Cycles Containing a Set of Elements in Cubic Graphs

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Abstract

In this paper, we obtain a necessary and sufficient condition for a 3-connected cubic graph to have a cycle containing any set of nine vertices and an edge. We also prove that in every 3-connected cubic planar graph any set of fourteen vertices and an edge is contained in a cycle. As there is a 3-connected cubic planar graph that has a set of fifteen vertices and an edge not lying on any cycle, the result is the best possible.

1 Introduction

We consider 3-connected cubic graphs. The connectivity of a graph G is denoted by $\kappa(G)$ and if $S \subset V(G)$ then $K(G-S)$ denotes the components of $G-S$. A component containing a cycle is called a *cyclic component*. A *cyclic cut set* is a cut set S with each component of $G-S$ cyclic. The *cyclic connectivity* of a graph is the size of a minimal cyclic cut set of the graph. The *coboundary* $B(H,J)$ of subgraphs H and J of G is the set of edges of G with one end in H and the other in J.

Let G be a graph and let R be a spanning subgraph of G. Define a graph H with $V(H) = K(R)$ in which distinct vertices $x, y \in V(H)$ are adjacent if there is an edge of G between the components x and y of R. This graph H is called a *contraction* of G and denoted $H = G/R$. Roughly speaking, each component of R is contracted to a single vertex in the contraction G/R, while keeping the adjacencies between components. If R is a spanning tree of G then $G/R = K_1$, and if $R = G - E(G)$ then $G/R = G$. These two contractions are called *trivial*.

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The subgraph R defines an equivalence relation on $V(G)$ where vertices of G are equivalent if they lie in the same component of R. This defines a function

$$\alpha : G \longrightarrow H.$$

This function is also called a *contraction* of G onto H. If such a contraction α maps a set A of elements of G to a set B of elements of H, then we write

$$\alpha : (G, A) \longrightarrow (H, B)$$

or $\alpha(G, A) = (H, B)$.

Let $e \in E(G)$. If e lies on every cycle containing A then e is called an *unavoidable* edge given A. If e is excluded by every cycle containing A then it is called a *forbidden* edge given A. In a hamiltonian graph G, an unavoidable edge given $V(G)$ is called an *a-edge* and a forbidden edge given $V(G)$ is called a *b-edge*.

Let $A, B \subset V(G) \cup E(G)$ and $A \cap B = \emptyset$. If $G - B$ has a cycle containing A, then we say that A is *cyclable* in $G - B$ and denote this fact by $A \in C(G - B)$. Let $p = (m, n; m', n')$. If for every A and B with $|V(A)| = m$, $|E(A)| = n$, $|V(B)| = m'$ and $|E(B)| = n'$ we have $A \in C(G - B)$, then we say that G is a *p-cyclable* graph. Denote this fact by $G \in C(m, n; m', n')$. The quadruple p of integers is called the *type* of the pair $M = (A, B)$. The parameter

$$\xi(G) = \max\{m : G \in C(m, 0; 0, 0)\}$$

has been called the cyclability of G and studied extensively. If $G \in C(m, 0; 0, 0)$ then we simply call G an *m-cyclable* graph. We may define parameters

$$\eta(G) = \max\{m : G \in C(m, 0; 0, 1)\}$$

and

$$\zeta(G) = \max\{m : G \in C(m, 1; 0, 0)\}.$$

These two parameters inform us about the unavoidable edges and forbidden edges of G.

A contraction $\alpha : G \rightarrow H$ is said to be a *p-primitive contraction* if (1) $A \notin C(G - B)$ implies $\alpha(A) \notin C(H - \alpha(B))$, (2) the pairs (A, B) and $(\alpha(A), \alpha(B))$ have the same type, and (3) $|V(H)|$ is the smallest with respect to (1) and (2). In this case, H is called a *p-primitive graph* and the pair $(\alpha(A), \alpha(B))$ is called a p-primitive pair of H.

Let $e \in E(G)$ and denote by $G + x_e$ the graph resulting from the subdivision of the edge e with a vertex $x_e \notin V(G)$. In Section 2 the integer $\eta(e) = \xi(G - e)$ assigned to e will frequently be used.

The following *nine point theorem* (see [15] and [16]) is well known.

Theorem 1.1 *Any 3-connected cubic graph is 9-cyclable.*

Let P be the Petersen graph. If there is a contraction $(G, A) \rightarrow (P, V(P))$ then clearly $A \notin C(G)$. As $|V(P)| = 10$, the above theorem is sharp. It was shown that this particular contraction determines noncyclable sets of ten (see [10]) and eleven (see [3]) vertices. Holton and the author [7] have recently shown that if G is a 3-connected cubic graph and $A \subset V(G)$ with $|A| = 12$, then either $A \in C(G)$ or there is a contraction $\alpha : (G, A) \rightarrow (P, V(P))$.

Results such as the following theorem (see [15]) have been frequently applied in the study of cyclability of graphs.

Theorem 1.2 *Let G be a 3-connected cubic graph and $A \subset V(G)$ with $|A| = 5$. Then for any $e \in E(G)$, $A \in C(G - e)$.*

This theorem cannot be improved without introducing exceptional graphs. The Petersen graph P can be presented by taking two disjoint 5-cycles [1, 2, 3, 4, 5, 1] and [6, 8, 10, 7, 9, 6] and joining a vertex u of the first cycle and a vertex v of the second if $u \equiv v \pmod 5$. A graph Q can be constructed using P. Subdivide the edge [3, 4] with a vertex 11 and the edge [7, 10] with a vertex 12 and introduce the new edge [11, 12]. The resulting graph is Q. Let $A_P = \{1, 3, 4, 6, 7, 10\}$ and $e_P \in \{[1, 6], [7, 10], [3, 4]\}$. Then $A_P \notin C(P - e_P)$. Also if $A_Q = A_P$ and $e_Q = [1, 6]$ then $A_Q \notin C(Q - e_Q)$. The *twisted cube* is the graph

$$W = [1, 2, 3, 4, 1] \cup [5, 6, 7, 8, 5] \cup \{[1, 6], [2, 5], [3, 7], [4, 8]\}.$$

Let $B_W = \{1, 2, 7, 8, [3, 4], [5, 6]\}$ and $B_P = \{2, 5, 8, 9, [3, 4], [7, 10]\}$. Then $B_W \notin C(W)$ and $B_P \notin C(P)$.

Theorem 1.2 has been extended to cycles containing six vertices and avoiding an edge in [10], where cycles through a set of four vertices and two edges were also studied.

