by

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Abstract: The infinite chessboard, or equivalently the lattice of points of integer coordinates in the plane, gives rise to paths consisting of steps between neighbouring lattice-points. On the chessboard the king is allowed horizontal, vertical and diagonal steps, one at a time. The number of possible paths from the origin to some arbitrary lattice point has been found by various authors. In the present work the number of paths modulo a prime p is explored. Under conditions placed on the allowable steps, the arrays representing the number of paths mod p show fractal structures which indicate that once these numbers are known for the fundamental region of lattice points with coordinates between 0 and p, there are simple formulae or algorithms for determining these numbers for arbitrary lattice points. The situation is similar to that known for binomial coefficients and summarized by Lucas' theorem.

INTRODUCTION

The language devised for arrays representing binomial coefficients modulo p, of the structure illustrated in Figure 1 below, will be used.

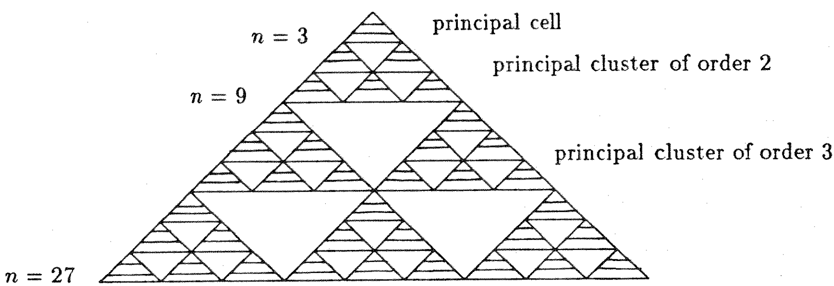

Figure 1: Binomial array $\binom{n}{k}$ (mod 3)

Detailed descriptions of these arrays can be found in [2] or [3].

The triangles shaded in the diagram are *cells*, *similar* to the principal cell in the sense that the entries in each are in constant ratio to the corresponding entries of the principal cell, determined by the entry at the *head* of the cell. In the same way, clusters are similar to the principal cluster of the same order.

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The inverted blank triangles represent *zero holes*, which are arrays consisting of zero entries exclusively.

Each *cluster* of order m consists of p *layers*, the layers containing clusters of order $m-1$, alternating with zero-holes.

Although the structure of the arrays to be discussed in the following differs in some respects from that of the binomial array, the terms cell, cluster, head of cell or cluster will be used, with specific notation for the entries introduced in each case.

2. KING'S WALK, PROGRESS RESTRICTED TO POSITIVE DIRECTION

In a recent paper, M. Razpet [1] deduced formulae for the possible number of walks $w(i,j)$ from the origin to a point (i,j) $(i,j \geq 0)$. For $w(i,j) \pmod p$ he obtained a result analogous to that of Lucas for binomials:
Let

$$\begin{aligned} i &= a_m p^m + \ldots + a_1 p + a_0, \\ j &= b_m p^m + \ldots + b_1 p + b_0 \end{aligned}$$

(where $0 \leq a_s, b_s < p$ for $0 \leq s \leq m$) be the expansions to base p of i and j.

Let

$$w(i,j) \pmod p = \overline{w}(i,j).$$

Then

$$\overline{w}(i,j) = \overline{w}(a_m, b_m)\,\overline{w}(a_{m-1}, b_{m-1}) \ldots \overline{w}(a_0, b_0) \tag{1}$$

Razpet obtained this identity by algebraic means and then produced computer outputs of $\overline{w}(i,j)$.

In this discussion we follow a somewhat different route. By establishing first the general formula for $w(i,j)$ from the recursion formula, we apply it to the principal cell: $0 \leq i,j < p$, and from this we show that a structure of cells and clusters is initiated. This implies the Lucas type relation (1). The cells and clusters in this situation will have square shapes instead of the triangular shapes of the binomial array, and the nature of the zero-holes will be different.

To fit with computer outputs, coordinate systems will be differently oriented from the usual way, as shown in Figure 2.

Since the king is allowed steps in the positive direction under the conditions imposed, the point (i,j) can be reached through the final steps indicated in the diagram. Hence

$$w(i,j) = w(i-1,j-1) + w(i-1,j) + w(i,j-1). \tag{2}$$

The general formula for $w(i,j)$ is devised from the recursion formula, by using *shift* operators defined by

$$\begin{aligned} \sigma_1 w(i,j) &= w(i-1,j-1) \\ \sigma_2 w(i,j) &= w(i-1,j) \\ \sigma_3 w(i,j) &= w(i,j-1). \end{aligned}$$

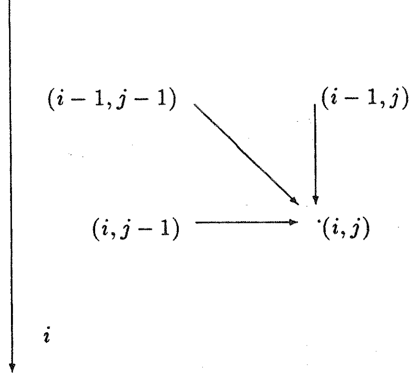

Figure 2

Thus (2) can be written in the form

$$(\sigma_1 + \sigma_2 + \sigma_3)\, w(i,j) = w(i,j). \tag{3}$$

We find first $w_k(i,j)$, the number of ways in which (i,j) can be reached in k steps from $(0,0)$.

Iterating (3), we obtain that in general for k steps terminating in (i,j)

$$w(i,j) = (\sigma_1 + \sigma_2 + \sigma_3)^k\, w(i,j) \tag{4}$$

and

$$w(i,j) = \sum_{k=\max(i,j)}^{i+j} w_k(i,j). \tag{5}$$

Denote by d, v, h the number of diagonal, vertical and horizontal steps, respectively, making up the path; we obtain for a path consisting of k steps

$$w(i,j) = \sum \frac{k!}{d!v!h!} \sigma_1^d \sigma_2^v \sigma_3^h w(i,j) \tag{6}$$

where

$$\sigma_1^d \sigma_2^v \sigma_3^h w(i,j) = w(i-(d+v), j-(d+h)). \tag{7}$$

Since we consider the case when k steps lead from $(0,0)$ to (i,j), we have

$$\sigma_1^d \sigma_2^v \sigma_3^h w(i,j) = w(0,0) = 1;$$

thus $i-(d+v) = 0$, $j-(d+h) = 0$, while $d+v+h = k$. These three equations determine precisely one value for the set (d,v,h), namely $v = k-j$, $h = k-i$, $d = i+j-k$, provided that $\max(i,j) \le k \le i+j$.

From (6) and (7)

$$w_k(i,j) = \frac{k!}{(k-i)!(k-j)!(i+j-k)!} = \binom{k}{i}\binom{i}{k-j}. \tag{8}$$

This formula can also be given a combinatorial interpretation by counting the ways in which k steps from the origin to point (i,j) can be distributed into horizontal, vertical and diagonal steps.

Hence, from (5),

$$w(i,j) = \sum_{k=\max(i,j)}^{i+j} \binom{k}{i}\binom{i}{k-j}. \tag{9}$$

In particular, for $i = 0$ or $j = 0$, $w(i,j) = 1$. Also $w(i,j) = w(j,i)$, as expected.

Next we investigate the $w(i,j)$ array (mod p). Since $\overline{w}(i,j) = w(i,j) \pmod p$, the principal cell is the array $(\overline{w}(i,j)|0 \le i,j < p)$. For the last row of the cell we have,from (9),

$$\overline{w}(p-1,j) = \sum_{k=p-1}^{p-1+j} \binom{k}{p-1}\binom{p-1}{k-j} \pmod p$$

Since

$$\binom{k}{p-1} \equiv 0 \pmod p \text{ for } p \le k < 2p-1,$$

$$\overline{w}(p-1,j) = \binom{p-1}{p-1-j} = \binom{p-1}{j} \equiv (-1)^j \pmod p,$$

and, by symmetry,

$$\overline{w}(i,p-1) = (-1)^i. \tag{10}$$

Denote the *principal cell* by $C[0,0]$, and, more generally, let

$$C[rp,sp] = (\overline{w}(i,j)|rp \le i < (r+1)p, sp \le j < (s+1)p)$$

(Here (rp,sp) represent the "head", the coordinates of the point at the top left corner.)

Theorem:

(i) $\overline{w}(rp,sp) = \overline{w}(r,s)$

(ii) $C[rp,sp]$ is a cell, that is

$$\overline{w}(rp+i,sp+j) = \overline{w}(r,s)\overline{w}(i,j). \tag{11}$$

Proof. We note first that if the rows and columns of the principal cell are extended to infinity, we obtain in both directions copies of the principal cell, as shown in Figure 3.

This follows immediately from the recursion formula (2) and (10); so $C[rp,0]$ and $C[0,sp]$ are cells for all r,s.

