Monogamous decompositions of complete bipartite graphs, symmetric H-squares, and self-orthogonal 1-factorizations

C.C. Lindner
Department of Discrete and Statistical Sciences
Auburn University
Auburn, AL 36849-5307, U.S.A.

A. Rosa
Department of Mathematics
McMaster University
Hamilton, Ontario, Canada L8S 4K1

Abstract

Necessary and sufficient conditions are obtained for the existence of the structures from the title of this article.

1. Introduction

It is well known that the complete bipartite graph $K_{m,n}$ can be decomposed into 4-cycles precisely when both m and n are positive even integers. While the necessity of this condition is obvious, it is almost equally easy to see how to obtain such a decomposition (and is a straightforward consequence of Sotteau's Theorem [S] anyway): arbitrarily partition the vertices of both partite classes into 2-subsets, and then form a 4-cycle (i.e., a $K_{2,2}$) for every pair of cells — one from each partite set — of such a partition in an obvious way.

The situation changes, however, if one considers decompositions of complete bipartite graphs into 4-cycles satisfying an additional "uniqueness" condition:

(U) If (a, x, b, y) and (a, u, b, v) are 4-cycles of the decomposition then $\{x, y\} = \{u, v\}$.

We call decompositions satisfying condition (U) *monogamous*. In Section 2, we discuss conditions for the existence of monogamous decompositions of $K_{m,n}$ into 4-cycles.

In Section 3, we define and discuss the existence of symmetric analogues of Howell designs, and provide various constructions for them. We also define, for a regular graph G, so-called self-orthogonal 1-factorizations of the graph $2G$, i.e. of the graph obtained by "doubling" each edge of G, and discuss the existence of decomposable and indecomposable self-orthogonal 1-factorizations.

In Section 4, a theorem concerning the relationship of the structures defined in previous sections is given. We also discuss some additional structures and further relationships.

2. Monogamous decompositions of $K_{m,n}$ into 4-cycles

Definition A decomposition of $K_{m,n}$ into 4-cycles is said to be *monogamous* if it satisfies the following condition:

(U) If (a, x, b, y) and (a, u, b, v) are 4-cycles in the decomposition then $\{x, y\} = \{u, v\}$.

In other words, there is at most one 4-cycle containing any pair of nonadjacent vertices.

Condition (U) forces an additional necessary condition for the existence of a decomposition of $K_{m,n}$ into 4-cycles.

Lemma 2.1. *If there exists a monogamous decomposition of $K_{m,n}$ $(m \leq n)$ into 4-cycles then $n \leq 2m - 2$.*

Proof. The total number of 4-cycles in any decomposition of $K_{m,n}$ into 4-cycles is $mn/4$, thus we must have $\binom{m}{2} \geq mn/4$. □

To avoid from now on the trivial case $m = n = 2$, we assume both $m, n \geq 4$.

Lemma 2.2. *There exists no monogamous decomposition of $K_{4,4}$ or of $K_{4,6}$ into 4-cycles.*

Proof The claim for $K_{4,4}$ is trivial. Let the bipartition of $K_{4,6}$ be $V_1 = \{a, b, c, d\}$, $V_2 = \{1, 2, 3, 4, 5, 6\}$. A monogamous decomposition of $K_{4,6}$ into 4-cycles must contain (w.l.o.g.) three 4-cycles $(a, 1, b, 2), (a, 3, c, 4), (a, 5, d, 6)$ which forces the two vertices b, c to occur in a 4-cycle $(b, 5, c, 6)$ violating condition (U). □

On the other hand, as the following example shows, there exists a monogamous decomposition (unique up to an isomorphism) of $K_{6,6}$ into 4-cycles. [In what follows we use nonnegative integers, and nonnegative integers primed, respectively, to denote the vertices of the two partite sets. We also omit, for the sake of brevity, round brackets and commas separating the vertices of the 4-cycle.]

Example 2.3. A monogamous decomposition of $K_{6,6}$ into 4-cycles.
$1\ 1'\ 2\ 2'$, $1\ 5'\ 4\ 6'$, $1\ 3'\ 6\ 4'$, $2\ 4'\ 3\ 5'$, $2\ 3'\ 5\ 6'$, $3\ 2'\ 4\ 3'$, $3\ 1'\ 6\ 6'$, $4\ 1'\ 5\ 4'$, $5\ 2'\ 6\ 5'$.