Theorem 1.3 *Let G be a 3-connected cubic graph, $A \subset V(G)$ with $|A| = 6$ and $e \in E(G)$. Then $A \in C(G - e)$ unless there is a contraction $\alpha : G \longrightarrow P$ such that $\alpha(A) = A_P$ and $\alpha(e) = e_P$, or a contraction $\beta : G \longrightarrow Q$ such that $\beta(A) = A_Q$ and $\beta(e) = e_Q$.*

Theorem 1.4 *Let G be a 3-connected cubic graph and let A be a set of four vertices and two edges of G. Then $A \in C(G)$ unless there is a contraction $\alpha : G \longrightarrow W$ such that $\alpha(A) = B_W$, or a contraction $\beta : G \longrightarrow P$ such that $\beta(A) = B_P$.*

Theorem 1.3 has been extended to $|A| = 7$ in [1]. We freely refer to [1] and do not discuss the details here. The theorem of [1] has ten families of primitive graphs. It is tedious to determine the unavoidable edges given a set of eight or more vertices. The following result in [6] motivates us to study the forbidden edges given a set of vertices.

Proposition 1.5 *Let G be a cubic graph and $A \subset V(G)$. If G has no forbidden edge given A then the unavoidable edges given A are independent.*

As we are often concerned about the nature of adjacencies of the unavoidable edges, this proposition enables us to study it using the information on forbidden edges. In P, take $uv \in E(P)$. If there is a contraction $\alpha : G \to P$ such that $\alpha(e) = uv \in E(P)$ and $\alpha(A) = V(P) - \{u,v\}$ then clearly $A \cup \{e\} \notin C(G)$. The converse of this assertion also holds [5].

Theorem 1.6 *Let G be a 3-connected cubic graph, $A \subset V(G)$, $e \in E(G)$ and $|A| = 8$. Then either $A \cup \{e\} \in C(G)$ or there is a contraction $\alpha : G \longrightarrow P$ such that $\alpha(e) = uv \in E(P)$ and $\alpha(A) = V(P) - \{u,v\}$.*

Corollary 1.7 *If G is a 3-connected cubic graph, then any set of seven vertices and an edge of G lies on a cycle.*

Corollary 1.8 *If G is a 3-connected cubic graph, then the unavoidable edges given any set of seven vertices of G are independent.*

Consider deleting an edge f from a 3-connected graph G. Then $G - f$ is 2-connected. Let $u, v \in V(G)$ and $e \in E(G)$. The edge e can be subdivided by a vertex x_e without altering the connectivity of $G - f$. The situation where $\{u, v, x_e\} \in C(G - f)$ was characterised in [18]. From this result $\{u, v, e\} \notin C(G - f)$ can be characterised. That is, there is a contraction $\alpha : G \to K_4$ such that $\alpha(\{u,v\}) = \{1,2\}$, $\alpha(e) = [3,4]$ and $\alpha(f) = [1,2]$. Here the complete graph K_4 is given in the obvious way by labelling the four vertices with integers $1, \ldots, 4$.

Proposition 1.9 *Let G be a 3-connected graph and $A = \{u,v\} \subset V(G)$. If $e, f \in E(G)$, then $\{e,u,v\} \in C(G - f)$ unless there is a contraction $\alpha : G \longrightarrow K_4$ such that $\alpha(A) = \{1,2\}$, $\alpha(e) = [3,4]$ and $\alpha(f) = [1,2]$.*

Let G be a 3-connected cubic graph and S be any cyclic edge cut of size 3 in G. Suppose that $K(G - S) = \{L, R\}$ and $L' = L \cup V(R)$, $R' = R \cup V(L)$. Then the graphs $H = G/R'$ and $J = G/L'$ are called the 3-*cut reductions* of G using S. Note that $G - S$ has precisely two components and the 3-cut reductions H and J are 3-connected cubic graphs with order at least 2 less than that of G. For a cubic graph G and $e = uv \in E(G)$ with $N(u) = \{u_1, u_2, v\}$ and $N(v) = \{u, v_1, v_2\}$, the graph

$$G_e = (G - \{u,v\}) \cup \{u_1u_2, v_1v_2\}$$

is called the *edge reduction* of G using the edge e. The edges u_1u_2 and v_1v_2 are called *the* two *new* edges in the reduction.

Let G be a cubic graph and $S = \{u_iv_i : i = 1,2,3,4\}$ be a cyclic cut set of four independent edges. Suppose that $K(G - S) = \{L, R\}$ and $p, q \notin V(G)$. Then the graph

$$L(u_1, u_2) = L \cup \{pq, pu_1, pu_2, qu_3, qu_4\}$$

is called the 4-*cut reduction* of G corresponding to the vertices u_1, u_2 using S. We call p and q the two new vertices in the reduction.

2 Forbidden Edges in Small Graphs

We consider cyclically 4-connected cubic graphs of order not exceeding 18. All graphs in this section can be found in the appendix.

If $|V(G)| \leq 14$ then G is contained in the catalogue produced in [8]. The graph $R = G(14.5)$ in the appendix is the only cyclically 4-connected cubic graph of order not exceeding 14 which has a b-edge. In R, let $B = \{k : 0 \leq k \leq 9\}$. Then $B \cup \{[12, 13]\} \notin C(R)$. Any set of nine vertices and an edge is contained in a cycle (i.e., $\zeta(R) = 9$).

Proposition 2.1 *Let G be a hamiltonian cyclically 4-connected cubic graph of order at most 14. Then $\zeta(R) = 9$ and for $G \neq R$, $\zeta(G) = |V(G)|$.*

All hamiltonian cyclically 4-connected cubic graphs on 16 and 18 vertices with b-edges were given in [17]. The b-edges were also listed. These graphs are labelled by $G(16.i)$, $i = 1, 2, 3$ and $G(18.i)$, $1 \leq i \leq 17$. We computed the parameter ζ for each of these graphs. This computation yields the following result.

Proposition 2.2 *Let G be a hamiltonian cyclically 4-connected cubic graph. Then*
(a) If $|V(G)| = 16$ then $\zeta(G) \geq 9$. More specifically,

$$\zeta(G(16.2)) = \zeta(G(16.3)) = 9, \zeta(G(16.1)) = 14$$

and for all other G, $\zeta(G) = 16$.
(b) If $|V(G)| = 18$ then $\zeta(G) \geq 9$. More specifically,

$$\zeta(G(18.i)) = 9 \text{ for } i \in \{1, 5, 6, 7, 8, 9, 10, 11\},$$
$$\zeta(G(18.2)) = 11, \zeta(G(18.13)) = \zeta(G(18.17)) = 12,$$
$$\zeta(G(18.12)) = \zeta(G(18.16)) = 13,$$
$$\zeta(G(18.i)) = 15 \text{ for } i \in \{3, 4, 14, 15\}$$

and for all other G, $\zeta(G) = 18$.

The Petersen graph P is the only nonhamiltonian cyclically 4-connected cubic graph on 10 vertices. There are precisely two nonhamiltonian cyclically 4-connected cubic graphs on 18 vertices. These three graphs are included in the appendix. The three graphs are the only nonhamiltonian cyclically 4-connected cubic graphs of order not exceeding 18. The parameter ζ for these three graphs can be computed easily.

Proposition 2.3 *$\zeta(P) = 7, \zeta(B_1) = 11$ and $\zeta(B_2) = 13$. The set*

$$S = \{1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 16, 18, [14, 15]\}$$

is a smallest noncyclable set of B_1. ∎

We now summarise this section.