Next assume that the structure is established down to the $(r-1)$-th horizontal layer (hence, by symmetry, also to the $(r-1)$-th vertical one). The induction can now proceed

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1				-1	1				-1	1				-1
1				1	1				1	1				1
1				-1	1				-1	1				-1
1	-1	1	-1	1	1	-1	1	-1	1	1	-1	1	-1	1
1	1	1	1	1										
1				-1										
1				1										
1				-1										
1	-1	1	-1	1										
1	1	1	1	1										
1				-1										
1				1										
1				-1										
1	-1	1	-1	1										

Figure 3

along the r-th layer by assuming that for $s > r$, $C[(r-1)p,(s-1)p]$, $C[(r-1)p, sp]$ and $C[rp,(s-1)p]$ are cells with head entries $\overline{w}(r-1,s-1]$, $\overline{w}[r-1,s]$ and $\overline{w}[r,s-1]$, respectively, with corresponding last rows and columns of alternating terms; so by (2) and the induction hypothesis

$$\overline{w}(rp, sp) = \overline{w}(r-1, s-1) + \overline{w}(r-1, s) + \overline{w}(r, s-1) = \overline{w}(r, s).$$

Also the first row and first column of $C[rp, sp]$ consist of identical entries. The rest follows.

Next we consider the array $(w(i,j)|0 \leq i,j < p^2)$, called the principal cluster of order 2, consisting of p layers of cells. We note here that the $C[(p-1)p, sp]$ and $C[rp,(p-1)p]$ cells have again alternating $+1$ and -1 entries in their last rows and columns.

We now apply an induction procedure similar to that used on cells, and proceed from clusters of order 2 to clusters of order 3 and so on, and thus establish the self similar structure of the $\overline{w}(i,j)$ array. □

Tables (1) and (2) show these arrays modulo 3 and 7. The zero-holes on these are really cells or clusters with heads = 0.

The extension of Lucas' theorem, relation (1) as established by Razpet, follows now directly from (11). We write the expansions of i and j to base p shortly as $[a_m, a_m, \ldots, a_0]$ and $[b_m, b_{m-1}, \ldots, b_0]$ respectively, and see that the entry $\overline{w}(i,j)$ is inside a nest of clusters of orders m, $m-1$, ..., 1 respectively, with heads $\overline{w}(a_m, b_m)$, $\overline{w}(a_{m-1}, b_{m-1})$, ..., $\overline{w}(a_1, b_1)$. Thus, from (11), the relation (1) follows.

3. KING WALKS WITH MINIMAL NUMBER OF STEPS

The assumption now is that the king moves from the origin $(0,0)$ to the point (n,k), the path being covered in a minimal number of steps; if we assume without loss of generality

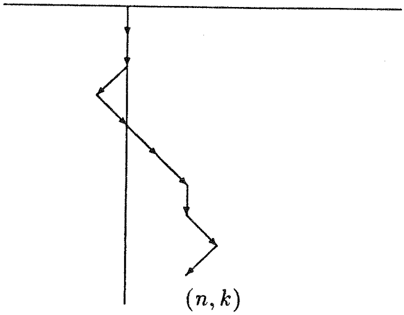

Figure 4

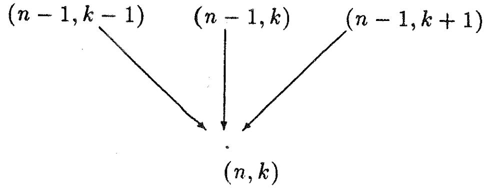

Figure 5

that $|k| \leq n$, then the minimality condition rules out horizontal steps, but diagonal steps may go in the negative direction as shown in Figure 4. Thus the last steps to reach the point (n, k) may come from one of the points shown in Figure 5.

Denote by $\left\{ {n \atop k} \right\}$ the number of possible paths from $(0,0)$ to (n,k) under the conditons imposed above. Figure 5 implies the recursion formula

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k+1} \right\} \tag{12}$$

with initial conditions $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {0 \atop i} \right\} = 0$ for $i \neq 0$.

The explicit formula for $\left\{ {n \atop k} \right\}$ may be deduced by the shift operator method, as in (2), but for the later part it is more convenient to use the generating functions

$$f_n(x) = \sum \left\{ {n \atop k} \right\} x^k. \tag{13}$$

The recursion (12) implies that $f_n(x) = (x + 1 + x^{-1})\, f_{n-1}(x)$ and it follows from this that

$$f_n(x) = \left(x + 1 + x^{-1}\right)^n. \tag{14}$$

$\left\{ {n \atop k} \right\}$ is then the coefficient of x^k in the expansion of (14); hence

$$\left\{ {n \atop k} \right\} = \sum \frac{n!}{i!j!(n-i-j)!}$$

where $i - j = k$, under the restriction that $i + j \leq n$.

Let $i + j = m$, where $k \leq m \leq n$; hence $i = \frac{m+k}{2}$, $j = \frac{m-k}{2}$ and since i and j are integers, $m \equiv k \pmod 2$. We then obtain the explicit formula

$$\left\{ {n \atop k} \right\} = \sum \frac{n!}{(n-m)!(\frac{1}{2}(n+m))!(\frac{1}{2}(n-m)!},$$

where $k \leq m \leq n$ and $m \equiv k \pmod 2$.

An alternative formula may be obtained by setting

$$x + 1 + x^{-1} = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 1.$$

Then

$$f_n(x) = \left(\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 1\right)^n = \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{2r}$$
$$= \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\sum_{s=0}^{2r}\binom{2r}{s}x^{s-r}.$$

The coefficient of x^k is then

$$\left\{ {n \atop k} \right\} = \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\binom{2r}{k+r}$$

An analogue for the convolution formula for binomial coefficients may be also obtained by writing, for some fixed m,

$$\left(x + x^{-1} + 1\right)^n = \left(x + x^{-1} + 1\right)^m \left(x + x^{-1} + 1\right)^{n-m}.$$

Hence

$$\left\{ {n \atop k} \right\} = \sum_s \left\{ {m \atop s} \right\}\left\{ {n-m \atop k-s} \right\} \tag{15}$$

Tables (3) and (4) show the interesting fractal structure of $\left\{ {n \atop k} \right\}$ mod 3 and $\left\{ {n \atop k} \right\}$ mod 7.

In the following, the structure of $\left\{ {n \atop k} \right\}$ will be analysed.

We begin by observing that

$$\left(x + 1 + x^{-1}\right)^p \equiv x^p + 1 + x^{-p} \pmod p.$$

Hence

$$\left(x + 1 + x^{-1}\right)^{np} \equiv \left(x^p + 1 + x^{-p}\right)^n \pmod p. \tag{16}$$

It follows from (16) that the expansion (mod p) contains only those powers of x where the index is an integer multiple of p; thus

$$\left\{ {np \atop r} \right\} \equiv 0 \pmod p \quad \text{whenever } p \not| \; r. \qquad (17).$$

Looking at coefficients in the expansion on the right hand side of (16), we see that

$$\left\{ {np \atop nk} \right\} \equiv \left\{ {n \atop k} \right\} \pmod p. \tag{18}$$

In what follows we use again the notation

$$\overline{\left\{ {n \atop k} \right\}} = \left\{ {n \atop k} \right\} \pmod p.$$

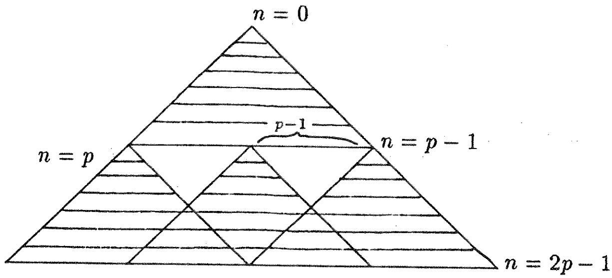

Figure 6

Next we consider the *principal cell*

$$\left(\overline{\left\{ n \atop k \right\}} \,\middle|\, 0 \le n < p \right).$$

From the formula for $\left\{ n \atop k \right\}$ it follows that

$$\left\{ n \atop n \right\} = \left\{ n \atop -n \right\} = 1,$$

$$\left\{ n \atop k \right\} = 0 \quad \text{for} \quad |k| > n,$$

and

$$\left\{ n \atop k \right\} = \left\{ n \atop -k \right\} \quad \text{for all} \quad n.$$

When $n = p$, it also follows that

$$\overline{\left\{ p \atop 0 \right\}} = 1.$$

So the principal cell is a triangular structure, symmetrical about the zero axis, and having 1 for its extremal entries, and

$$1 \;\; 0 \;\; 0 \;\; \ldots \;\; 1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; 1$$

for the entries of the row to follow (the p-th row of the $\overline{\left\{ n \atop k \right\}}$ array).

Figure 6 shows the structure of the principal cell and the rows which follow immediately.

It follows from the recursion formula (12) that the zero entries of row p initiate zero-holes, and hence the recursion formula also implies that the first rows of the principal cells are reproduced in the shaded arrays headed by $\overline{\left\{ p \atop -p \right\}}, \overline{\left\{ p \atop 0 \right\}}, \overline{\left\{ p \atop p \right\}} = 1$. However, the depth of the zero-holes is only $\frac{1}{2}(p-1)$, since the distance between non-zero entries decreases by one at each end with each additional row. Thus the $\overline{\left\{ n \atop k \right\}}$ array is made more complex by the *overlap* of the cells.

