ON POTENTIALLY P-GRAPHIC DEGREE SEQUENCES

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ABSTRACT

A sequence $\pi = (d_1, d_2, \ldots, d_n)$ of positive integers is said to be **graphic** if there exists a simple graph G such that π is the degree sequence of G. For a specified property P of graphs, a sequence $\pi = (d_1, d_2, \ldots, d_n)$ of positive integers is said to be **potentially P-graphic** if π is graphic and there exists a realization of π with the property P.

In this paper we characterize potentially P-graphic sequences where P is one of the following properties :

(i) connected and each block is a clique on k vertices.

(ii) connected and each block is a clique on k_1 or k_2 vertices.

1. INTRODUCTION

In this paper we consider finite undirected graphs without loops or multiple edges. For a graph G, let V(G) and E(G) denote the vertex and edge sets respectively. Let $deg_G(u)$ denote the degree of vertex u of G, that is, the number of edges in G incident at u. A sequence $\pi = (d_1, d_2, \ldots, d_n)$ of positive integers is called the **degree sequence** of graph G if the vertices of G can be labelled $u_1, u_2, \ldots, u_n$ such that $deg_G(u_i) = d_i$, $1 \le i \le n$. A sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be **graphic** if there exists a graph G such that π is the degree sequence of G. Then G is called a **realization** of π. All sequences in this paper are non-negative integer sequences in non-increasing order. Let $\omega(G)$ denote the number of connected components of G.

Let P be an invariant property of graphs. A graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be **potentially P-graphic** if there exists a realization of π with property P and it is said to be **forcibly P-graphic** if every realization of π has property P. Many authors have characterized potentially P-graphic and forcibly P-graphic sequences for various properties P. Some of the properties P considered in the literature are : k-edge connectedness (Edmonds [3]), k-vertex connectedness (Wang and Kleitman [8]), k-factorability (Rao and Rao [6], Kundu [4]) and many more. For a good survey of these problems the reader is referred to S.B.Rao [7].

A nontrivial connected graph with no cut vertices is called a **block**. A **block of a graph** G is a subgraph of G that is a block and it is maximal with respect to this property.

Consider the property P : G is connected and each block is a clique. For this property P, the problem of characterizing the potentially P-graphic sequences was suggested by A.Ramachandra Rao [5]. We define **property** $\mathbf{P(k_1, k_2, \ldots, k_r)}$ to be : G is connected and each block is a clique on k vertices where k is one of the specified r distinct positive integers $k_1, k_2, \ldots, k_r$. In this paper we characterize potentially $P(k_1, k_2, \ldots, k_r)$-graphic sequences for r = 1 and 2. In section 2 we present some of the results for general $P(k_1, k_2, \ldots, k_r)$. In section 3 we discuss the main results for $P(k_1, k_2, \ldots, k_r)$, r = 1 and 2.

2. RELATED RESULTS

We assume that $k_1, k_2, \ldots, k_r$ are r distinct integers greater than or equal to 2. Let $\pi = (d_1, d_2, \ldots, d_n)$ be a potentially $P(k_1, k_2, \ldots, k_r)$-graphic sequence and G a realization of π with the property $P(k_1, k_2, \ldots, k_r)$. With reference to G , we introduce the following notation :

b – Number of blocks of G;

b_i – Number of blocks of order k_i, $1 \leq i \leq r$;

$\alpha_i(v)$ - Number of blocks of order k_i containing the vertex v of G, $1 \le i \le r$;

$b(v)$ - Number of blocks of G containing the vertex v;

n_i - Number of vertices with degree $(k_i - 1)$, $1 \le i \le r$;

G_i - The subgraph of G such that $V(G_i) = \{ u / u$ belongs to a block of order $k_i \}$ and $E(G_i) = \{ e / e$ belongs to a block of order $k_i \}$.