Proposition 2.4 *Let G be a cyclically 4-connected cubic graph with $|V(G)| \leq 18$. Then $\zeta(P) = 7$ and for every $G \neq P$, $\zeta(G) \geq 9$.* ∎

3 Primitive Graphs

Let G be a 3-connected cubic graph and let $A \subset V(G)$, $e \in E(G)$ and $|A| = 8$. Theorem 1.6 asserts that $A \cup \{e\} \notin C(G)$ if and only if there is a contraction

$$\alpha : G \longrightarrow P$$

such that $\alpha(e) = uv \in E(P)$ and $\alpha(A) = V(P) - \{u, v\}$. Let $|A| = 9$. If there is a contraction $\alpha : G \to P$ such that $\alpha(e) = uv \in E(P)$ and $\alpha(A) \supseteq V(P) - \{u, v\}$ then certainly $A \cup \{e\} \notin C(G)$. In this case, we call $A \cup \{e\}$ *derived* in G. The graph pair $(G, A \cup \{e\})$ is a *derived* pair.

Let $M = A \cup \{e\}$ and suppose that (G, M) is not derived. We construct primitive graph pairs for $M \notin C(G)$.

Let $\alpha(A) = V(P) - u$ and $\alpha(e) = u$. Then clearly $A \cup \{e\} \notin C(G)$. This primitive pair is denoted by

$$(H_1, M_1) = (P, V(P)).$$

For K_4 with $V(K_4) = \{1', 2', 3', 4'\}$, we know that

$$\{1', 2', [3', 4']\} \notin C(K_4 - [1', 2'])$$

by Proposition 1.9. We also know that for P, $\{k \in V(P) : 3 \le k \le 10\} \cup \{[1, 2]\} \notin C(P)$. Let $H = (K_4 - 1') \cup (P - 1) \cup \{[2, 2'], [5, 4'], [6, 3']\}$. Take a connected graph L with $u_i \in V(L)$, $i = 1, 2, 3, 4$ such that the graph

$$H_2 = (H - \{[5, 4'], [6, 3']\}) \cup \{[5, u_1], [6, u_2], [3', u_3], [4', u_4]\}$$

is a 3-connected cubic graph. Then

$$M_2 = \{k : 3 \le k \le 10\} \cup \{2', [3', 4']\} \notin C(H_2).$$

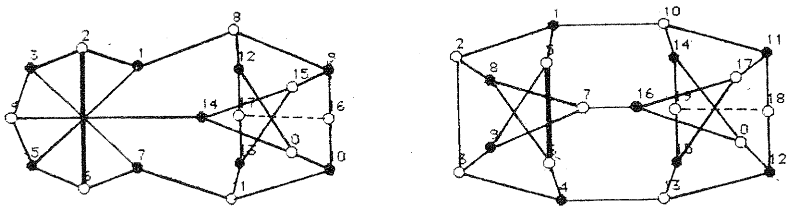
We have thus found another primitive graph pair.

We display four other primitive graphs in Figure 1. They are labelled by (H_k, M_k), $k = 3, 4, 5, 6$. That $M_k \notin C(H_k)$ for $k = 3, 4, 5, 6$ can be seen by Theorem 1.3 and Theorem 1.4.

The family of the six primitive graphs constructed above is denoted by $\mathbf{P}$. If (G, M) is derived or contractible to a graph pair in the primitive family $\mathbf{P}$, then clearly $M \notin C(G)$. One of the main objectives of this paper is to prove the converse.

4 Application of a Computer

We perform the inverse of an edge reduction on a primitive graph. Is it possible that in this way, we produce a primitive graph? In this section, we describe a way of deciding this on a computer. We call a possible inverse of an edge reduction an *extension*. For $A \subset V(G)$ and $e = xy \in E(G)$, if $x, y \notin A$ then we say that e is *A-free*.



H_4 : the first graph with the edge [16, 17]. $H_3 = (H_4)_{[16,17]}$. H_6: the second graph with the edge [18, 19]. $H_5 = (H_4)_{[18,19]}$.

Figure 1: Primitive graphs

Assume now that G is a cyclically 4-connected cubic graph. Let $A \subset V(G)$ with $|A| = 9$. Suppose that G has an A-free edge $f = xy$ and G_f is the edge reduction of G using f. Let

$$\alpha : (G_f, A \cup \{e\}) \longrightarrow (H_k, M_k) \in \mathbf{P}$$

be the primitive contraction. Denote by

$$S(v) = \alpha^{-1}(v) = \{w \in V(G_f) : \alpha(w) = v\}$$

the preimages of the vertices of H_k under the contraction α and $T(v) = \langle S(v) \rangle$ be the connected subgraph induced by $S(v)$. Since G is cyclically 4-connected, f must be incident with a vertex in each such subgraph. Let t be the number of such nontrivial induced subgraphs. Three cases occur. (1) $t = 0$, (2) $t = 1$ and (3) $t = 2$. Now the computation is performed as follows.

Let J be a candidate of G_f and let $g, h \in E(J)$. Subdivide the edges g and h with vertices x_g and x_h respectively. ,Then the graph

$$G^* = Ext(J; g, h) = (J + x_g + x_h) \cup \{x_g x_h\}$$

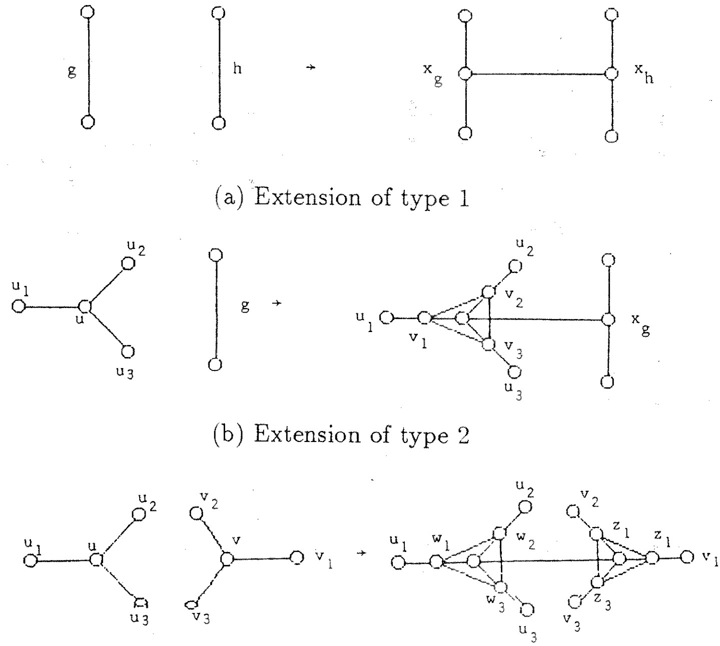
is called an *extension* of type 1.