On the other hand, relations (17) and (18) ensure the fractal structure of the array, with cells and zero-holes being initiated in every p-th row, while in row p^2 we have

$$\overline{\begin{Bmatrix} p^2 \\ kp \end{Bmatrix}} = \overline{\begin{Bmatrix} p \\ k \end{Bmatrix}} = 0 \quad \text{if} \quad p \not| \, k,$$

while

$$\overline{\begin{Bmatrix} p^2 \\ p^2 \end{Bmatrix}} = \overline{\begin{Bmatrix} p^2 \\ -p^2 \end{Bmatrix}} = \overline{\begin{Bmatrix} p^2 \\ 0 \end{Bmatrix}} = 1;$$

so we have a structure similar to that of row p, with the two strings of zero-entries being of length $p^2 - 1$. Thus, down to $n = p^2 + \frac{p^2-1}{2}$, the principal cluster $\left(\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}} \,\middle|\, n < p^2 \right)$ is reproduced in the three arrays headed by $\overline{\begin{Bmatrix} p^2 \\ -p^2 \end{Bmatrix}}, \overline{\begin{Bmatrix} p^2 \\ p^2 \end{Bmatrix}}$ and $\overline{\begin{Bmatrix} p^2 \\ 0 \end{Bmatrix}}$.

As the array is developed further, clusters of higher order arise, also overlapping, so on first sight the array becomes a maze of overlapping clusters and subclusters.

In what follows, it will be shown that all $\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}}$ entries are determined by the principal cell, and while we cannot find a formula as simple as that of Lucas for binomials, and for arrays described in Section 2, an algorithm can be developed to calculate $\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}}$ in the general case.

The following theorem refers to two neighbouring cells, headed by entries in row ap, where $a > 0$. Because of the symmetry of the array and of the principal cell, we shall be looking only at the positive sides of the array and cells.

Theorem: Let the coordinates of the head of a cell be (ap, bp), $(a, b > 0)$, and the coordinates of the right hand neighbour $(ap, (b+1)p)$, and denote the entries at the heads by ℓ, r respectively. Consider the point P having coordinates

$$n = ap + i, \quad k = bp + j \qquad (0 \leq i, j < p).$$

Then

$$\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}} = \ell \overline{\begin{Bmatrix} i \\ j \end{Bmatrix}} + r \overline{\begin{Bmatrix} i \\ p-j \end{Bmatrix}}. \tag{19}$$

Proof: Figure 7 illustrates the region defined by the conditions imposed on i and j, with the positive side of the cell of reference, showing the entry ℓ at its head, overlapping with the negative side of its right neighbour.

We begin the proof for the "free part" of the array, when $0 \leq i \leq \frac{p-1}{2}$.

We consider three cases:

(i) $0 \leq j \leq i$

P is in the free part of the left cell; hence by similarity

$$\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}} = \ell \overline{\begin{Bmatrix} i \\ j \end{Bmatrix}}$$

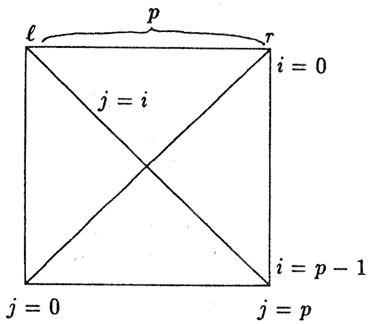

Figure 7

while $p - j > i$. Hence

$$\overline{\begin{Bmatrix} i \\ p-j \end{Bmatrix}} = 0;$$

so (19) follows.

(ii) $i < j < p - i$, so $p - j > i$.

In this case

$$\overline{\begin{Bmatrix} i \\ j \end{Bmatrix}} = 0 \quad \text{and} \quad \overline{\begin{Bmatrix} i \\ p-j \end{Bmatrix}} = 0.$$

Indeed, the point is between the two cells, inside the zero hole; hence the relation is true.

(iii) $p - i \leq j < p$; hence $p - j \leq i$.

The point is in the free part of the right hand cell (on the negative side), its coordinates relative to the head being $(i, -(p-j))$.

Now by similarity and symmetry

$$\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}} = r \overline{\begin{Bmatrix} i \\ p-j \end{Bmatrix}}$$

and, since $j \geq p - i > i$,

$$\overline{\begin{Bmatrix} i \\ j \end{Bmatrix}} = 0.$$

The relation is verified again.

For $i > \frac{p-1}{2}$ we proceed by induction. We note first that the relation holds also for the axes of the two cells, namely, $j = 0$ and $j = p$, by reasoning as in cases (i) and (iii). For a point $P(n,k)$ in the lower half of the rectangle, we have the recursion formula (12). Since we may assume that $0 < j < p$, since the cases $j = 0$ and $j = p$ are settled, we assume that

(19) is valid for all three terms $\left\{{n-1 \atop k-1}\right\}$, $\left\{{n-1 \atop k}\right\}$ and $\left\{{n-1 \atop k+1}\right\}$. From (12) we have

$$\overline{\left\{{n \atop k}\right\}} = \left(\ell\overline{\left\{{i-1 \atop j-1}\right\}} + r\overline{\left\{{i-1 \atop p-(j-1)}\right\}}\right) + \left(\ell\overline{\left\{{i-1 \atop j}\right\}} + r\overline{\left\{{i-1 \atop p-j}\right\}}\right)$$
$$+ \left(\ell\overline{\left\{{i-1 \atop j+1}\right\}} + r\overline{\left\{{i-1 \atop p-(j+1)}\right\}}\right). \tag{20}$$

Applying the recursion formula for the principal cell, we have

$$\overline{\left\{{i \atop j}\right\}} = \overline{\left\{{i-1 \atop j-1}\right\}} + \overline{\left\{{i-1 \atop j}\right\}} + \overline{\left\{{i-1 \atop j+1}\right\}}.$$

The corresponding recursion holds for $\overline{\left\{{i \atop p-j}\right\}}$; hence by rearranging the terms in (20), we verify the relation (19). □

Alternatively, (19) may be proved by the convolution identity (15):

$$\begin{aligned}
\overline{\left\{{n \atop k}\right\}} &= \sum_s \overline{\left\{{ap \atop s}\right\}} + \overline{\left\{{i \atop k-s}\right\}} \\
&= \overline{\left\{{ap \atop bp}\right\}\left\{{i \atop j}\right\}} + \overline{\left\{{ap \atop bp+p}\right\}\left\{{i \atop j-p}\right\}} \\
&= \ell\overline{\left\{{i \atop j}\right\}} + r\overline{\left\{{i \atop j-p}\right\}}.
\end{aligned}$$

Note 1. For all i in the region

$$j < p-i \Rightarrow \overline{\left\{{i \atop p-j}\right\}} = 0$$

and

$$j > i \Rightarrow \overline{\left\{{i \atop j}\right\}} = 0;$$

hence the region of overlap is defined by $p-i \leq j \leq i$ (this implies that $i > \frac{p}{2}$).

Note 2. As remarked earlier, the principal cell may have zero entries and so some cells may have zero heads. Trivially, the theorem is still valid for these cases, including the case when the left cell is "extreme"and the right cell is out of the array (hence all its entries are zero).

As in binomial arrays, the last row of the principal cell has a simple structure. For $p = 2$ and $p = 3$, the situations are trivial, the $(p-1)$-st rows being 1 1 1 and 1 2 0 2 1, respectively.

Let $p > 3$. Now

$$\overline{\left\{{p \atop -p}\right\}} = \overline{\left\{{p \atop 0}\right\}} = \overline{\left\{{p \atop p}\right\}} = 1 \quad \text{and} \quad \overline{\left\{{p \atop k}\right\}} = 0 \quad \text{for} \quad p \nmid k.$$

Thus, for $0 < i < p-2$,

$$\overline{\left\{ {p-1 \atop i-1} \right\}} + \overline{\left\{ {p-1 \atop i} \right\}} + \overline{\left\{ {p-1 \atop i+1} \right\}} = \overline{\left\{ {p-1 \atop i} \right\}} + \overline{\left\{ {p-1 \atop i+1} \right\}} + \overline{\left\{ {p-1 \atop i+2} \right\}} = 0;$$

hence

$$\overline{\left\{ {p-1 \atop i+2} \right\}} = \overline{\left\{ {p-1 \atop i-1} \right\}}$$

in the above range. Thus $\overline{\left\{ {p-1 \atop i} \right\}}$ is periodic with period = **3**. Furthermore

$$\overline{\left\{ {p \atop 0} \right\}} = \overline{\left\{ {p-1 \atop -1} \right\}} + \overline{\left\{ {p-1 \atop 0} \right\}} + \overline{\left\{ {p-1 \atop 1} \right\}} = 1$$

or

$$\overline{\left\{ {p-1 \atop 0} \right\}} + 2\overline{\left\{ {p-1 \atop 1} \right\}} = 1. \tag{21}$$

If $p \equiv 1 \pmod 3$, then by periodicity

$$\overline{\left\{ {p-1 \atop 0} \right\}} = \overline{\left\{ {p-1 \atop p-1} \right\}} = 1$$

and from (21)

$$\overline{\left\{ {p-1 \atop 1} \right\}} = 0;$$

so for $i \geq 0$ the entries form the sequence $1 \quad 0 \quad -1 \quad \ldots \quad 1$. If $p \equiv -1 \pmod 3$ then

$$\overline{\left\{ {p-1 \atop 1} \right\}} = \overline{\left\{ {p-1 \atop p-1} \right\}} = 1$$

and from (21)

$$\overline{\left\{ {p-1 \atop 0} \right\}} = -1.$$

In this case the entries for $i \geq 0$ are $-1 \quad 1 \quad 0 \quad -1 \quad 1 \quad 0 \quad \ldots \quad 1$.