Some of the following facts are immediate and the rest can be found in any text book (Bondy and Murty [2], Berge [1]). For a graph G on n vertices with degree sequence $\pi = (d_1, d_2, \ldots, d_n)$ and property $P(k_1, k_2, \ldots, k_r)$, we have the following :

Fact 1. $b(v) = \sum_{i=1}^{r} \alpha_i(v)$.

Fact 2. $n = 1 + \sum_{i=1}^{r} b_i(k_i - 1)$.

Fact 3. $\sum_{j=1}^{n} d_j = \sum_{i=1}^{r} b_i \, k_i(k_i - 1)$.

Fact 4. For a vertex v with degree d_j,

$$d_j = \deg(v) = \sum_{i=1}^{r} [(k_i - 1)\, \alpha_i(v)].$$

Fact 5. $b = 1 + \sum_{v} (b(v) - 1)$.

Fact 6. $\sum_{i=1}^{r} b_i = 1 + \sum_{v} \left(\left(\sum_{i=1}^{r} \alpha_i(v) \right) - 1 \right)$.

Fact 7. There exists at least one i such that $n_i \ge (k_i - 1)$, $1 \le i \le r$.

Fact 8. $\sum_{v \in V(G_i)} (\alpha_i(v) - 1) = b_i - \omega(G_i) \le b_i - 1$, $1 \le i \le r$.

From the definition of $P(k_1,k_2,\ldots,k_r)$ the following can be easily observed.

Fact 9. If π is potentially $P(k_1,k_2,\ldots,k_q)$-graphic then π is potentially $P(k_1,k_2,\ldots,k_m)$-graphic, where $m > q$.

3. MAIN RESULTS

In the following theorem we characterize potentially $P(k)$ degree sequences.

Theorem 1. Let $k \geq 2$ be an integer and $\pi = (d_1,d_2,\ldots,d_n)$ be a non-increasing sequence of positive integers. π is potentially $P(k)$ - graphic if and only if there exist positive integers b and α_j, $1 \leq j \leq n$ such that

(i) $n = 1 + b(k - 1)$.

(ii) $\sum_{j=1}^{n} d_j = k(n - 1)$.

(iii) $d_j = \alpha_j(k - 1)$, $1 \leq j \leq n$ where $1 \leq \alpha_j \leq b$.

Proof. Let π be potentially $P(k)$-graphic . Then there exists a graph G, a realization of π with property $P(k)$. Now invoking Facts 2,3,4 and 7 we have the necessary conditions (i) to (iii).

The proof of sufficiency is by induction on b. Now let π satisfy the conditions (i) to (iii). For the basis of induction ,if b = 1 it is easy to see from conditions (i) to (iii) that k = n and $d_j = n - 1$, $1 \leq j \leq n$. Thus K_n the complete graph on n vertices has the degree sequence π and the property $P(k)$, proving the basis of induction. Now we make the induction hypothesis that the suffciency is true for b - 1 and prove it for $b \geq 2$. The proof of the inductive step depends on a procedure of laying off $(k - 1)$ vertices of degree $(k - 1)$ and reducing the degree of a suitable vertex by $(k - 1)$ such that the new degree sequence π' satisfies conditions (i) to (iii).

With reference to π let $n_1 = |\{j \mid d_j = k-1;\ 1 \le j \le n\}|$. Note that $n_1 < n$ since $b \ge 2$. Using conditions (iii) in (ii) and rewriting it we get,

$$(k-1)\left(\sum_{j=1}^{n}(\alpha_j - 1)\right) + (k-1) = (n - n_1) + (n_1 - 1) \tag{1}$$

Note that,

$$\sum_{j=1}^{n}(\alpha_j - 1) = \sum_{j=1}^{n-n_1}(\alpha_j - 1) \ge (n - n_1) \tag{2}$$

Combining (1) and (2) we can conclude that $n_1 \ge k$.