Let $u \in V(J), g \in E(J)$ and $N_J(u) = \{u_i : i = 1, 2, 3\}$. Subdivide the edge g with a vertex x_g and the edge uu_i with a vertex v_i. Then an *extension* of type 2 is the graph

$$G^* = Ext(J; u, g) = (J + x_g + v_1 + v_2 + v_3) \cup \{ux_g, v_1v_2, v_1v_3, v_2v_3\}.$$

Let $u, v \in V(J)$ and $w_i, z_i \notin V(J)$, $i = 1, 2, 3$. Assume that $N_J(u) = \{u_i : i = 1, 2, 3\}$ and $N_J(v) = \{v_i : i = 1, 2, 3\}$. Subdivide uu_i with w_i and subdivide vv_i with z_i, $i = 1, 2, 3$. Then an *extension* of type 3 is the graph

$$G^* = Ext(J; u, v) = (J + \sum_{i=1}^{3} w_i + \sum_{i=1}^{3} z_i) \cup \{uv, w_1w_2, w_1w_3, w_2w_3, z_1z_2, z_1z_3, z_2z_3\}.$$



(a) Extension of type 1

(b) Extension of type 2

(c) Extension of type 3

Figure 2: Extensions

This is illustrated in Figure 2.

We omit J in $Ext(J; a, b)$ when the graph J is clear from the context. On a computer, these extensions have been constructed and certain cycle properties have been verified.

If (G_f, M) is derived then by Theorem 1.6, there is a contraction $\alpha : G_f \longrightarrow P$ such that $\alpha(e) = uv \in E(P)$ and $\alpha(A) = V(P) - \{u, v\}$. Return the edge f to the graph. Then the proof that $M \in C(G)$ is exactly the same as the corresponding part of the proof of Theorem 1.6 (see [5]).

Assume that there is a contraction $\alpha : J \longrightarrow H \in \mathbf{P}$. Then $G = G^*$.

(1) Extensions of type 1. Then $G_f = J$. Take each $H \in \mathbf{P}$, perform all nonisomorphic extensions of H. The edge e will then be subdivided in each of the resulting graphs. We then check whether the graph is hamiltonian. If the graph is hamiltonian then $M \in C(G)$ and if the resulting graph is nonhamiltonian we determine $\zeta(G)$.

$H = H_1$. Recall the labelling of P and assume that the vertex 1 is replaced by a component K. Denote the neighbours of 2, 5 and 6 in K by $2'$, $5'$, and $6'$. Without loss of generality the graph G is obtained by the edge extension involving an edge g in K and either the edge [2, 3] or the edge [3, 4]. (i) $G = Ext(g, [2,3])$. Then let

$x \in g$ and $y \in [2,3]$. The 4-cut reduction $K(2,x)$ is a 3-connected cubic graph since $5' \neq 6'$ and $2' \neq x$. Then $\kappa(K(2,x) - q) \geq 2$, and there is a cycle D in $K(2,x) - q$ that contains e and p. Then the cycle

$$(D - p) \cup [2', 2, 7, 10, 5, 4, 9, 6, 8, 3, y, x]$$

contains $A \cup \{e\}$. Hence $M \in C(G)$. (ii) $G = Ext(g, [3,4])$. The argument is precisely the same as that of (i). This time y lies on the edge [3, 4] and the cycle

$$(D - p) \cup [2', 2, 7, 10, 5, 4, 9, 6, 8, 3, y, x]$$

shows that $M \in C(G)$.

$H = H_2$. In the extension G, the edge $f = xy$ must join an edge on the triangle and an edge incident with $\{k : 2 \leq k \leq 10\}$ but different from the edges forming any cyclic cut set of size 3. Now precisely the same argument as in the case of $H = H_1$ shows that M is cyclable in G.

$H = H_3$. Any cyclically 4-connected edge extension of H is a cubic graph on 18 vertices and we have already discussed these graphs in Section 1.

$H = H_4$. If (g,h) is not any of $(e_f, [9,15])$, $(e_f, [14,15])$ and $(e_f, [14,16])$ then for each $G = Ext(g,h)$, e is not a b-edge of G. $\zeta(e) \geq 15$ in $Ext(e_f, [9,15])$, $\zeta(e) \geq 14$ in $Ext(e_f, [14,15])$ and $\zeta(e) \geq 12$ in $Ext(e_f, [14,16])$. Here e_f denotes the edge corresponding to e in the extension.

$H = H_5$ and $G = Ext(g,h)$. Then e_f is not a b-edge of G unless $g = [7,8]$ or $[7,9]$ and $h = [11,17]$ or $[12,18]$. The automorphism

$$\sigma = (1,4)(2,3)(5,6)(8,9)(10,13)(11,12)(14,15)(17,18)$$

interchanges [7, 8] and [7, 9] and fixes the edge $e_f = [5,6]$. Hence we consider only $g = [7,8]$. In both $Ext([7,8],[11,17])$ and $Ext([7,8],[12,18])$, $\zeta(e_f) \geq 11$.

$H = H_6$ and $G = Ext(g,h)$. Then e is not a b-edge of G unless $g = [7,8]$ or $[7,9]$ and $h \in \{[11,17],[12,18],[14,18],[15,17],[16,17],[16,18]\}$. The automorphism

$$\sigma = (1,4)(2,3)(5,6)(8,9)(10,13)(11,12)(14,15)(17,18)$$

interchanges [7, 8] and [7, 9] and fixes the edge $e_f = [5,6]$. Hence we consider $g = [7,8]$. In each case, $\zeta(e) \geq 13$ in G. Hence any edge extension of each of the graphs in **P**has a cycle containing M.

(2) Extensions of type 2. We may assume that $x \in T$, the nontrivial subgraph and y is the midpoint of an edge g of H. We replace the subgraph T with a copy of K_4, perform the extension of type 2, subdivide the edge e and store the resulting graph G^*. Since α is a primitive contraction, $\alpha(M) \subset H$ and $|\alpha(A)| = 9$.

Therefore $|A \cap T| \leq 1$. We now show that any hamiltonian cycle of G^* corresponds to a cycle containing $M = A \cup \{e\}$ in G.

Proposition 4.1 *If $G^* = Ext(u,g)$ is hamiltonian then $M \in C(G)$.*

Proof. Let $B(T, G-T) = \{u_iv_i : i = 1,2,3,4\}$ be the coboundary of T and $G-T$ with $u_i \in T$. Let C be any hamiltonian cycle of G^*. Then $C \cap (G^* - E(K_4))$ is the union of paths having nonempty even intersection with the set $\{u_i : i = 1,2,3,4\}$. By permuting the labels when necessary we may consider the following two cases.

(1) $C \cap (G^* - E(K_4))$ is a single (u_1, u_2)-path π. Consider the 3-connected cubic 4-cut reduction $T(u_1, u_2)$ with two new vertices $p, q \notin V(G)$. Since $|A \cap T| \leq 1$, $T(u_1, u_2)$ has a cycle D which contains $A \cap T \cup \{p\}$ and avoids q. Now $\pi \cup (D - p)$ is a cycle in G containing M.