It follows from this and (19) that the entries in row $p-1$ of Figure 7 are

$$\ell, r, -(\ell + r), \ell, r, -(\ell + r), \ldots, \ell, r, \quad \text{if} \quad p \equiv 1 \pmod 3$$

and

$$-\ell, \ell + r, -r, -\ell, \ldots, \ell + r, \ldots, \ell + r, -r, \quad \text{if} \quad p \equiv -1 \pmod 3.$$

The algorithm to find $\overline{\left\{ {n \atop k} \right\}}$ in the general case can now be developed as follows.

Let

$$n = a_m p^m + a_{m-1} p^{m-1} + \ldots + a_0$$

$$k = b_m p^m + b_{m-1} p^{m-1} + \ldots + b_0$$

be the expansions to base p of n and k, where $0 \leq a_s, b_s < p$ for $0 \leq s \leq m$ and $a_m > 0$, $b_m \leq a_m$.

Let

$$A_1 = \sum_{t=1}^{m} a_t p^{t-1}, \quad B_1 = \sum_{t=1}^{m} b_t p^{t-1}.$$

Then

$$n = A_1 p + a_0, \quad k = B_1 p + b_0.$$

The heads of the cells related to the entry $\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}}$ (referring to Figure 7) are

$$\ell = \overline{\begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix}} \quad \text{and} \quad r = \overline{\begin{Bmatrix} A_1 \\ B_1 + 1 \end{Bmatrix}}$$

by (18). Hence, by (19),

$$\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}} = \overline{\begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_0 \\ b_0 \end{Bmatrix}} + \overline{\begin{Bmatrix} A_1 \\ B_1 + 1 \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_0 \\ p - b_0 \end{Bmatrix}}.$$

Thus, if $\overline{\begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix}}$ and $\overline{\begin{Bmatrix} A_1 \\ B_1 + 1 \end{Bmatrix}}$ are known, then $\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}}$ can be found.

More generally, let

$$A_s = \sum_{t=s}^{m} a_t p^{t-s} \quad \text{and} \quad B_s = \sum_{t=s}^{m} b_t p^{t-s}.$$

Then $\overline{\begin{Bmatrix} A_{s+1} \\ B_{s+1} \end{Bmatrix}}$ and $\overline{\begin{Bmatrix} A_{s+1} \\ B_{s+1} + 1 \end{Bmatrix}}$ determine $\overline{\begin{Bmatrix} A_s \\ B_s \end{Bmatrix}}$ where $A_s = A_{s+1}p + a_s$ and $B_s = B_{s+1}p + b_s$. So

$$\overline{\begin{Bmatrix} A_s \\ B_s \end{Bmatrix}} = \overline{\begin{Bmatrix} A_{s+1} \\ B_{s+1} \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_s \\ b_s \end{Bmatrix}} + \overline{\begin{Bmatrix} A_{s+1} \\ B_{s+1} + 1 \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_s \\ p - b_s \end{Bmatrix}} \tag{22}$$

and $\overline{\begin{Bmatrix} A_s \\ B_s + 1 \end{Bmatrix}}$ is determined similarly. We note that

$$A_m = a_m, \quad B_m = b_m, \quad B_m + 1 = b_m + 1.$$

Hence, beginning with this, we have

$$\overline{\begin{Bmatrix} A_{m-1} \\ B_{m-1} \end{Bmatrix}} = \overline{\begin{Bmatrix} a_m \\ b_m \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_{m-1} \\ b_{m-1} \end{Bmatrix}} + \overline{\begin{Bmatrix} a_m \\ b_m + 1 \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_{m-1} \\ p - b_{m-1} \end{Bmatrix}}$$

$$\overline{\begin{Bmatrix} A_{m-1} \\ B_{m-1} + 1 \end{Bmatrix}} = \overline{\begin{Bmatrix} a_m \\ b_m \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_{m-1} \\ b_{m-1} + 1 \end{Bmatrix}} + \overline{\begin{Bmatrix} a_m \\ b_m + 1 \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_{m-1} \\ p - (b_{m-1} + 1) \end{Bmatrix}}$$

Proceeding step by step, using (22), we evaluate $\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}}$. The algorithm has been programmed, and gives fast results for large values of n and k. There is a closed formula consisting of sums of products (most of which are zero in practice), which we do not write down.

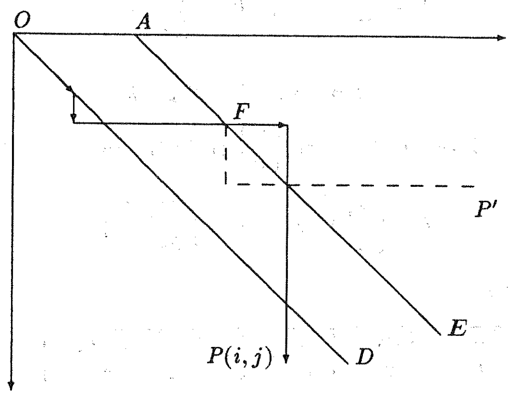

Figure 8

The algorithm simplifies to the familiar Lucas formula

$$\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}} = \overline{\begin{Bmatrix} a_m \\ b_m \end{Bmatrix}}\ \overline{\begin{Bmatrix} a_{m-1} \\ b_{m-1} \end{Bmatrix}} \cdots \overline{\begin{Bmatrix} a_0 \\ b_0 \end{Bmatrix}}$$

if and only if the second term in (19) is 0; this means, geometrically, that the coordinates (a_s, b_s) determine a point in the free part of the cell, or in the zero-hole. This happens *if and only if for all s in the expansions,* $a_s + b_s < p$.

4. RESTRICTED PATHS

In both cases discussed in this paper, further restraints may be placed on the king's route. When progress is allowed only in the positive direction (Section 2) one possible restraint proposed in [1] is to cut the chessboard diagonally, that is, to allow only positions (i, j) where $j \le i$. We denote by $u(i, j)$ the number of possible paths from $(0, 0)$ to (i, j) where each step, vertical, horizontal or diagonal is positively directed and $y \le x$ for each point (x, y) along the path. Tables (5) and (6) are computer outputs showing arrays of $u(i, j)$ mod 3 and $u(i, j)$ mod 7 respectively.

Let $\overline{u}(i, j) = u(i, j) \pmod p$. It is easy to evaluate $\overline{u}(i, j)$ for all values of (i, j), but the array has not the cell cluster structure in the sense that the array of the $\overline{w}(i, j)$ has. It has, however, some outstanding regularity features (inherited from the $\overline{w}(i, j)$ array).

A formula for the $u(i, j)$ function will be found first. The well known technique of *reflection* is used here.

On Figure 8, OD represents the line bisecting diagonally the chessboard, and the line AE is drawn parallel to it, originating at $A(0, 1)$.

The figure represents an "illegal" path, getting to (i, j) through some points in the "forbidden" half of the board. The point where the path first meets the line AE is F.

Leaving the section from 0 to F unchanged, reflect the section from F to P about AE. The reflection is shown by the dotted line ending at P'. The coordinates of P' are $(j-1, i+1)$. The reflection determines a one-to-one map of any illegitimate path to (i,j) to another path to $(j-1, i+1)$. Thus $u(i,j)$ is obtained by eliminating the illegal routes, and so

$$u(i,j) = w(i,j) - w(j-1, i+1). \tag{23}$$

Thus the problem of determining $\overline{u}(i,j)$ is reduced to finding $\overline{w}(i,j)$ and $\overline{w}(i+1, j-1)$.

Two noticeable features of the $\overline{u}(i,j)$ array are:
(i) the apparent preservation of the zero-holes of the $w(i,j)$ array,
(ii) the string of zeros running along diagonally at equal intervals.

Interpretation of (i) and (ii).
(i) From (23) and the symmetry of $\overline{w}(i,j)$ array it follows that

$$\overline{u}(i,j) = \overline{w}(i,j) - \overline{w}(i+1, j-1). \tag{24}$$

The zero-holes of the $\overline{w}(i,j)$ array represent square shaped cells or clusters containing $p \times p$, $p^2 \times p^2$, ...zero-entries. Exclude the first column and last row of the zero-hole. Then, for any entry (i,j) of the remaining array, $\overline{w}(i,j) = \overline{w}(i+1, j-1) = 0$; hence $\overline{u}(i,j) = 0$. Thus the $\overline{u}(i,j)$ array contains zero-holes beginning in rows divisible by p, p^2, ...; these holes are generally square shaped and of size $(p-1) \times (p-1)$, $(p^2-1) \times (p^2-1)$, ... At the cut-off edge, the squares are cut to triangular shape.

