A note on the blocking sets in the large Mathieu design $S(5,8,24)$

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Abstract

We simplify the classification of blocking sets in the Steiner system $S(5,8,24)$ obtained by Beradi and Eugeni. We show that every blocking set in $S(5,8,24)$ is contained in precisely two blocks.

1 Introduction

First of all, we introduce some definitions and terminologies.

Definition 1.1. A *Steiner system* $S(t,k,v)$ is a pair $(\mathcal{S},\mathcal{B})$, where $\mathcal{S}$ is a v-set of elements called points, $\mathcal{B}$ is a family of k-sets called blocks, such that any fixed t-set is contained in exactly one element of $\mathcal{B}$.

Definition 1.2. A set of points of a Steiner system is called a *blocking set* if it contains no block, but intersects every block.

Definition 1.3. A blocking set C is said to be of *index* t if C is contained in t blocks. The index of C is denoted by $i(C)$.

Definition 1.4. A blocking set C is said to be *irreducible* if for any $x \in C$, the set $C - \{x\}$ is not a blocking set. Otherwise, C is said to be *reducible* .

Let B, B' be two blocks in $S(5,8,24)$ with $|B \cap B'| = 2$. We define

$$M := B \triangle B'; M_0 := B \triangle B' - \{a\}; I := B \cup B' - \{u,v\}; R := B \cup B' - \{z,a\}$$

where $u \in B - B'$, $v \in B' - B$, $a \in B \triangle B'$, $z \in B \cap B' = \{x,y\}$.

In [2] L. Berardi and F. Eugeni have proved the following theorem which gives the complete classification of the blocking sets in $S(5,8,24)$.

Theorem 1.1. *Let C be a blocking set in $S(5,8,24)$. Then $11 \le |C| \le 13$. Moreover,*

1. $|C| = 11$ *implies that* $C = M_0$ *and* $i(M_0) = 2$.

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2. $|C| = 12$ *and* C *irreducible imply that* $C = I$ *and* $i(I) = 2$.

3. $|C| = 12$ *and* C *reducible imply that* $C = M_0 \cup \{x\}$, $x \notin M_0$. *Moreover, if* $i(C) = 2$, *then either* $C = M$ *or* $C = R$.

4. $|C| = 13$ *implies that* C *is reducible and* C *is the complement of* M_0. *Moreover, if* $i(C) = 2$, *then* $C = B \cup B' - \{a\}$, *where* B, B' *are two blocks with* $|B \cap B'| = 2$ *and* $a \in B \cap B'$.

In this paper, we prove the following theorem, which improves the results in theorem 1.1.

Theorem 1.2. *Let* C *be a blocking set in* $S(5, 8, 24)$. *Then* $11 \leq |C| \leq 13$ *and* $i(C) = 2$. *Moreover,*

1. *If* $|C| = 11$, *then* $C = M_0$.

2. *If* $|C| = 12$ *and* C *is irreducible, then* $C = I$

3. *If* $|C| = 12$ *and* C *is reducible, then* $C = M$ *or* R

4. *If* $|C| = 13$, *then* $C = \mathcal{S} - M_0 = B \cup B' - \{z\}$ *is reducible, where* B, B' *are two blocks with* $|B \cap B'| = 2$ *and* $z \in B \cap B'$.

2 Some results

Let $r_s(s = 0, 1, \cdots, t)$ be the number of blocks containing a fixed s-set, then

$$r_s = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$$

Let E be a c-set in $S(t, k, v)$. Denote by x_i the number of blocks i-secant to E. Let $T = \{m_1, m_2, \cdots, m_h\}$ be a set of integers with $0 \leq m_1 < m_2 < \cdots < m_h$.

A set E is said to be of type $(m_1, m_2, \cdots, m_h)$, if $x_i \neq 0$ if and only if $i \in T$. We have the following identities:

$$\sum_{i=s}^{k} \binom{i}{s} x_i = r_s \binom{c}{s}, s = 0, 1, \cdots, t. \tag{2.1}$$

In the case of $S(5, 8, 24)$, if E is a blocking set, then (2.1) implies:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 &= 759 \\ x_2 + 2x_3 + 3x_4 + 4x_5 + 5x_6 + 6x_7 &= g_1 \\ x_3 + 3x_4 + 6x_5 + 10x_6 + 15x_7 &= g_2 \\ x_4 + 4x_5 + 10x_6 + 20x_7 &= g_3 \\ x_5 + 5x_6 + 15x_7 &= g_4 \\ x_6 + 6x_7 &= g_5 \end{aligned} \tag{2.2}$$

where

$$
\begin{aligned}
g_1 &= 253c - 759 \\
2g_2 &= 77c(c-1) - 2g_1 \\
6g_3 &= 21c(c-1)(c-2) - 6g_2 \\
24g_4 &= 5c(c-1)(c-2)(c-3) - 24g_3 \\
120g_5 &= c(c-1)(c-2)(c-3)(c-4) - 120g_4.
\end{aligned}
\tag{2.3}
$$

The following lemmas are quoted from [1, 2].