(2) $C \cup (G^* - E(K_4))$ is the disjoint union of an (u_1, u_2)-path π and a (u_3, u_4)-path π'. By Theorem 1.2, $T(u_1, u_2)$ has a cycle containing $A \cap T \cup \{p, q\}$ avoiding the edge pq. Then $\pi \cup \pi' \cup (D - \{p, q\})$ is a cycle in G through M. ∎

By this result, $M \in C(G)$ can be proved by the hamiltonicity of G^*. If G^* is hamiltonian then M is cyclable in G. If G^* is not hamiltonian then we compute $\zeta(e)$ in G^*. If M is cyclable in G^* then it is certainly cyclable in G. The result of computing is as follows. For $H = H_i$ $(i = 1,2,3)$, each G^* is hamiltonian. For $H = H_4$, each extension G^* is hamiltonian except $G^* = Ext(u, [2,6])$ for $u \in \{14, 15, 16\}$. In $G^* = Ext(14, [2,6])$, $\zeta(e) \geq 10$, in $G^* = Ext(15, [2,6])$, $\zeta(e) \geq 13$ and in $G^* = Ext(16, [2,6])$, $\zeta(e) \geq 13$. For $H = H_5$, each $G^* = Ext(u, g)$ has a hamiltonian cycle through e. For $H = H_6$, each $G^* = Ext(u, g)$ has a hamiltonian cycle through e except $Ext(16, [7,8]) = Ext(16, [7,9])$ for which $\zeta(e) \geq 14$. Hence in each case $M \in C(G)$.

(3) Extensions of type 3. Let the two nontrivial components be T_1 and T_2. In this case the edge e is also subdivided and both T_1 and T_2 are replaced by a copy of K_4. The resulting graph is denoted by G^*. The proof of the following statement is similar to that of Proposition 4.1.

Proposition 4.2 *If $G^* = Ext(u, v)$ is hamiltonian then $M \in C(G)$.*

The number of cases is comparably small. The cases can be analysed both on a piece of paper and using a computer. In each case, we show that $M = A \cup \{e\}$ is cyclable in G. We now summarise this section.

Proposition 4.3 *Let G be any cyclically 4-connected cubic graph, $A \subset V(G)$, $|A| = 9$ and $e \in E(G)$. If*

(a) $(G, A \cup \{e\})$ is not derived, and

(b) G has an A-free edge $f \neq e$ such that the f-reduction G_f of G is contractible to a primitive graph pair $(H_k, M_k) \in \mathbf{P}$,

then $M \in C(G)$. ∎

5 Cubic Graphs

In this section, we prove that any set of nine vertices and an edge in a 3-connected cubic graph, that is not derived, is contained in a cycle if and only if the graph pair is not mapped onto a primitive graph pair given in Section 3 under a contraction.

Theorem 5.1 *Let G be a 3-connected cubic graph and let M be any set of nine vertices and an edge of G. Then either $M \in C(G)$, or (G, M) is derived, or there is a contraction*

$$\alpha : (G, M) \longrightarrow (H_k, M_k) \in \mathbf{P}.$$

Proof. The proof is by induction on the order of the graph. For $|V(G)| \leq 12$, the truth of this statement is established by considering the graphs catalogued in [8]. Suppose that G is a 3-connected cubic graph with $|V(G)| \geq 14$ and the statement holds for all 3-connected cubic graphs with fewer vertices. Consider the following two cases.

(1) G has a cyclic 3-edge cut $S = \{u_i v_i : i = 1, 2, 3\}$. Let H and J be the 3-cut reductions defined in Section 1, with e in H. Denote the new vertex in H adjacent to u_1, u_2 and u_3 by u and the new vertex in J adjacent to v_1, v_2 and v_3 by v. Let $a = |A \cap V(H)|$.

(1.1) $a = 0$. Suppose first that $e \notin S$. By the main theorem of [10], either $(A \cap V(J)) \cup \{v\} \in C(J)$ or there is a contraction $\alpha : (J, (A \cap V(J)) \cup \{v\}) \rightarrow (P, V(P))$. If $(A \cap V(J)) \cup \{v\} \in C(J)$ then let D be the cycle containing $(A \cap V(J)) \cup \{v\}$. Suppose that D avoids the edge vv_3. Since $\kappa(H - uu_3) \geq 2$, $H - uu_3$ has a cycle C that contains $\{u, e\}$. But

$$C' = (C - u) \cup (D - v) \cup \{u_1 v_1, u_2 v_2\}$$

is a cycle containing $A \cup \{e\}$ in G. If there is a contraction

$$\alpha : (J, (A \cap V(J)) \cup \{v\}) \rightarrow (P, V(P)),$$

then let α' be defined by $\alpha'(x) = \alpha(x)$, for each $x \in J - v$ and $\alpha'(x) = \alpha(v)$ for each $x \in H - u$. Then α' is a primitive contraction of (G, M) onto $(H_1, M_1) \in \mathbf{P}$. If $e \in S$ then (G, M) is derived.

(1.2) $a = 1$. Let $e \notin S$ and $A \cap V(H) = \{x\}$. Assume that for any $i \in \{1, 2, 3\}$ the edge uu_i can be avoided by a cycle in H through $\{e, u, x\}$. By the nine point theorem there is a cycle D in J through $(A \cap V(J)) \cup \{v\}$. Suppose that $vv_3 \notin D$. Then let C be a cycle in H through $\{e, u, x\}$ avoiding uu_3. Now C' in (1.1) is a cycle in G through M. Hence, suppose that one of the edges in $\{uu_i : i = 1, 2, 3\}$ is unavoidable given $\{e, u, x\}$. Let this edge be uu_1. By Proposition 1.9, there is a contraction $\alpha_H : H \rightarrow K_4$ such that $\alpha_H(\{u, x\}) = \{1, 2\}$, $\alpha_H(e) = [3, 4]$ and $\alpha_H(uu_1) = [1, 2]$. Also for each $i \in \{2, 3\}$, uu_i can be avoided by a cycle in H through $\{e, u, x\}$. If there is a cycle D in J through $(A \cap V(J)) \cup \{vv_1\}$, then suppose that D avoids vv_3. Now let C be a cycle in H through $\{e, u, x\}$ avoiding uu_3. Again C' in (1.1) is a cycle in G containing M. Suppose then that vv_1 cannot be contained in a cycle of J through $A \cap V(J)$. Then by Theorem 1.6, there is a contraction

$$\alpha_J : (J, (A \cap V(J)) \cup \{vv_1\}) \rightarrow (P, (V(P) - \{u, v\}) \cup \{uv\}).$$

Let α be the mapping whose restriction to $H-u$ is α_H and that to $J-v$ is α_J. Then α is a contraction from (G, M) onto (H_2, M_2).

If $e \in S$ then let $e = u_1v_1$ and consider the edge uu_1 instead of e. Then the discussion is similar but in this case $M \in C(G)$.