(ii) Consider the point (d,d) on the diagonal of some cell in the $\overline{w}(i,j)$ array. By the symmetry of the cell, $\overline{w}(d, d+1) = \overline{w}(d+1, d)$, as long as both $(d, d+1)$ and $(d+1, d)$ belong to the same cell. (This is not the case when (d,d) is in the last row.)

Hence by (24), $\overline{u}(d, d+1) = \overline{w}(d, d+1) - \overline{w}(d+1, d) = 0$, provided that $p \nmid (d+1)$.

This accounts for the diagonal strings of zeros, cut at points where i is a multiple of p.

Next we deal with restrictions on the minimal route (Section 3). Outputs are shown on tables (7) and (8), of the positive half of the array mod 3 and mod 7. Here the restriction consists of cutting away the negative part of the chessboard, while still allowing diagonal steps in the negative direction, as long as the whole path is confined to the positive side.

Let AD, shown on Figure 9, be the line $y+1=0$, and OFP an illegal path, with F the first intersection of this path with AD, and the path FP' the reflection of the FP section in AD. The coordinates of P' are $(n, -(k+2))$.

Denote by $\begin{bmatrix} n \\ k \end{bmatrix}$ the number of restricted paths between 0 and P. All illegal paths ending at P are bijectively mapped to P'. Excluding these, we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix} - \begin{Bmatrix} n \\ -(k+2) \end{Bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix} - \begin{Bmatrix} n \\ k+2 \end{Bmatrix} \tag{25}$$

Thus

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} - \left(\begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k+2 \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k+3 \end{Bmatrix} \right)$$

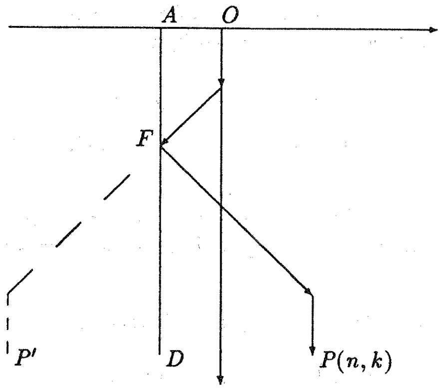

Figure 9

$$= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k+1 \end{bmatrix} \quad (k \geq 0)$$

with

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1.$$

Let

$$\overline{\begin{bmatrix} n \\ k \end{bmatrix}} = \begin{bmatrix} n \\ k \end{bmatrix} \pmod{p}.$$

By (25),

$$\overline{\begin{bmatrix} n \\ k \end{bmatrix}} = \overline{\begin{Bmatrix} n \\ k \end{Bmatrix}} - \overline{\begin{Bmatrix} n \\ k+2 \end{Bmatrix}} \tag{26}$$

From (26) it follows that

(i) $\overline{\begin{bmatrix} np \\ k \end{bmatrix}} = 0$ unless $p|k$ or $p|(k+2)$

(ii) $\overline{\begin{bmatrix} np \\ kp \end{bmatrix}} = \overline{\begin{Bmatrix} np \\ kp \end{Bmatrix}} - \overline{\begin{Bmatrix} np \\ kp+2 \end{Bmatrix}} = \overline{\begin{Bmatrix} np \\ kp \end{Bmatrix}} = \overline{\begin{Bmatrix} n \\ k \end{Bmatrix}} \quad (k \geq 0).$

Note that, in general, $\overline{\begin{bmatrix} np \\ kp \end{bmatrix}} \neq \overline{\begin{bmatrix} n \\ k \end{bmatrix}}$, and this means that the cell cluster structure of the $\overline{\begin{Bmatrix} n \\ k \end{Bmatrix}}$ array is not inherited.

However, an interesting feature of the array is the regularly spaced vertical columns of zeros, interrupted in the rows just above the rows where $n \equiv 0 \pmod{p}$. Consider

$$\overline{\begin{bmatrix} A_1 p + a_0 \\ B_1 p + b_0 \end{bmatrix}}, \quad \text{where} \quad b_0 = p-1 \quad \text{and} \quad 0 \leq a_0 < p-1.$$

Then, by (26),

$$\overline{\begin{bmatrix} A_1p + a_0 \\ B_1p + p - 1 \end{bmatrix}} = \overline{\begin{Bmatrix} A_1p + a_0 \\ (B_1 + 1)p - 1 \end{Bmatrix}} - \overline{\begin{Bmatrix} A_1p + a_0 \\ (B_1 + 1)p + 1 \end{Bmatrix}}$$

The two terms on the right hand side represent entries in a cell where the head is

$$\begin{Bmatrix} A_1p \\ (B_1 + 1)p \end{Bmatrix}.$$

If $a_0 < p - 1$, the entries are in the *free* part of the cell which is symmetrical; hence the terms are equal. Thus under the above conditions

$$\overline{\begin{bmatrix} A, p + a_0 \\ B, p + b_0 \end{bmatrix}} = 0.$$

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2. M. Sved, "Geometry of Combinatorial Arithmetic", Ars Combinatoria 21-A (1986), 271–298.
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```
 ARRAY DISPLAYING NO. OF WALKS MODULO      3
A, B, C,D       ARE        1    1    1    0


 0 111111111111111111111111111111111111111111111
 1 1.21.21.21.21.21.21.21.21.21.21.21.21.21.21.2
 2 121121121121121121121121121121121121121121121
 3 111...222111...222111...222111...222111...222
 4 1.2...2.11.2...2.11.2...2.11.2...2.11.2...2.1
 5 121...212121...212121...212121...212121...212
 6 111222111111222111111222111111222111111222111
 7 1.22.11.21.22.11.21.22.11.21.22.11.21.22.11.2
 8 121212121121212121121212121121212121121212121
 9 111111111.........222222222111111111.........
10 1.21.21.2.........2.12.12.11.21.21.2.........
11 121121121.........212212212121121121.........
12 111...222.........222...111111...222.........
13 1.2...2.1.........2.1...1.21.2...2.1.........
14 121...212.........212...121121...212.........
15 111222111.........222111222111222111.........
16 1.22.11.2.........2.11.22.11.22.11.2.........
17 121212121.........212121212121212121.........
18 111111111222222222111111111111111111222222222
19 1.21.21.22.12.12.11.21.21.21.21.21.22.12.12.1
20 121121121212212212121121121121121121212212212
21 111...222222...111111...222111...222222...111
22 1.2...2.12.1...1.21.2...2.11.2...2.12.1...1.2
23 121...212212...121121...212121...212212...121
24 111222111222111222111222111111222111222111222
25 1.22.11.22.11.22.11.22.11.21.22.11.22.11.22.1
26 121212121212121212121212121121212121212121212
27 111111111111111111111111111..................
28 1.21.21.21.21.21.21.21.21.2..................
29 121121121121121121121121121..................
30 111...222111...222111...222..................
31 1.2...2.11.2...2.11.2...2.1..................
32 121...212121...212121...212..................
33 111222111111222111111222111..................
34 1.22.11.21.22.11.21.22.11.2..................
35 121212121121212121121212121..................
36 111111111.........222222222..................
37 1.21.21.2.........2.12.12.1..................
38 121121121.........212212212..................
39 111...222.........222...111..................
40 1.2...2.1.........2.1...1.2..................
41 121...212.........212...121..................
42 111222111.........222111222..................
43 1.22.11.2.........2.11.22.1..................
44 121212121.........212121212..................
```

TABLE 1

TABLE 2

ARRAY DISPLAYING NO. OF WALKS MODULO 7
A, B, C,D ARE 1 1 1 0