Now define $\pi' = (d_1', d_2', \ldots, d_{n'}')$ where $n' = n-k+1$ as follows :

$$d_j' = \begin{cases} d_j, & 1 \le j \le n - n_1 - 1 \\ d_j - (k-1), & j = n - n_1 \\ d_j, & n - n_1 + 1 \le j \le n'. \end{cases}$$

Note that π' is well defined since $n_1 \ge k$. It is easy to verify that π' satisfies conditions (i) to (iii) with $b' = b - 1$. Thus using the induction hypothesis π' is potentially $P(k)$ -graphic. Now let G' be a realization of π' with property $P(k)$. From G' obtain a G by attaching a K_k to a vertex of degree $(d_{n-n_1} - (k-1))$ in G'. It is easy to see that G has the degree sequence π and the property $P(k)$. This completes the proof of the inductive step and hence the theorem.

□

We note that Theorem 1 covers the following known results :

(i) When $k = 2$, it characterizes the degree sequence of a tree on n vertices (Bondy and Murty [2], p. 27).

(ii) When $k = 3$, it characterizes the degree sequence of a connected graph in which each block is a triangle (A. Ramachandra Rao [5]).

In theorem 2, we characterize potentially $P(k_1,k_2)$-graphic sequences.

Theorem 2. Let $k_1 \ge k_2 \ge 2$ be integers and $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of positive integers. π is potentially $P(k_1,k_2)$ - graphic if and only if there exist non-negative integers b_1 , b_2 ; α_j , β_j , for $1 \le j \le n$ such that

(i) $n = 1 + b_1(k_1 - 1) + b_2(k_2 - 1)$.

(ii) $\sum_{j=1}^{n} d_j = b_1\, k_1(k_1 - 1) + b_2\, k_2(k_2 - 1)$.

(iii) $d_j = \alpha_j\,(k_1 - 1) + \beta_j(k_2 - 1)$, $0 \le \alpha_j \le b_1$, $0 \le \beta_j \le b_2$,

$\alpha_j + \beta_j \ge 1$; for $1 \le j \le n$.

(iv) $\sum_{j=1}^{n}(\alpha_j + \beta_j - 1) = b_1 + b_2 - 1$.

(v) $\sum_{\substack{j=1 \\ \alpha_j \ge 1}}^{n} (\alpha_j - 1) \le b_1 - 1$ and $\sum_{\substack{j=1 \\ \beta_j \ge 1}}^{n} (\beta_j - 1) \le b_2 - 1$.

Proof. Let π be potentially $P(k_1,k_2)$-graphic. Then there exists a realization G of π with property $P(k_1,k_2)$. Using this graph G, define b_i = the number of blocks of order k_i, $1 \le i \le 2$. For a vertex v with degree d_j , define $\alpha_j = \alpha_1(v)$ and $\beta_j = \alpha_2(v)$ where $\alpha_i(v)$ = the number of blocks of order k_i containing the vertex v, $1 \le i \le 2$. Then conditions (i) and (ii) follow from Facts 2 and 3 respectively. The condition (iii) is immediate from Facts 4, 7 and the fact that $\alpha_i(v) \le b_i$, for $1 \le i \le 2$. The conditions (iv) and (v) follow from the Facts 6 and 8 respectively. This completes the necessity part of the proof.

The proof of sufficiency is by induction on $b_1 + b_2$. Assume that the π satisfies the conditions (i) to (v). For the basis of induction assume that $b_1 + b_2 \le 2$. Now if either $b_1 = 0$ or $b_2 = 0$, then the condition (i) to (v) reduce to conditions (i) to (iii) of Theorem 1 and thus π is potentially $P(k_1)$-graphic and hence $P(k_1,k_2)$-graphic, using Fact 9. Next if $b_1 = 1$ and $b_2 = 1$, from condition (iii) we note that the possible degrees in π are $k_1 + k_2 - 2$, $k_1 - 1$ and $k_2 - 1$. Further, using conditions (i), (ii) and (iv) we can prove that

$$d_i = \begin{cases} k_1 + k_2 - 2, & \text{if } i = 1 \\ k_1 - 1\ , & \text{if } 2 \le i \le k_1 \\ k_2 - 1\ , & \text{if } k_1 + 1 \le i \le n. \end{cases}$$

Now construct a graph G by taking a K_{k_1} and to exactly one of these k_1 vertices attach a K_{k_2}. It is easy to show that G is a realization of π with property $P(k_1,k_2)$ and thus π is potentially $P(k_1,k_2)$-graphic.