Example 2.4. A monogamous decomposition of $K_{6,10}$ into 4-cycles.
$1\ 1'\ 2\ 2'$, $1\ 3'\ 3\ 4'$, $1\ 5'\ 4\ 6'$, $1\ 7'\ 5\ 8'$, $1\ 9'\ 6\ 10'$, $2\ 5'\ 3\ 9'$, $2\ 3'\ 4\ 7'$, $2\ 4'\ 5\ 10'$, $2\ 6'\ 6\ 8'$, $3\ 8'\ 4\ 10'$, $3\ 1'\ 5\ 6'$, $3\ 2'\ 6\ 7'$, $4\ 2'\ 5\ 9'$, $4\ 1'\ 6\ 4'$, $5\ 3'\ 6\ 5'$.

One can adapt the starter method (see,e.g., [DS]) to generate monogamous decompositions of $K_{m,m}$ into 4-cycles. Indeed, let $Z_m \times i, i = 1, 2$ be the partite sets of $K_{m,m}$, and let $\mathcal{B}$ be a set of 4-cycles such that every mixed difference occurs on the edges of the 4-cycles of $\mathcal{B}$ exactly once *and* every (pure) difference between nonadjacent vertices in each of the 4-cycles of $\mathcal{B}$ occurs at most once. Then $\mathcal{B}$ is a set of starter 4-cycles for a monogamous decomposition of $K_{m,m}$ into 4-cycles (modulo m).

Examples of some starter sets for monogamous 4-cycle decompositions of $K_{m,m}$ are given below.

Example 2.5.

$K_{8,8}: 0\ 0'\ 1\ 2',\ 0\ 5'\ 2\ 6'$ (mod 8)
$K_{10,10}: 0\ 2'\ 1\ 4',\ 0\ 8'\ 2\ 9',\ 0\ 5'\ 5\ 0'$ (mod 10)
$K_{12,12}: 0\ 2'\ 1\ 6',\ 0\ 10'\ 2\ 11',\ 0\ 4'\ 4\ 7'$ (mod 12)
$K_{14,14}: 0\ 1'\ 5\ 4',\ 0\ 2'\ 8\ 3',\ 0\ 6'\ 1\ 12',\ 0\ 7'\ 7\ 0'$ (mod 14)

Similarly, a further adaptation of the starter method using two infinite points can be used to generate directly monogamous 4-cycle decompositions of $K_{m,m+2}$.

Example 2.6. Starters for monogamous decompositions into 4-cycles (A, B are the two infinite points):

$K_{8,10}: 0\ 2'\ 1\ 5',\ 0\ 3'\ 4\ 7'$ (mod 8)
$0\ 0'\ 2\ A,\ 1\ 1'\ 3\ A,\ 4\ 4'\ 6\ A,\ 5\ 5'\ 7\ A,$
$2\ 2'\ 4\ B,\ 3\ 3'\ 5\ B,\ 6\ 6'\ 0\ B,\ 7\ 7'\ 1\ B$
$K_{10,12}: 0\ 3'\ 2\ 4',\ 0\ 6'\ 1\ 9'$ (mod 10)
$2i\ (2i)'\ 2i+3\ A,\ 2i+1\ (2i+1)'\ 2i+4\ B$ (mod 10), $0 \le i \le 9$.

To give a general construction of monogamous 4-cycle decompositions, even for $K_{m,m}$, by means of a formula for starters appears elusive — a situation not unlike that for Room squares, for example. Nevertheless, it turns out that the existence of a pair of orthogonal latin squares of order m guarantees the existence of a monogamous decomposition of $K_{2m,2m}$ into 4-cycles. Rather than provide a proof at this point, we postpone it until Section 3 when this implication will become apparent. In fact, in Section 3 a definitive existence theorem for monogamous decompositions of $K_{m,n}$ into 4-cycles will be proved:

Theorem 2.7. *A monogamous decomposition of the complete bipartite graph $K_{m,n}$ ($m \le n$) into 4-cycles exists if and only if $n \le 2m-2, m, n \ge 6$.*

3. Symmetric analogue of Howell designs

Consulting the surveys [R], [DS] seems to suggest that the following symmetric analogue of Room squares, or rather, more generally, Howell designs has not been considered in the literature. [For a definition of a Howell design H$(s, 2n)$, where $n \le s \le 2n-1$, see [DS]; in particular, a Howell design H$(2n-1, 2n)$ is a Room square RS$(2n)$.]