```
 0 11111111111111111111111111111111111111111111111111111111111111111111111111111111111111
 1 135.246135.246135.246135.246135.246135.246135.246135.246135.246135.246135.246135.246135.2
 2 156465115646511564651156465115646511564651156465115646511564651156465115646511564651156465115646
 3 1.4.3.61.4.3.61.4.3.61.4.3.61.4.3.61.4.3.61.4.3.61.4.3.61.4.3.61.4.3.61.4.3.61.4.3.61.4.3
 4 126362112636211263621126362112636211263621126362112636211263621126362112636211263621126362112636
 5 145.236145.236145.236145.236145.236145.236145.236145.236145.236145.236145.236145.236145.2
 6 161616116161611616161161616116161611616161161616116161611616161161616116161611616161161616116161
 7 11111113333333555555.......22222224444444666666611111113333333555555.......22222224444
 8 135.246321.654514.362.......263.415456.123642.531135.246321.654514.362.......263.415456.1
 9 1564651314541354262445.......235153246323646213126156465131454135426245.......235153246323
10 1.4.3.63.5.2.45.6.1.2.......2.1.6.54.2.5.36.3.4.11.4.3.63.5.2.45.6.1.2.......2.1.6.54.2.5
11 12636213642463532123 5.......24565424135314651415612636213642463532123 5.......245654241353
12 145.236351.624564.312.......213.465426.153632.541145.236351.624564.312.......213.465426.1
13 1616161343434352525 25.......252525243434346161616161616134343435252525.......25252524343 4
14 111111155555556666666444444466666665555555111111111111115555555666666644444446666666655555
15 135.246514.362642.531456.123642.531514.362135.246135.246514.362642.531456.123642.531514.3
16 156465154262456213126463236462131265426245156465115646515426245621312646323646213126542 62
17 1.4.3.65.6.1.26.3.4.14.2.5.36.3.4.15.6.1.21.4.3.61.4.3.65.6.1.26.3.4.14.2.5.36.3.4.15.6.1
18 126362153212356514156413531465141565321235126362112636215321235651415641353146514156532 12
19 145.236564.312632.541426.153632.541564.312145.236145.236564.312632.541426.153632.541564.3
20 161616152525256161616434343461616165252525161616116161615252525616161643434346161616525 25
21 1111111.......4444444.......3333333.......66666661111111.......4444444.......3333333.....
22 135.246.......456.123.......321.654.......642.531135.246.......456.123.......321.654.....
23 1564651.......4632364.......3145413.......62131261564651.......4632364.......3145413.....
24 1.4.3.6.......4.2.5.3.......3.5.2.4.......6.3.4.11.4.3.6.......4.2.5.3.......3.5.2.4.....
25 1263621.......4135314.......3642463.......65141561263621.......4135314.......3642463.....
26 145.236.......426.153.......351.624.......632.541145.236.......426.153.......351.624.....
27 1616161.......4343434.......3434343.......61616161616161.......4343434.......3434343.....
28 111111122222226666666333333366666662222222111111111111112222222666666633333336666666622222
29 135.246263.415642.531321.654642.531263.415135.246135.246263.415642.531321.654642.531263.4
30 156465123515326213126314541362131262351532156465115646512351532621312631454136213126235 15
31 1.4.3.62.1.6.56.3.4.13.5.2.46.3.4.12.1.6.51.4.3.61.4.3.62.1.6.56.3.4.13.5.2.46.3.4.12.1.6
32 126362124565426514156364246365141562456542126362112636212456542651415636424636514156245 65
33 145.236213.465632.541351.624632.541213.465145.236145.236213.465632.541351.624632.541213.4
34 161616125252526161616343434361616162525252161616116161612525252616161634343436161616252 52
35 11111114444444555555.......22222223333333666666611111114444444555555.......22222233333
36 135.246456.123514.362.......263.415321.654642.531135.246456.123514.362.......263.415321.6
37 156465146323645426245.......235153231454136213126156465146323645426245.......235153231454
38 1.4.3.64.2.5.35.6.1.2.......2.1.6.53.5.2.46.3.4.11.4.3.64.2.5.35.6.1.2.......2.1.6.53.5.2
39 126362141353145321235.......245654236424636514156126362141353145321235.......245654236424
40 145.236426.153564.312.......213.465351.624632.541145.236426.153564.312.......213.465351.6
41 161616143434345252525.......252525234343436161616161616143434345252525.......25252523434 3
42 111111166666661111111666666611111116666666111111111111116666666111111166666661111111166666
43 135.246642.531135.246642.531135.246642.531135.246135.246642.531135.246642.531135.246642.5
44 15646516213126156465162131261564651621312615646511564651621312615646516213126156465162131
45 1.4.3.66.3.4.11.4.3.66.3.4.11.4.3.66.3.4.11.4.3.61.4.3.66.3.4.11.4.3.66.3.4.11.4.3.66.3.4
46 126362165141561263621651415612636216514156126362112636216514156126362165141561263621651 41
47 145.236632.541145.236632.541145.236632.541145.236145.236632.541145.236632.541145.236632.5
48 161616161616161616161616161616161616161616161616116161616161616161616161616161616161616161616
49 111111111111111111111111111111111111111111111111111111113333333333333333333333333333333333
50 135.246135.246135.246135.246135.246135.246135.246321.654321.654321.654321.654321.654321.6
51 156465115646511564651156465115646511564651156465131454133145413314541331454133145413314 54
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53 126362112636211263621126362112636211263621126362136424633642463364246336424633642463364 24
54 145.236145.236145.236145.236145.236145.236145.236351.624351.624351.624351.624351.624351.6
55 161616116161611616161161616116161611616161161616134343433434343343434334343433434343343 43
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59 1.4.3.63.5.2.45.6.1.2.......2.1.6.54.2.5.36.3.4.13.5.2.42.1.6.51.4.3.6.......6.3.4.15.6.1
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61 145.236351.624564.312.......213.465426.153632.541351.624213.465145.236.......632.541564.3
62 161616134343435252525.......252525243434346161616343434325252521616161.......616161652525
63 111111155555556666666444444466666665555555111111333333311111114444444555555544444444 11111
64 135.246514.362642.531456.123642.531514.362135.246321.654135.246456.123514.362456.123135.2
65 156465154262456213126463236462131265426245156465131454131564651463236454262454632364156 46
66 1.4.3.65.6.1.26.3.4.14.2.5.36.3.4.15.6.1.21.4.3.63.5.2.41.4.3.64.2.5.35.6.1.24.2.5.31.4.3
67 126362153212356514156413531465141565321235126362136424631263621413531453212354135314126 36
68 145.236564.312632.541426.153632.541564.312145.236351.624145.236426.153564.312426.153145.2
69 161616152525256161616434343461616165252525161616134343431616161434343452525254343434161 61
70 1111111.......4444444.......3333333.......66666663333333.......5555555.......2222222.....
71 135.246.......456.123.......321.654.......642.531321.654.......514.362.......263.415.....
72 1564651.......4632364.......3145413.......62131263145413.......5426245.......2351532.....
73 1.4.3.6.......4.2.5.3.......3.5.2.4.......6.3.4.13.5.2.4.......5.6.1.2.......2.1.6.5.....
74 1263621.......4135314.......3642463.......65141563642463.......5321235.......2456542.....
75 145.236.......426.153.......351.624.......632.541351.624.......564.312.......213.465.....
76 1616161.......4343434.......3434343.......61616163434343.......5252525.......2525252.....
77 111111122222226666666333333366666662222222111111133333336666666444444422222224444444466666
78 135.246263.415642.531321.654642.531263.415135.246321.654642.531456.123263.415456.123642.5
79 156465123515326213126314541362131262351532156465131454136213126463236423515324632364621 31
80 1.4.3.62.1.6.56.3.4.13.5.2.46.3.4.12.1.6.51.4.3.63.5.2.46.3.4.14.2.5.32.1.6.54.2.5.36.3.4
81 126362124565426514156364246365141562456542126362136424636514156413531424565424135314651 41
82 145.236213.465632.541351.624632.541213.465145.236351.624632.541426.153213.465426.153632.5
83 161616125252526161616343434361616162525252161616134343436161616434343425252524343434616 16
84 11111114444444555555.......22222223333333666666633333335555555111111.......66666622222
85 135.246456.123514.362.......263.415321.654642.531321.654514.362135.246.......642.531263.4
86 156465146323645426245.......235153231454136213126314541354262451564651.......621312623515
87 1.4.3.64.2.5.35.6.1.2.......2.1.6.53.5.2.46.3.4.13.5.2.45.6.1.21.4.3.6.......6.3.4.12.1.6
88 126362141353145321235.......245654236424636514156364246353212351263621.......651415624565
```