(1.3) $a \in \{2,3\}$. Let $e \in S$. By Corollary 1.7, for any $i \in \{1,2,3\}$, there is a cycle D in J which contains $(A \cap V(J)) \cup \{vv_i\}$. Assume that none of vv_i, is an unavoidable edge given $(A \cap V(J)) \cup \{v\}$. Then by Corollary 1.7, there is a cycle C in H through $(A \cap V(H)) \cup \{e, u\}$, which must avoid one of the edges uu_i $(i = 1,2,3)$. We choose D to avoid the corresponding edge of vv_i. The two paths $C-u$ and $D-v$, and a pair of suitable edges from S, give rise to a desired cycle in G. Suppose that there is an unavoidable edge in vv_i given $(A \cap V(J)) \cup \{v\}$. Then let it be vv_1. Then by Proposition 1.5, any one of vv_2 and vv_3 can be avoided by a cycle containing $(A \cap V(J)) \cup \{v\}$ in J. In H there is a cycle C which contains $(A \cap V(H)) \cup \{e, uu_1\}$ by Theorem 1.4. Suppose that C excludes uu_3. Since vv_1 is unavoidable in J given $(A \cap V(J)) \cup \{v\}$, vv_3 can be avoided by a cycle D containing $(A \cap V(J)) \cup \{vv_1\}$. Once again C' in (1.1) is a cycle in G containing $A \cup \{e\}$. If $e \in S$ then let $e = u_1v_1$ and consider uu_1 in H.

(1.4) $a = 4$. Let $e \in S$. Assume that for any $i \in \{1,2,3\}$, there is a cycle in J through $(A \cap V(J)) \cup \{v\}$ avoiding vv_i. By Corollary 1.7, there is a cycle C in H containing $(A \cap V(H)) \cup \{e, u\}$. Such a cycle C must exclude one of uu_i, say uu_3. Let D be a cycle in J through $(A \cap V(J)) \cup \{v\}$ excluding vv_3. Then C' in (1.1) is a cycle containing $A \cup \{e\}$ in G. Hence, one of the edges $\{vv_i : i = 1,2,3\}$ is unavoidable given $(A \cap V(J)) \cup \{v\}$ in J. Let this edge be vv_1. By Theorem 1.3, there is a contraction

$$\alpha_J : (J, (A \cap V(J)) \cup \{vv_1\}) \longrightarrow (P, A_P \cup \{e_P\}) \text{ or } (Q, A_Q \cup \{e_Q\}),$$

For each $i \in \{2,3\}$, the edge vv_i can be avoided by a cycle through $(A \cap V(J)) \cup \{v\}$. If there is a cycle C in H containing $(A \cap V(H)) \cup \{e, uu_1\}$, then let $uu_3 \notin C$. J has a cycle D containing $(A \cap V(J)) \cup \{v\}$ which avoids the edge vv_3. Then C' in (1.1) is again a cycle of G containing M. If uu_1 cannot be contained in a cycle through $(A \cap V(H)) \cup \{e\}$, then by Theorem 1.4, there is a contraction

$$\alpha_H : (H, (A \cap V(H)) \cup \{e, uu_1\}) \longrightarrow (W, B_W) \text{ or } (P, B_P),$$

Low let α be a mapping whose restriction to $H-u$ is α_H and to $J-v$ is α_J. Then α is a primitive contraction of (G, M) onto (H_k, M_k) for some $k \in \{3,4,5,6\}$. If $e \in S$ then let $e = u_1v_1$ and repeat the above argument for vv_1 instead of e. In this case the contraction α_H does not exist. Hence $M \in C(G)$.

In the following four cases, whether $e \in S$ or not does not affect our discussion.

(1.5) $a \in \{5,6\}$. By Corollary 1.7, there is a cycle C in H containing $(A \cap V(H)) \cup \{e, u\}$. Suppose that C avoids the edge uu_3. By Theorem 1.2 there is a cycle D in J which contains $(A \cap V(J)) \cup \{v\}$ and avoids the edge vv_3. Then C' in (1.1) is a desired cycle in G.

(1.6) $a = 7$. Since $A \cup \{e\}$ is not derived in G, $(A \cap V(H)) \cup \{u, e\}$ is not derived in H. Hence by Theorem 1.6 there is a cycle C in H containing $(A \cap V(H)) \cup \{u, e\}$. Suppose that C avoids the edge uu_3. By Theorem 1.2 there is a cycle D in J which contains $(A \cap V(J)) \cup \{v\}$ and avoids the edge vv_3. Then C' in (1.1) is a cycle in G containing M.

(1.7) $a = 8$. Since $A \cup \{e\}$ is not derived in G, $(A \cap V(H)) \cup \{u, e\}$ is not derived in H. We apply the inductive hypothesis to H. If there is a cycle C in H containing $(A \cap V(H)) \cup \{u, e\}$. Then assume that C avoids the edge uu_3. By Theorem 1.2, there is a cycle D in J which contains $(A \cap V(J)) \cup \{v\}$ and avoids the edge vv_3. Then $(C - u) \cup (D - v) \cup \{u_1v_1, u_2v_2\}$ is a desired cycle in G. Assume now that $(A \cap V(H)) \cup \{u, e\}$ is neither cyclable nor derived in H. Then by the inductive assumption there is a contraction

$$\alpha : (H, (A \cap V(H)) \cup \{u, e\}) \longrightarrow (H_k, M_k) \in \mathbf{P}.$$

Let α' be defined by $\alpha'(x) = \alpha(x)$ for each $x \in H - u$ and $\alpha'(x) = \alpha(u)$ for each $x \in J - v$. Then α' is a desired primitive contraction.

(1.8) $a = 9$. We apply the inductive assumption to (H, M). Assume that $M \in C(H)$ and let C be a cycle in H through M. If $u \notin C$, then C itself is a cycle of G through M. If $u \in C$, then let $uu_3 \notin C$. There is a (v_1, v_2)-path π in the 2-connected graph $J - v$. But $(C - u) \cup \{u_1v_1, u_2v_2\} \cup \pi$ is a cycle of G through M. Suppose then that the graph pair (H, M) is contractible to a primitive pair in $\mathbf{P}$. Then let α denote this contraction. Define α' as $\alpha'(x) = \alpha(x)$ for all $x \in H - u$ and $\alpha'(x) = \alpha(u)$ for all $x \in J - v$. Then α' is a contraction of (G, M) onto the corresponding primitive pair in $\mathbf{P}$.

(2). Suppose that G has no cyclic 3-edge cut. Hence G is cyclically 4-connected, and any edge reduction of G is 3-connected.

(2.1) Assume that there is an edge $f \neq e$ which is A-free. Suppose e and f are independent. Let G_f be the f-reduction of G. Then by the inductive assumption either there is a cycle C in G_f containing M or (G_f, M) is derived or it is contractible to a primitive pair as in the statement of the theorem. If $M \in C(G_f)$ then $M \in C(G)$. If (G_f, M) is derived or it is contractible, then by Proposition 4.3, $M \in C(G)$. If e and f are adjacent then there is an edge e_f in G_f corresponding to e. We apply the inductive assumption to $(G_f, A \cup \{e_f\})$. By Proposition 4.3, $M \in C(G)$.