Definition. An SH-square SH(m, n) is an $m \times m$ square array such that

(i) every cell is either empty or contains a 2-subset of an n-element set N;
(ii) every diagonal cell is empty;
(iii) every element of N is contained in exactly one cell of every row (every column); and
(iv) if the 2-subset $\{x, y\}$ of N is contained in an (off-diagonal) cell (a, b) then it is also contained in the cell (b, a)
(v) every 2-subset of N is contained in 0 or 2 cells.

["SH" stands for "symmetric Howell".]

The *dual* of an SH(m,n) is obtained as follows: if the cell (a,b) (and also (b,a), of course) contains the 2-subset $\{x,y\}$ then place the 2-subset $\{a,b\}$ in the cell (x,y) (and in the cell (y,x)). Clearly, the dual is an SH(n,m).

It folllows easily that for an SH(m,n) to exist, both m and n must be even, and

(*) $\frac{n+2}{2} \leq m \leq 2n-2$.

Some examples of SH-squares follow.

Example 3.1.

–	12	–	56	–	34
12	–	45	–	36	–
–	45	–	23	–	16
56	–	23	–	14	–
–	36	–	14	–	25
34	–	16	–	25	–

SH(6,6)

Example 3.2

–	12	34	56	78	90
12	–	59	37	40	68
34	59	–	80	16	27
56	37	80	–	29	14
78	40	16	29	–	35
90	68	27	14	35	–

SH(6,10)

Example 3.3

–	–	–	78	26	45	–	–	13	–
–	–	–	–	18	37	56	–	24	–
–	–	–	–	–	12	48	67	–	35
78	–	–	–	–	–	23	15	–	46
26	18	–	–	–	–	–	34	57	–
45	37	12	–	–	–	–	–	68	–
–	56	48	23	–	–	–	–	–	17
–	–	67	15	34	–	–	–	–	28
13	24	–	–	57	68	–	–	–	–
–	–	35	46	–	–	17	28	–	–

SH(10,8)

Howell designs are instrumental in settling the existence question for SH-squares.

Theorem 3.4. *An SH-square SH(m,n) exists if and only if both $m,n \geq 6$ are even, and the necessary condition (*) is satisfied.*

Proof. It is an easy exercise to see that neither SH(4,4) nor SH(4,6) exist. Assume now m,n are as above, and $m \geq n$. If there exists a Howell design H$(\frac{m}{2},n)$, say A, then

$$\begin{pmatrix} E & A \\ A^T & E \end{pmatrix}$$

is an SH(m,n) (here A^T is the transpose of A, and E is an empty $\frac{m}{2} \times \frac{m}{2}$ array). It is well known (see, e.g., [DS]) that a Howell design H(s,n) exists if and only if $\frac{n}{2} \leq s \leq 2n-1$ except when $(s,n) \in \{(2,4),(3,4),(5,6),(5,8)\}$ in which case H(s,n) does not exist. This settles the existence of SH-squares in all cases when $m \geq n$ except for an SH(10,6) and an SH(10,8). The dual of the former is given in Example 3.2, and the latter is given in Example 3.3. Considering now the dual handles the cases when $m \leq n$, and the proof is complete. □

Theorem 3.5. *A monogamous decomposition of $K_{m,n}$ into 4-cycles exists if and only if there exists an SH-square SH(m,n).*

Proof. If (a,x,b,y) is a 4-cycle in a monogamous decomposition of $K_{m,n}$ into 4-cycles where $a,b \in V_1$, $x,y \in V_2$, $|V_1| = m, |V_2| = n$, then place $\{x,y\}$ in the cells (a,b) and (b,a) of an $m \times m$ array A. Condition (U) guarantees that at most one 2-subset occupies any one off-diagonal cell while the fact that the vertex a is adjacent to every vertex of V_2 in exactly one 4-cycle guarantees that the a-th row of A is latin. Clearly, A is symmetric and its diagonal is empty, so A is an SH(m,n). The converse is equally transparent. □

Theorem 2.7 now follows readily from Theorem 3.4 and Theorem 3.5.