```
KING-WALK MODULO     3
A, B, C,D      ARE      1    1    0

 0  1
 1  111
 2  12.21
 3  1..1..1
 4  111111111
 5  12.......21
 6  1..2.....2..1
 7  111222...222111
 8  12.12.12.21.21.21
 9  1........1........1
10  111......111......111
11  12.21....12.21....12.21
12  1..1..1..1..1..1..1..1..1
13  111111111111111111111111111
14  12.........................21
15  1..2.......................2..1
16  111222.....................222111
17  12.12.12...................21.21.21
18  1........2.................2........1
19  111......222...............222......111
20  12.21....21.12.............21.12....12.21
21  1..1..1..2..2..2...........2..2..2..1..1..1
22  111111111222222222.........222222222111111111
23  12.......12.......12.......21.......21.......21
24  1..2.....1..2.....1..2.....2..1.....2..1.....2..1
25  111222...111222...111222...222111...222111...222111
26  12.12.12.12.12.12.12.12.12.21.21.21.21.21.21.21.21.21
27  1..........................1..........................1
28  111........................111........................111
29  12.21......................12.21......................12.21
30  1..1..1....................1..1..1....................1..1..1
31  111111111..................111111111..................111111111
32  12.......21................12.......21................12.......21
33  1..2.....2..1..............1..2.....2..1..............1..2.....2..1
34  111222...222111............111222...222111............111222...222111
35  12.12.12.21.21.21..........12.12.12.21.21.21..........12.12.12.21.21.21
36  1........1........1........1........1........1........1........1........1
37  111......111......111......111......111......111......111......111......111
38  12.21....12.21....12.21....12.21....12.21....12.21....12.21....12.21....12.21
39  1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1..1
40  111111111111111111111111111111111111111111111111111111111111111111111111111111111
41  12...............................................................................21
42  1..2.............................................................................2..1
43  111222...........................................................................222111
44  12.12.12.........................................................................21.21.21
```

TABLE 3

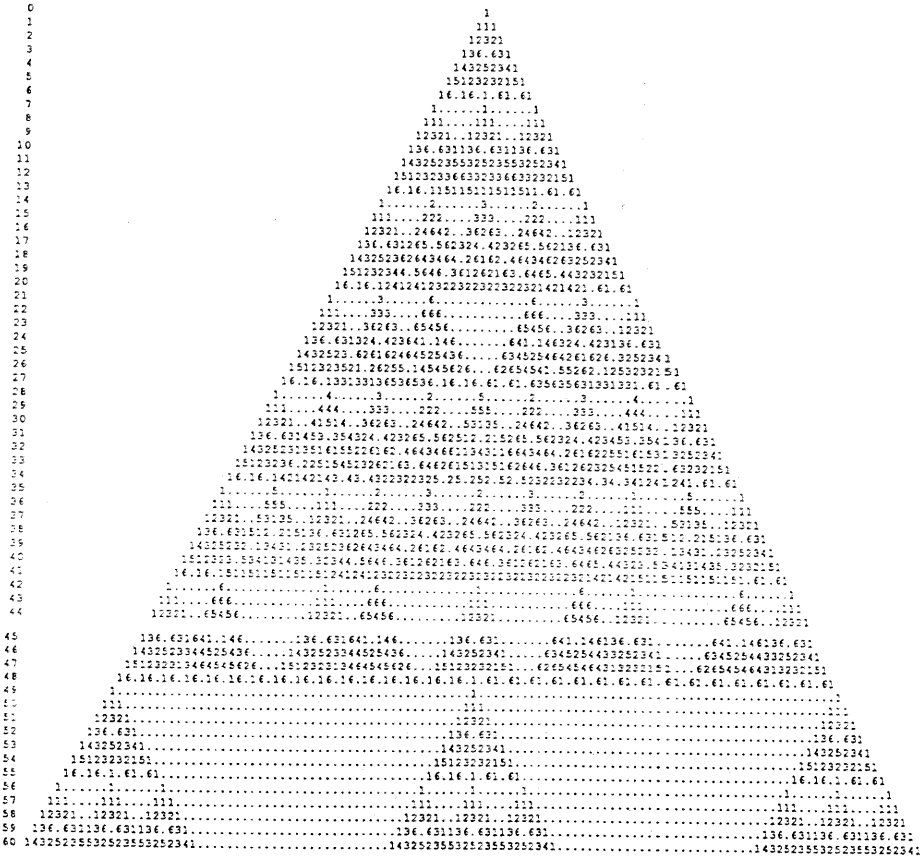

TABLE 4

KING-WALK MODULO 7
A, B, C, D ARE 1 1 0 1

```
RESTRICTED WALK MODULO        3
A, B, C,D        ARE         1     1     1     0


 0 1
 1 12
 2 11.
 3 1.11
 4 12.2.
 5 11.211
 6 1.1..2.
 7 12.11.22
 8 11.1.1.1.
 9 1.12.12.11
10 12..2..2.2.
11 11..211.21..
12 1.11..2.22...
13 12.2..21.1....
14 11.211..12.....
15 1.1..2..11......
16 12.11.22.2.......
17 11.1.1.1.211111111
18 1.12.12.1..21.21.2.
19 12..2..2.11..1..1.22
20 11..211.2.2..122.1.1.
21 1.11..2.21.22..1.12.11
22 12.2..21..1.1..12..2.2.
23 11.211..112.122..221.211
24 1.1..2..1..2..1..2..1..2.
25 12.11.22.11.22.11.22.11.22
26 11.1.1.1.1.1.1.1.1.1.1.1.1.
27 1.12.12.12.12.12.12.12.12.11
28 12..2..2..2..2..2..2..2..2.2.
29 11..211.221..211.221..211.21..
30 1.11..2.2..11..2.2..11..2.22...
31 12.2..21.22.2..21.22.2..21.1....
32 11.211..1.1.211..1.1.211..12.....
33 1.1..2..12.1..2..12.1..2..11......
34 12.11.22..2.11.22..2.11.22.2.......
35 11.1.1.1..212121211.2.2.2.21........
36 1.12.12.11........2.21.21.22.........
37 12..2..2.2........21..1..1.1..........
38 11..211.21........22..122.12...........
39 1.11..2.22........2.22..1.11............
40 12.2..21.1........21.1..12.2.............
41 11.211..12........22.122..21..............
42 1.1..2..11........2.2..1..22...............
43 12.11.22.2........21.22.11.1................
44 11.1.1.1.211111111..1.1.1.12.................
```

TABLE 5

```
0 1
1 12
2 146
3 1621
4 11256
5 136632
6 15.5.5.
7 1.531644
8 12.155126
9 146.621454
10 1621.142463
11 11256.543614
12 136632.224426
13 15.5.5.2656565
14 1.531646.415264
15 12.155143.544515
16 146.621662.235431
17 1621.144235.561156
18 11256.565342.446521
19 136632.412213.141145
20 15.5.5.42525256424242
21 1.531641......4.654324
22 12.15516......41.46643.
23 146.6211......423.3144..
24 1621.146......4314.423...
25 11256.51......44163.64....
26 136632.6......453351.3.....
27 15.5.5.6555555245454514444444
28 1.531643.362511.246132.123452
29 12.155114.233245.622646.311335
30 146.621315.542131.156254.463231
31 1621.141542.214156.634463.353156
32 11256.532435.316521.223614.126521
33 136632.165564.441145.424426.341145
34 15.5.5.1151515231313154343436646464
35 1.531645.654322.415265......2.362515
36 12.155131.466413.54452......24.233216
37 146.621523.314262.2355......215.542554
38 1621.143314.421235.562......2542.211463
39 11256.554163.625342.45......22435.353614
40 136632.353351.112213.2......265564.124426
41 15.5.5.3454545614141461111134141412545454
42 1.53164..24613..53164..24613..53164..24613.
43 12.155155.622622.155155.622622.155155.622622
44 146.621.31.156.46.621.31.156.46.621.31.156.46
45 1621.145156.632621.145156.632621.145156.632621
46 11256.5.6521.2.1256.5.6521.2.1256.5.6521.2.1256
47 136632.541145.236632.541145.236632.541145.236632
48 15.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.5.
49 1.531642.531642.531642.531642.531642.531642.531644
50 12.1551.2.1551.2.1551.2.1551.2.1551.2.1551.2.155126
51 146.621246.621246.621246.621246.621246.621246.621454
52 1621.14.621.14.621.14.621.14.621.14.621.14.621.142463
53 11256.521256.521256.521256.521256.521256.521256.543614
54 136632..36632..36632..36632..36632..36632..36632.224426
55 15.5.5..353535513131336161611464646662424244.2.2.241.1.1.
56 1.531644.123451.415265......2.362516.654323.246132.123455
57 12.155126.311323.54452......24.233221.466425.622646.311364
58 146.621454.463162.2355......215.542623.314331.156254.463515
59 1621.142463.352235.562......2542.212314.422156.634463.356542
60 11256.543614.115342.45......22435.364163.636521.223614.152435
61 136632.224426.212213.2......265564.253351.241145.424426.665564
62 15.5.5.2656565.36363613333335636363556565651.5.5.521.1.1.6414141
63 1.531646.415264.246131.654323.246134.415266.531644.123454.531645
64 12.155143.544515.622651.466425.622663.544532.155126.311352.155131
65 146.621662.235431.156323.314331.156462.235646.621454.463446.621523
66 1621.144235.561156.635314.422156.636235.563621.142463.355621.143314
67 11256.565342.446521.234163.636521.245342.461256.543614.141256.554163
68 136632.412213.141145.553351.241145.612213.336632.224426.536632.353351
69 15.5.5.42525256424242.34343465353532252525251.2.2.241.1.1.5616161456565656
70 1.531641......4.654324......3.123453......6.246132.123453......5.415265
71 12.15516......41.46643......36.31134......65.622646.31134......53.54452.
72 146.6211......423.3144......354.4633......631.156254.4633......562.2355..
73 1621.146......4314.423......3463.354......6156.634463.354......5235.562...
74 11256.51......44163.64......33614.13......66521.223614.13......55342.45....
75 136632.6......453351.3......324426.4......641145.424426.4......512213.2.....
76 15.5.5.6555555245454514444444.5454541555555554.5.5.521.1.1.41111116515151355555555
77 1.531643.362511.246132.123452.246131.362513.531644.123452.246123.654326.362516
78 12.155114.233245.622646.311335.622634.233262.155126.311335.622651.466454.233221
79 146.621315.542131.156254.463231.156115.542346.621454.463231.156323.314615.542623
80 1621.141542.214156.634463.353156.633542.216621.142463.353156.635314.425542.212314
81 11256.532435.316521.223614.126521.212435.331256.543614.126521.234163.662435.364163
82 136632.165564.441145.424426.341145.365564.636632.224426.341145.553351.565564.253351
83 15.5.5.1151515231313154343436646464.262665.2.2.241.1.1.3313131623232315252524454545
84 1.531645.654322.415265......2.362515.123452.246132.123451.415266.531641......6.246131
85 12.155131.466413.54452......24.233216.311335.622646.311323.544532.15516......65.622634
86 146.621523.314262.2355......215.542554.463231.156254.463162.235646.6211......631.156115
87 1621.143314.421235.562......2542.211463.353156.634463.352235.563621.146......6156.633542
88 11256.554163.625342.45......22435.353614.126521.223614.115342.461256.51......66521.212435
89 136632.353351.112213.2......265564.124426.341145.424426.212213.336632.6......641145.365564
90 15.5.5.3454545614141461111134141412545454..5.5.521.1.1.25151514353535433333325353536151515
```

TABLE 6

RESTRICTED WALK MODULO 7

A, B, C, D ARE 1 1 1 0

TABLE 7

RESTRICTED KING-WALK MODULO 3

A, B, C, D ARE 1 1 0 1