Lemma 2.1. [2] *Let B, B' be two blocks in $S(5,8,24)$. Then*

1. *The type of B is $(0,2,4,8)$ with*

$$x_0 = 30, x_2 = 448, x_4 = 280, x_8 = 1.$$

2. *If $|B \cap B'| = 4$, then $B \triangle B'$ is a block.*

3. *If $|B \cap B'| = 2$, then $M = B \triangle B'$ is a set of type $(2,4,6)$ with*

$$x_2 = x_6 = 132, x_4 = 495.$$

4. *If $|B \cap B'| = 0$, then $B \triangle B' = B \cup B'$ is a set of type $(0,4,6,8)$ with*

$$x_0 = 1, x_4 = 280, x_6 = 448, x_8 = 30.$$

5. *Let E be a set. Then $S - E$ is a block if and only if $E = B \cup B'$, $B \cap B' = \emptyset$.*

6. *Let F be a 4-set, $F \cap B = \emptyset$, then there exists a block B' such that $F \subseteq B'$ and $B \cap B' = \emptyset$.*

By 1, 3 and 4 of lemma 2.1 we get the following corollaries respectively.

Corollary 2.1. *No blocking set can be contained in one block.*

Corollary 2.2. *The sets M, M_0 are blocking sets in $S(5,8,24)$.*

Corollary 2.3. *Let C be a blocking set. If $C \subseteq B \cup B'$, then $|B \cap B'| \neq 0$.*

Fix a point x in $S(t,k,v)$ and set

$$\mathcal{B}_x = \{B - \{x\} | x \in B, B \in \mathcal{B}\}.$$

The pair $(S - \{x\}, \mathcal{B}_x)$ is a Steiner system $S(t-1, k-1, v-1)$, which is said to be the contraction of $S(t,k,v)$ at x. For $S(4,7,23)$ we have

Lemma 2.2. [1] *Let B, B' be two blocks in $S(4,7,23)$ with $B \cap B' = \{x\}$, then for any $u \in B - B'$ and $v \in B' - B$, there exists a block B'' in $S(4,7,23)$ such that $B'' \cap (B \cup B') = \{u, v\}$.*

Corollary 2.4. *Let B, B' be two blocks in $S(5,8,24)$ with $B \cap B' = \{x, y\}$ and let $u \in B - B'$, $v \in B' - B$. Then $(B \cup B') - \{y, u, v\}$ is not a blocking set.*

3 Proof of the theorem 1.2

From now on, C will be used to denote a blocking set in $S(5,8,24)$.

Lemma 3.1, proposition 3.1, proposition 3.2 and proposition 3.3 are proved in [2]; we quote them here for our convenience.

Lemma 3.1. $11 \leq |C| \leq 13$.

Proposition 3.1. *If* $|C| = 11$, *then* $C = M_0$ *has no* 7*-secant block.*

Proposition 3.2. I *is an irreducible blocking set.*

Proposition 3.3. R *is a reducible blocking set.*

By (2.2), if $|C| = 12$, then

$$x_1 = x_7, x_2 = x_6 = 132 - 6x_7, x_3 = x_5 = 15x_7, x_4 = 495 - 20x_7. \quad (3.1)$$

The following proposition plays a crucial role in our proof.

Proposition 3.4. *Let* $|C| = 12$.

1. *If* C *has a* 7*-secant block, then* $C = R$ *or* I.
2. *If* C *has no* 7*-secant block, then* $C = M$.

Proof. Let B be a block 7-secant to C, B' be a block containing the five points in $C - B$, then $|B \cap B'| = 2$. Let $B \cap B' = \{x, y\}$. Since $|B \cap C| = 7$, $\{x, y\} \cap C \neq \emptyset$. If $x \in C$, $y \notin C$, then $C = R$. If $x, y \in C$, then $C = I$.

If C has no 7-secant block, then by 3.1 C is of type $(2, 4, 6)$. Let B be a block 6-secant to C, let five of the six points in $C - B$ be contained in block B', then B' contains another point in C. We claim that this point must be the remaining point in $C - B$. Suppose this point is in B, then by lemma 2.1. 1 $|B \cap B'| = 2$. Let $B \cap B' = \{x, y\}$, $x \in C$, $y \notin C$, $u \in B - B'$, $v \in B' - B$, $w \in C - (B \cup B')$. Since C is of type $(2, 4, 6)$, the set $C - \{w\} = (B \cup B') - \{y, u, v\}$ is a blocking set, contradiction. So B, B' are blocks 6-secant to C, $C = B \triangle B' = M$. □

Proposition 3.5 and proposition 3.7 had been proved in [2], but using proposition 3.4, we can simplify the proof.