This completes the proof of the basis of induction. Make the induction hypothesis that the sufficiency is true for $b_1 + b_2 - 1$ and prove it for $b_1 + b_2 \geq 3$. We prove this inductive step by cases.

Case A : either $b_1 = 0$ or $b_2 = 0$.

In this case, conditions (i) to (v) reduce to conditions (i) to (iii) of Theorem 1. Now using Theorem 1 and Fact 9 we conclude that π is potentially $P(k_1,k_2)$-graphic.

Case B : $b_1 \geq 1$ and $b_2 \geq 1$.

Using conditions (i) and (iv) we can easily show that

$$n_1 \geq (k_1 - 1) \text{ or } n_2 \geq (k_2 - 1)$$

where, $n_1 = |\{j \,/\, \alpha_j = 1,\ \beta_j = 0 \,;\ 1 \leq j \leq n\}|$

and $n_2 = |\{j \,/\, \alpha_j = 0\ ,\ \beta_j = 1;\ 1 \leq j \leq n\}|$.

Subcase (i): $n_2 \geq (k_2 - 1)$.

Now in order to accomplish the laying off procedure we prove the existence of a degree d_t such that $d_t = \beta_t(k_2 - 1)$ with $\beta_t \geq 2$ or $d_t = \alpha_t(k_1 - 1) + \beta_t(k_2 - 1)$ with $\alpha_t \geq 1$ and $\beta_t \geq 1$. Suppose such a d_t does not exist. Then condition (iv) can be rewritten as

$$\sum_{\substack{j \\ \alpha_j \geq 1 \\ \beta_j \geq 1}} (\alpha_j + \beta_j - 1) + \sum_{\substack{j \\ \alpha_j \geq 1 \\ \beta_j = 0}} (\alpha_j - 1) + \sum_{\substack{j \\ \alpha_j = 0 \\ \beta_j \geq 1}} (\beta_j - 1) = b_1 + b_2 - 1.$$

That is , $$\sum_{\substack{j \\ \alpha_j \geq 1 \\ \beta_j = 0}} (\alpha_j - 1) = b_1 + b_2 - 1 .$$

Thus we get $$\sum_{\alpha_j \geq 1} (\alpha_j - 1) = \sum_{\substack{\alpha_j \geq 1 \\ \beta_j = 0}} (\alpha_j - 1) = b_1 + b_2 - 1$$

and this contradicts condition (v). Thus the existence of a d_t as required is proved. From the set of such d_t's, choose the one with the largest β_t. Now construct a degree sequence π' with $b_1 + b_2$ reduced by one by laying off $(k_2 - 1)$ vertices of degree $(k_2 - 1)$ and reducing the degree d_t by $(k_2 - 1)$. More precisely define $\pi' = (d_1', d_2', \ldots, d_{n'}')$ where $n' = n - k_2 + 1$ as follows :

$$d_j' = \begin{cases} d_j , & j \neq t, \ 1 \leq j \leq n' \\ d_j - (k_2 - 1), & j = t. \end{cases}$$

With reference to the sequence π', let us define b_1', b_2'; α_j', β_j', $1 \leq j \leq n'$ as follows :

$b_1' = b_1$, $\quad \alpha_j' = \alpha_j$, $1 \leq j \leq n'$

$b_2' = b_2 - 1$, $\quad \beta_j' = \beta_j$, $1 \leq j \leq n'$ and $j \neq t$,

$\beta_t' = \beta_t - 1$

Now using the facts $n' = n - k_2 + 1$, $\alpha_j = 0$ and $\beta_j = 1$ for $n' + 1 \leq j \leq n$, it is easy to verify that π' satisfies conditions (i) to (v). Now by the induction hypothesis obtain a realization G' of π' with the property $P(k_1,k_2)$. From G' construct a graph G by attaching a K_{k_2} to a vertex of degree d_t' in G'. Note that G is a realization of π and also has property $P(k_1,k_2)$. Thus π is potentially $P(k_1,k_2)$-graphic.