(2.2) Suppose now that every edge other than e has an end vertex in A. Then $|V(G)| \leq 18$, and the proof is completed by Proposition 2.4. ∎

We note that the nine point theorem, the main theorem of [10] and Theorem 1.6 can be proved as corollaries to this theorem. The adjacency of unavoidable edges can be determined using this theorem. For the cyclability of a set of ten vertices and an edge, an infinite family of primitive graphs can be constructed (see [4]).

6 Cubic Planar Graphs

We show that every 3-connected cubic planar graph has a cycle containing any specified set of fourteen vertices and an edge.

It is not difficult to show that every 3-connected cubic planar graph has a cycle containing any specified set of five vertices and two edges.

Theorem 6.1 *Every 3-connected cubic planar graph has a cycle containing any specified set of five vertices and two edges.*

Proof. Let $A \subset V(G)$ and $e, f \in E(G)$. Subdivide e and f with vertices x and y. Then $H = G \cup \{x, y, xy\}$ is a 3-connected cubic graph and the edge reduction H_{xy} is planar. By the main theorem of [1], there is a cycle C in H containing $A' = A \cup \{x, y\}$ avoiding the edge xy unless H is contractible to a graph in the primitive family of [1]. But no edge reduction of any such graph is planar. Hence the theorem is proved. ∎

The pentagonal prism T_5 is obtained by taking two disjoint pentagons [1, 2, 3, 4, 5, 1] and [6, 7, 8, 9, 10, 6] and joining a vertex u of the first pentagon and a vertex v of the second if $v \equiv u \pmod 5$. Take $A = \{1, 3, 4, 6, 8, 9\}$, $e = [2, 7]$ and $f = [5, 10]$. Then there is no cycle in G that contains $A \cup \{e, f\}$. Tutte's first example of a nonhamiltonian 3-connected cubic planar graph was contructed using this fact. The graph T_5 shows that Theorem 6.1 is the best possible. We have not yet investigated the situation for $|A| \geq 6$.

It was shown that in any 3-connected cubic planar graph any set of nine vertices is contained in a cycle which avoids any specified edge (see [11] or [13]).

Theorem 6.2 *Let G be a 3-connected cubic planar graph and $A \subset V(G)$ with $|A| = 9$. Then for any $e \in E(G)$, $A \in C(G - e)$.*

The main result in this section is the following.

Theorem 6.3 *Every 3-connected cubic planar graph has a cycle containing any set of fourteen vertices and an edge.*

Proof. The proof is again by induction on the order of G. It goes exactly the same as that of Theorem 5.1. In this case, however, the argument is much simpler. First let $|V(G)| \leq 22$. Then the assertion was established by the fact that G is hamiltonian and it has no b-edge [9]. Suppose then that G is any 3-connected cubic planar graph with $|V(G)| \geq 24$ and the statement holds for every 3-connected cubic planar graph of order smaller than that of G.

(1) Assume that G has a cyclic 3-edge cut $S = \{u_1v_1, u_2v_2, u_3v_3\}$. Let H and J be the usual 3-cut reductions with e in H.

(1.1) Let $A \subset V(J)$. By the inductive assumption, we have a cycle D in J that contains $(A \cap V(J)) \cup \{vv_1\}$. Suppose that D also uses the edges vv_2. There is a cycle C in $H - uu_3$ which contains $\{u, e\}$ since $\kappa(H - uu_3) \geq 2$. Then

$$C' = (C - u) \cup (D - v) \cup \{u_1v_1, u_2v_2\}$$

is a cycle containing $A \cup \{e\}$.

(1.2) $1 \leq |A \cap V(H)| \leq 5$. If for every $i \in \{1, 2, 3\}$ the edge vv_i can be avoided by a cycle in J containing $(A \cap V(J)) \cup \{v\}$, then let C be a cycle in H through $(A \cap V(H)) \cup \{e, u\}$ which exists by the inductive assumption. Assume that $uu_3 \notin C$. Then let D be a cycle of J containing $(A \cap V(J)) \cup \{v\}$ avoiding vv_3. Then the cycle C' in (1.1) is a cycle of G that contains $A \cup \{e\}$ in this case. Hence assume that one of the edges vv_i $(i = 1, 2, 3)$ is unavoidable given $(A \cap V(J)) \cup \{v\}$. Let vv_1 be such an unavoidable edge. Then any one of the two edges vv_2 and vv_3 can be avoided by a cycle in J through $(A \cap V(J)) \cup \{v\}$ by Proposition 1.5. By Theorem 6.1, there is a cycle C in H that contains $(A \cap V(H)) \cup \{e, uu_1\}$. Suppose that $uu_3 \notin C$ and let D be a cycle of J through $(A \cap V(J)) \cup \{v\}$ avoiding the edge vv_3. Then again C' is a cycle of G containing $A \cup \{e\}$.

(1.3) $6 \leq |A \cap V(H)| \leq 13$. By the inductive hypothesis, there is a cycle C in H that contains $(A \cap V(H)) \cup \{e, u\}$. Such a cycle C must exclude one of uu_i, say uu_3. By Theorem 6.2, J has a cycle D that contains $(A \cap V(J)) \cup \{v\}$ excluding vv_3. Then C' is a cycle containing $A \cup \{e\}$ in G.

(1.4) $A \subset V(H)$. By the inductive hypothesis, there is a cycle C in H that contains $(A \cap V(H)) \cup \{e\}$. If $u \notin C$, then this is the required cycle in G. If $u \in C$ then suppose that $uu_3 \notin C$. Since $J - v$ is connected, it has a (v_1, v_2)-path π. But

$$(C - u) \cup \{u_1v_2, u_1v_2\} \cup \pi$$

is a desired cycle in G.

(2). Suppose then that G has no cyclic 3-edge cut. Hence G is cyclically 4-connected, and any edge reduction of G is 3-connected.

(2.1) Assume that there is an edge $f \neq e$ which is A-free. Suppose e and f are independent. Let G_f be the f-reduction of G. Then by the inductive assumption there is a cycle C in G_f that contains $A \cup \{e\}$ which is the required cycle in G. If e and f are adjacent, then there is an edge e_f in the f-reduction G_f of G, that corresponds to e. We apply the inductive assumption to G_f for $A \cup \{e_f\}$.

(2.2) Suppose then that every edge other than e has an end vertex in A. Then $|V(G)| \leq 28$. But $|V(G)| \geq 24$. Hence, $24 \leq |V(G)| \leq 28$.

Let $|V(G)| = 24$. Then there is only one 3-connected cubic planar graph that has a b-edge. For this graph the assertion holds. If $|V(G)| = 26$, then there are seven 3-connected cubic planar graphs that have b-edges. For these graphs, the assertion holds. Finally, if $|V(G)| = 28$, then G is bipartite and by [12], G is hamiltonian and has no b-edge. ∎

We note that the 3-connected cubic planar graph of order 24 has a set of fifteen vertices and an edge that is not cyclable (see [14]). This shows that Theorem 6.3

is sharp. We have made no attempt to determine cyclable sets of fifteen or more vertices and an edge in 3-connected cubic planar graphs.

