' respectively ($k = 1, 2, \ldots, 2p(p+q')$). When $h_k = h_{k+1}$, join h_k in V_1 and j_k, j_{k+1} in V_2 with a directed path, either $j_k h_k j_{k+1}$ if $h_k \in R_i'$ or $j_{k+1} h_k j_k$ if $h_k \in R_i''$. Let h_k, j_k be the k-th elements of I'' and J'' respectively ($k = 1, 2, \ldots, 2q'(p+q')$). When $j_k = j_{k+(p+q')/2}$, join h_k, $h_{k+(p+q')/2}$ in V_1 and j_k in V_2 with a directed path, either $h_{k+(p+q')/2} j_k h_k$ if $j_k \in T_i'$ or $h_k j_k h_{k+(p+q')/2}$ if $j_k \in T_i''$. Construct the digraph $\overrightarrow{F}$ with the two vertex sets $\{h_k\}$ and $\{j_k\}$ and this directed path set. Then $\overrightarrow{F}$ is a $\overrightarrow{P_3}$-factorization. This digraph is called the $\overrightarrow{P_3}$-factor constructed from the two sequences I and J.

Construct r_1 sequences I_i where $I_i = I + (i-1)m_0 \pmod{m}$ $(1 \le i \le r_1)$. Construct r_2 sequences J_j where $J_j = J + (j-1)n_0 \pmod{n}$ $(1 \le j \le r_2)$. Construct the $r_1 r_2$ $\overrightarrow{P_3}$-factors $\overrightarrow{F}_{ij}$ from I_i and J_j $(1 \le i \le r_1,\ 1 \le j \le r_2)$. Then it is easy to see that the $\overrightarrow{F}_{ij}$ are arc-disjoint and their union is a $\overrightarrow{P_3}$-factorization of $K^*_{m,n}$.

By applying Theorem 2.2 with Lemmas 2.7 to 2.9, it can be seen that when the parameters m and n satisfy conditions (1)–(4), the digraph $K^*_{m,n}$ has a $\overrightarrow{P_3}$-factorization. This completes the proof of Theorem 1.1

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