Subcase (ii) : $n_2 < (k_2 - 1)$.

In this case note that $n_1 \geq (k_1 - 1)$. Now the laying off procedure will be accomplished using degrees $(k_1 - 1)$ and the arguments can be provided along the same lines as in the subcase (i). This completes the sufficiency proof in all the cases and hence the theorem is proved. □

A. Ramachandra Rao characterized potentially Q-graphic sequences (Theorem 1, [5]) where property Q is : the graph is connected and each block is an edge or a triangle. Note that property Q is equivalent to property P(3,2) . Thus Theorem 2 covers the characterization of potentially Q-graphic sequences.

In the following we provide examples of sequences to demonstrate that none of the five conditions of Theorem 2 is redundant. In these examples, we express the sequence using the convention that an integer

d_i occurring s_i times in the sequence is represented by $d_i^{s_i}$.

1. $\pi = (10,4,2^{12})$ for $k_1 = 5$, $k_2 = 3$:
   Note that n = 14. Consider $b_1 = 1$, $b_2 = 3$, $\alpha_1 = 1$, $\beta_1 = 3$, $\alpha_2 = 1$, $\beta_2 = 0$, $\alpha_3 = 0$, and $\beta_3 = 1$. (Here α_i and β_i are used to express d_i in condition (iii) for $1 \le i \le 3$ where $d_1 = 10$, $d_2 = 4$ and $d_3 = 2$.). It is easy to see that they satisfy conditions (ii) to (v) but not (i).
2. $\pi = (10,4^2,2^{12})$ for $k_1 = 5$, $k_2 = 3$:
   Note that $b_1 = 1$, $b_2 = 5$, $\alpha_1 = 1$, $\beta_1 = 3$, $\alpha_2 = 0$, $\beta_2 = 2$, $\alpha_3 = 0$ and $\beta_3 = 1$ satisfy conditions (i),(iii),(iv) and (v), but not (ii).
3. $\pi = (6,3^6,2^3)$ for $k_1 = 4$, $k_2 = 3$:
   It is easy to show that $b_1 = 1$, $b_2 = 3$, $\alpha_1 = 1$, $\beta_1 = 3$, $\alpha_2 = 1$, $\beta_2 = 0$, $\alpha_3 = 0$ and $\beta_3 = 1$ satisfy conditions (i), (ii),(iv) and (v), but not (iii).
4. $\pi = (6,3^6,2^3)$ for $k_1 = 4$, $k_2 = 3$:
   In this case $b_1 = 1$, $b_2 = 3$, $\alpha_1 = 0$, $\beta_1 = 3$, $\alpha_2 = 1$, $\beta_2 = 0$, $\alpha_3 = 0$, and $\beta_3 = 1$ satisfy conditions(i),(ii),(iii) and (v), but not (iv).
5. $\pi = (5,4^2,3^7,2)$ for $k_1 = 4$, $k_2 = 3$:
   Now $b_1 = 2$, $b_2 = 2$, $\alpha_1 = 1$, $\beta_1 = 1$, $\alpha_2 = 0$, $\beta_2 = 2$, $\alpha_3 = 1$, $\beta_3 = 0$, $\alpha_4 = 0$ and $\beta_4 = 1$ satisfy conditions (i) to (iv) but not (v).

One can easily verify that there is no graph with the property $P(k_1,k_2)$, realizing any of the above sequences. This establishes the fact that none of the five conditions of Theorem 2 is redundant.

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