Corollary 6.4 *Let G be a 3-connected cubic planar graph and $A \subset V(G)$ with $|A| = 14$. If e and f are two unavoidable edges given A then e and f are independent.*

Proof. This follows from Theorem 6.3 and Proposition 1.5. ∎

Employing Theorem 6.3, we have the following result. This result is significant since there are 3-connected cubic planar graphs that are not 24-cyclable.

Theorem 6.5 *If every cyclically 4-connected cubic planar graph G with $|V(G)| \leq 44$ is 23-cyclable, then every 3-connected cubic planar graph is 23-cyclable.*

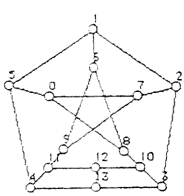
The proof of this theorem is similar to that of Theorem 6.3.

7 Appendices

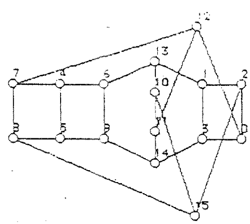
The graphs will be labelled by the elements of the ring $Z_{|V(G)|}$ and we do not distinguish 0 and $|V(G)|$.

7.1 Cyclically 4-connected Cubic Hamiltonian Graphs of Order 14, 16 and 18 That Contain b-edges

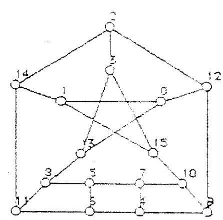
The b-edges are listed after the labels of the graphs.



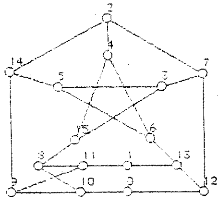
$G(14.1)$. [12, 13]



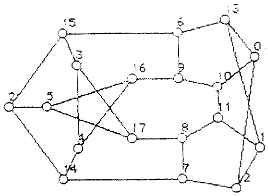
$G(16.1)$. [4, 5]



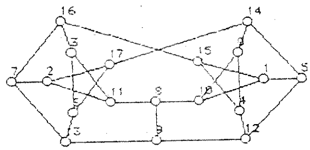
$G(16.2)$. [4, 7], [5, 6]



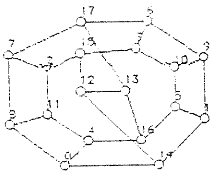
$G(16.3)$. [0, 1]



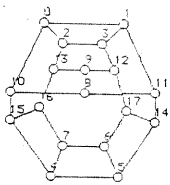
$G(18.1)$. [0, 13], [1, 12], [10, 11]



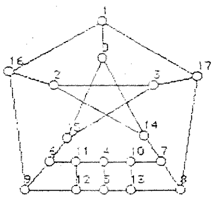
$G(18.2)$. [8, 9], [14, 17], [15, 16]



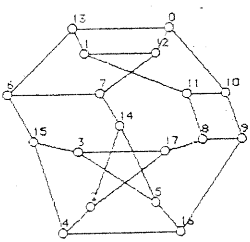
$G(18.3)$. [8, 11], [9, 10]



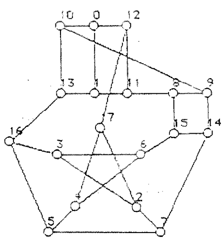
$G(18.4)$. [4, 7], [5, 6]



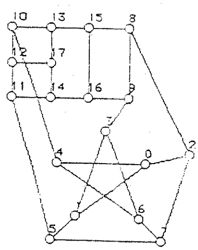
$G(18.5)$. [4, 5], [10, 13], [11, 12]



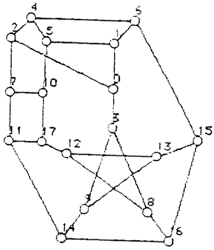
$G(18.6)$. [10, 11]

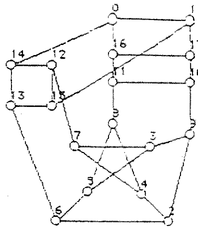


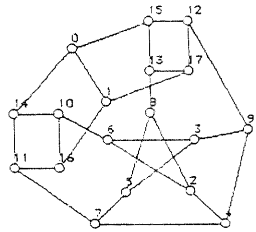
$G(18.7)$. [8, 9]



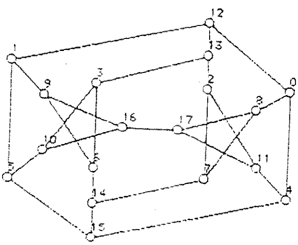
$G(18.8)$. [15, 16]

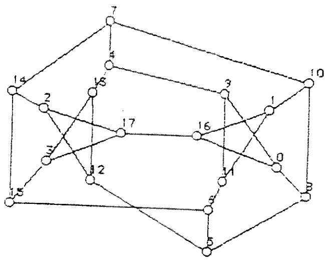


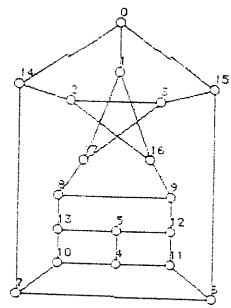




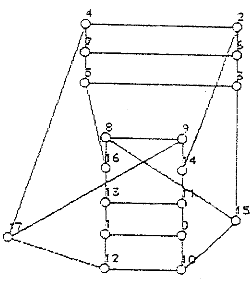
$G(18.9)$. [7, 10] $G(18.10)$. [1, 16] $G(18.11)$. [1, 16]

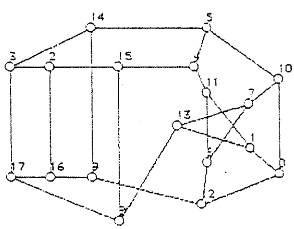


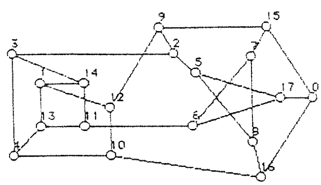




$G(18.12)$. [16, 17] $G(18.13)$. [16, 17] $G(18.14)$. [4, 5]

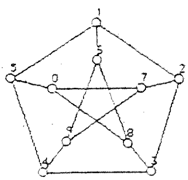
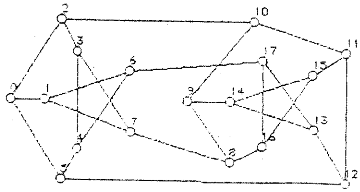
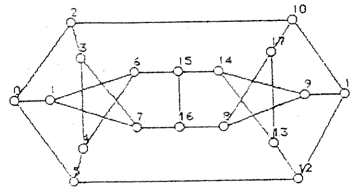






$G(18.15)$. [1, 16] $G(18.16)$: [16, 17] $G(18.17)$. [11, 13].

7.2 The Three Nonhamiltonian Cyclically 4-connected Cubic Graphs of Order up to 18

The Petersen graph P B_1 B_2

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