```
 0 1
 1 11
 2 221
 3 12.1
 4 ...11
 5 ..1221
 6 .1.22.1
 7 11.11.11
 8 222222221
 9 1......2.1
10 11....22.11
11 221..211.221
12 12.12.12.12.1
13 ............11
14 ...........1221
15 ..........1.22.1
16 .........11.11.11
17 ........1222222221
18 .......1.2......2.1
19 ......11.22....22.11
20 .....122.112..211.221
21 ....1.21.21.22.12.12.1
22 ...11.......11.......11
23 ..1221.....1221.....1221
24 .1.22.1...1.22.1...1.22.1
25 11.11.11.11.11.11.11.11.11
26 222222222222222222222222221
27 1........................2.1
28 11......................22.11
29 221....................211.221
30 12.1..................2.12.12.1
31 ...11................22.......11
32 ..1221..............2112.....1221
33 .1.22.1............2.11.2...1.22.1
34 11.11.11..........22.22.22.11.11.11
35 222222221........211111111.222222221
36 1......2.1......2.1......2.1......2.1
37 11....22.11....22.11....22.11....22.11
38 221..211.221..211.221..211.221..211.221
39 12.12.12.12.12.12.12.12.12.12.12.12.12.1
40 .......................................11
41 ......................................1221
42 .....................................1.22.1
43 ....................................11.11.11
44 ...................................1222222221
45 ..................................1.2......2.1
46 .................................11.22....22.11
47 ................................122.112..211.221
48 ...............................1.21.21.22.12.12.1
49 ..............................11.......11.......11
50 .............................1221.....1221.....1221
51 ............................1.22.1...1.22.1...1.22.1
52 ...........................11.11.11.11.11.11.11.11.11
53 ..........................1222222222222222222222222221
54 .........................1.2........................2.1
55 ........................11.22......................22.11
56 .......................122.112....................211.221
57 ......................1.21.21.2..................2.12.12.1
58 .....................11.......22................22.......11
59 ....................1221.....2112..............2112.....1221
60 ...................1.22.1...2.11.2............2.11.2...1.22.1
61 ..................11.11.11.22.22.22..........22.22.22.11.11.11
62 .................122222222.111111112........211111111.222222221
63 ................1.2......1.2......1.2......2.1......2.1......2.1
64 ...............11.22....11.22....11.22....22.11....22.11....22.11
65 ..............122.112..122.112..122.112..211.221..211.221..211.221
66 .............1.21.21.21.21.21.21.21.21.22.12.12.12.12.12.12.12.12.1
67 ............11.........................11.........................11
68 ...........1221.......................1221.......................1221
69 ..........1.22.1.....................1.22.1.....................1.22.1
70 .........11.11.11...................11.11.11...................11.11.11
71 ........1222222221.................1222222221.................1222222221
72 .......1.2......2.1...............1.2......2.1...............1.2......2.1
73 ......11.22....22.11.............11.22....22.11.............11.22....22.11
74 .....122.112..211.221...........122.112..211.221...........122.112..211.221
75 ....1.21.21.22.12.12.1.........1.21.21.22.12.12.1.........1.21.21.22.12.12.1
76 ...11.......11.......11.......11.......11.......11.......11.......11.......11
77 ..1221.....1221.....1221.....1221.....1221.....1221.....1221.....1221.....1221
78 .1.22.1...1.22.1...1.22.1...1.22.1...1.22.1...1.22.1...1.22.1...1.22.1...1.22.1
79 11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11.11
80 222222222222222222222222222222222222222222222222222222222222222222222222222222221
```

TABLE 8

RESTRICTED KING-WALK MODULO 7

A, B, C,D ARE 1 1 0 1

```
 0 1
 1 11
 2 221
 3 4531
 4 25241
 5 .24.51
 6 2662661
 7 1....6.1
 8 11..66.11
 9 221655.221
10 452523.4531
11 245235.25241
12 644331..24.51
13 3.43.412662661
14 3....5.2....6.1
15 33..55.22..66.11
16 663533.442655.221
17 51.446.135623.4531
18 6651.3.42.645.25241
19 535643.661312..24.51
20 16.16.256356322662661
21 .....1.6....4.3....6.1
22 ....11.66..44.33..66.11
23 ...122.556411.663655.221
24 ..1354.321462.511.23.4531
25 .14252.56.451.6.2355.25241
26 15.42..443236.615363..24.51
27 66266241421425.52.5232662661
28 5....5.2....4.3....3.4....6.1
29 55..55.22..44.33..33.44..66.11
30 335533.442411.663366.114655.221
31 646646.133.62.515515.264123.4531
32 322223.4.6211.644446.154.65.25241
33 5.66.5.431242.3.55.3.632344..24.51
34 5455452.16.165313313224124142662661
35 2....4.3....5.2....6.1....2.5....6.1
36 22..44.33..55.22..66.11..22.55..66.11
37 442411.663533.442655.221244.335655.221
38 133.62.51.446.135623.455.31.64.223.4531
39 4.6211.6651.3.42.645.2.3144.3364.5.25241
40 431242.535643.661312.254121.656325..24.51
41 .16.16.16.16.25635634.43.43.43.43.52662661
42 1...........1.6....6.1...........1.6....6.1
43 11.........11.66..66.11.........11.66..66.11
44 221.......122.556655.221.......122.556655.221
45 4531.....1354.323323.4531.....1354.323323.4531
46 25241...14252.511115.25241...14252.511115.25241
47 .24.51.15.42..6.33.6..24.51.15.42..6.33.6..24.51
48 2662662662662662662662662662662662662662662662661
49 1..............................................6.1
50 11............................................66.11
51 221..........................................655.221
52 4531........................................6423.4531
53 25241......................................63525.25241
54 .24.51....................................62.35...24.51
55 2662661..................................6115115.2662661
56 1....6.1................................6.1....6.1....6.1
57 11..66.11..............................66.11..66.11..66.11
58 221655.221............................655.221655.221655.221
59 452523.4531..........................6423.452523.452523.4531
60 245235.25241........................63525.245235.245235.25241
61 644331..24.51......................62.35..644331.644331..24.51
62 3.43.412662661....................611511563.43.4.3.43.412662661
63 3....5.2....6.1..................6.1....5.2....4.3....5.2....6.1
64 33..55.22..66.11................66.11..55.22..44.33..55.22..66.11
65 663533.442655.221..............655.221533.442411.663533.442655.221
66 51.446.135623.4531............6423.451246.133.62.51.446.135623.4531
67 6651.3.42.645.25241..........63525.231.53.4.6211.6651.3.42.645.25241
68 535643.661312..24.51........62.35..564611.431242.535643.661312..24.51
69 16.16.256356322662661......611511554124125.16.16.16.16.256356322662661
70 .....1.6....4.3....6.1....6.1....4.3....1.6...........1.6....4.3....6.1
71 ....11.66..44.33..66.11..66.11..44.33..11.66.........11.66..44.33..66.11
72 ...122.556411.663655.221655.221411.663122.556.......122.556411.663655.221
73 ..1354.321462.511.23.452523.45.662.513654.3246.....1354.321462.511.23.4531
74 .14252.56.451.6.2355.245235.2245.1.623.12.52536...14252.56.451.6.2355.25241
75 15.42..443236.615363.644331.414261.145433..53.26.15.42..443236.615363..24.51
76 66266241421425.52.5223.43.4552.52.25365363511511.66266241421425.52.5232662661
77 5....5.2....4.3....2.5....2.5....4.3....5.2....2.5....5.2....4.3....3.4....6.1
78 55..55.22..44.33..22.55..22.55..44.33..55.22..22.55..55.22..44.33..33.44..66.11
79 335533.442411.663244.335244.335411.663533.442244.335533.442411.663366.114655.221
80 646646.133.62.514231.643431.645362.51.446.131131.646646.133.62.515515.264123.4531
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