Thus, returning to our remark in Section 2 before Theorem 2.7, we note that a pair of orthogonal latin squares of order m yields a Howell design H$(m,2m)$ which, as indicated in the proof of Theorem 3.4 yields an SH$(2m,2m)$, from which a monogamous decomposition of $K_{2m,2m}$ arises as decribed in Theorem 3.5.

For instance, the SH(6,10) of Example 3.2 gives rise to the following monogamous decomposition of $K_{6,10}$ into 4-cycles:

$1\ 1'\ 2\ 2'$, $1\ 3'\ 3\ 4'$, $1\ 5'\ 4\ 6'$, $1\ 7'\ 5\ 8'$, $1\ 9'\ 6\ 0'$
$2\ 5'\ 3\ 9'$, $2\ 3'\ 4\ 7'$, $2\ 4'\ 5\ 0'$, $2\ 6'\ 6\ 8'$, $3\ 8'\ 4\ 0'$
$3\ 1'\ 5\ 6'$, $3\ 2'\ 6\ 7'$, $4\ 2'\ 5\ 9'$, $4\ 1'\ 6\ 4'$, $5\ 3'\ 6\ 5'$.

4. Self-orthogonal 1-factorizations

A 1-factorization of a (regular) graph G is a partition of the edge-set of G into 1-factors (=perfect matchings). Two 1-factorizations $\mathcal{F}, \mathcal{H}$ of G are *orthogonal* if any two edges of G belong to distinct 1-factors of $\mathcal{H}$ whenever they belong to the same 1-factor of $\mathcal{F}$. It is well known that a Howell design H(s,n) is equivalent to a pair of orthogonal 1-factorizations of some regular graph of degree s with n vertices [DS], [RS].

Given a regular graph G, call a 1-factorization $\mathcal{F}$ of the graph $2G$ (every edge of G is "doubled") *self-orthogonal* if any two distinct 1-factors of $\mathcal{F}$ have at most one edge in common. Clearly, a union of two orthogonal 1-factorizations of G yields a self-orthogonal 1-factorization of $2G$ but the converse is not true in general. A self-orthogonal 1-factorization $\mathcal{F}$ of $2G$ which can be partitioned into two orthogonal 1-factorizations of G is *decomposable*, otherwise, it is *indecomposable*. The above equivalence can therefore be restated as follows: a Howell design $\mathrm{H}(s,n)$ is equivalent to a decomposable self-orthogonal 1-factorization of $2G$, for some regular graph G of degree s with n vertices.

It is now clear that an SH-square $\mathrm{SH}(m,n)$ is equivalent to a (decomposable or indecomposable) self-orthogonal 1-factorization of $2G$, for some regular graph G of degree $\frac{m}{2}$ with n vertices.

The above remarks together with Theorem 3.5 prove:

Theorem 4.1. *The following are equivalent:*

(i) *a monogamous decomposition of $K_{m,n}$ into 4-cycles;*
(ii) *an SH-square $SH(m,n)$;*
(iii) *a self-orthogonal 1-factorization of $2G$, for some regular graph G of degree $\frac{m}{2}$ with n vertices.*

Let us call the graph G in (iii) above the *underlying graph* of the corresponding SH-square $\mathrm{SH}(m,n)$. So, for example, the underlying graph of the SH(6,6) (easily seen to be) unique up to an isomorphism is $K_{3,3}$ while the underlying graph of the SH(6,10) from Example 3.2 is the Petersen graph P. The corresponding self-orthogonal 1-factorization of $2P$ is, of course, indecomposable, as are those of the graphs $2G$ corresponding to the starter generated monogamous decompositions into 4-cycles given in Examples 2.5 and 2.6. It remains an open problem to determine the spectrum of SH-squares $\mathrm{SH}(m,n)$ corresponding to *indecomposable* self-orthogonal 1-factorizations of $2G$ where G is its underlying graph.

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