Proposition 3.5. *If* $|C| = 12$, C *is irreducible, then* $C = I$.

Proof. Since C is irreducible, $x_7 = x_1 \geq 12$. By proposition 3.4 $C = I$. □

Proposition 3.6. *If* $|C| = 12$ *and* C *is reducible, then* $C = M$ *or* R.

Proof. If C has a 7-secant block, then $C = R$; if C has no 7-secant block, then $C = M$. □

Proposition 3.7. *Let* A *be one of the* 12*-sets* I, M *and* R. *Then* $S - A$ *is isomorphic to* A.

Proof. Let $A = M$. Since M is of type $(2,4,6)$, so is $\mathcal{S} - M$. By proposition 3.4 $\mathcal{S} - M = M$.

Let $A = R = B \cup B' - \{z, a\}$, where $|B \cap B'| = 2$, $a \in B \triangle B'$, $z \in B \cap B'$. Since R has a 7-secant block and $R \cup \{a\}$ is a blocking set, $\mathcal{S} - R$ is reducible and also has a 7-secant block. By proposition 3.6 and proposition 3.4, $\mathcal{S} - R = R$.

Let $A = I$. Suppose $\mathcal{S} - I$ is reducible, then $\mathcal{S} - I = R$, but $\mathcal{S} - (\mathcal{S} - I) = I$, so $I = \mathcal{S} - R = R$, contradiction. Therefore $\mathcal{S} - I$ is irreducible and $\mathcal{S} - I = I$. □

Proposition 3.8. *If* $|C| = 13$, *then* $C = \mathcal{S} - M_0 = B \cup B' - \{z\}$ *is reducible, where* B, B' *are blocks with* $|B \cap B'| = 2$ *and* $z \in B \cap B'$.

Proof. Since $|\mathcal{S} - C| = 11$, we have $\mathcal{S} - C = M_0$. But M_0 has no 7-secant block, so C has no 1-secant block. This means that C is reducible.

The fact that M_0 has a 1-secant block means that C has 7-secant blocks. Let B be a 7-secant block to C and let five of the six points in $C - B$ be contained in block B'.

We claim that the remaining one point $w \in C - B$ is still in B'.

Suppose $w \notin B'$, we may assume that $B \cap B' \neq \emptyset$. (If $B \cap B' = \emptyset$, then B' contains three points in $\mathcal{S} - (C \cup B)$. Since there are six blocks that contain five points in $C - B$, any two of these only intersect at four points in $C - B$, and there are only ten points in $\mathcal{S} - (C \cup B)$, so at least one of these blocks will intersect B. We can label this block as B'.). Then $|B \cap B'| = 2$. Let $B \cap B' = \{x, y\}$, since B is 7-secant to C, $\{x, y\} \cap C \neq \emptyset$.

If $x, y \in C$, then $C - \{w\} = I$. But on the other hand, $M_0 \cup \{w\}$ is reducible, so $\mathcal{S} - I = M_0 \cup \{w\} = R$, contradiction.

If $x \in C$, $y \notin C$, let $v \in B' - (C \cup B)$, $B = \{x, y, a_1, a_2, a_3, a_4, a_5, a_6\}$. By lemma 2.2 we know that there is a block B_i that contains a_i, v, y such that $B_i \cap (C - \{w, a_i\}) = \emptyset$, $i = 1,2,3,4,5,6$. Since C has no 1-secant block, $w \in B_i$, $i = 1,2,3,4,5,6$. Let $D_i = B_i - \{v, y, w, a_i\}$, then $|D_i| = 4$, $D_i \subseteq \mathcal{S} - (C \cup \{x, y\})$, $|D_i \cap D_j| = 1$, $i \neq j$. Since $|\mathcal{S} - (C \cup \{v, y\})| = 9$, we have $D_1 \cap D_i \neq D_1 \cap D_j$, if $i \neq j$. Hence $|D_1| = 5$, contradiction.

Now we have proved that $w \in B'$. From $C \subseteq (B \cup B')$ we know that $|B \cap B'| = 2$. Let $B \cap B' = \{x, y\}$, since $|B \cap C| = 7$, this means that $|(B \cap B') \cap C| = 1$, so $C = B \cup B' - \{z\}$, $z \in B \cap B'$. □

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References

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