Bigraph-factorization of symmetric complete bipartite multi-digraphs

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Abstract

We show that a necessary and sufficient condition for the existence of a $K_{p,q}$ - factorization of the symmetric complete bipartite multi-digraph $\lambda K^*_{m,n}$ is (i) $m = n \equiv 0 \pmod{p}$ for $p = q$ and (ii) $m = n \equiv 0$ (mod $d(p'+q')p'q'/e$) for $p \neq q$, where $d = (p,q)$, $p' = p/d$, $q' = q/d$, $e = (\lambda, p'q')$.

1. Introduction

The *symmetric complete bipartite multi-digraph* $\lambda K^*_{m,n}$ is the symmetric complete bipartite digraph $K^*_{m,n}$ in which every arc is taken λ times. Let $K_{p,q}$ denote the *complete bipartite digraph* in which all arcs are directed away from p start-vertices to q end-vertices. A spanning subgraph F of $\lambda K^*_{m,n}$ is called a $K_{p,q}$ - *factor* if each component of F is isomorphic to $K_{p,q}$. If $\lambda K^*_{m,n}$ is expressed as an arc-disjoint sum of $K_{p,q}$ - factors, then this sum is called a $K_{p,q}$ - *factorization* of $\lambda K^*_{m,n}$. In this paper, it is shown that a necessary and sufficient condition for the existence of such a factorization is (i) $m = n \equiv 0 \pmod{p}$ for $p = q$ and (ii) $m = n \equiv 0$ (mod $d(p'+q')p'q'/e$) for $p \neq q$, where $d = (p,q)$, $p' = p/d$, $q' = q/d$, $e = (\lambda, p'q')$.

Let K_{n_1,n_2}, $K^*_{n_1,n_2}$, K_{n_1,n_2,n_3}, and $K^*_{n_1,n_2,n_3}$ denote the complete bipartite graph, the symmetric complete bipartite digraph, the complete tripartite graph, and the symmetric complete tripartite digraph, respectively. Let $\hat{C}_k$, $\hat{S}_k$, $\hat{P}_k$, and $\hat{K}_{p,q}$ denote the cycle or the directed cycle, the star or the directed star, the path or the directed path, and the complete bipartite graph or the complete bipartite digraph, respectively, on two partite sets V_i and V_j. Then the problems of giving the necessary and sufficient conditions of $\hat{C}_k$ - factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, and $K^*_{n_1,n_2,n_3}$ have been completely solved by Enomoto, Miyamoto and Ushio[3] and Ushio[12]. $\hat{S}_k$ - factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, and $K^*_{n_1,n_2,n_3}$ have been studied by Du[2], Ushio and Tsuruno[9], Ushio[14], and Wang[18]. Recently, Martin[5,6] and Ushio[11] gave necessary and sufficient conditions for $\hat{S}_k$ - factorization of K_{n_1,n_2} and $K^*_{n_1,n_2}$. $\hat{P}_k$ - factorization of K_{n_1,n_2} and $K^*_{n_1,n_2}$ have been studied by Ushio and Tsuruno[8], and Ushio[7,10]. $\hat{K}_{p,q}$ - factorization of K_{n_1,n_2} has been studied by Martin[5]. Ushio[13] gave necessary and sufficient conditions for $\hat{K}_{p,q}$ - factorization of $K^*_{n_1,n_2}$. For graph theoretical terms, see [1,4].

2. $K_{p,q}$ - factor of $\lambda K^*_{m,n}$

The following theorem is on the existence of a $K_{p,q}$ - factor of $\lambda K^*_{m,n}$.

Theorem 1. $\lambda K^*_{m,n}$ has a $K_{p,q}$ - factor if and only if for $p = q$ (i) $m = n \equiv 0$ (mod p), and for $p \neq q$ (ii) $m + n \equiv 0$ (mod $p + q$), (iii) $pm - qn \equiv 0$ (mod $p^2 - q^2$), (iv) $pn - qm \equiv 0$ (mod $p^2 - q^2$), (v) $pm \geq qn$ and (vi) $pn \geq qm$.

Proof. (Necessity) Suppose that $\lambda K^*_{m,n}$ has a $K_{p,q}$ - factor F. Let t be the number of components of F. Then $t = (m+n)/(p+q)$. Among these t components, let t_1 and t_2 be the number of components whose start-vertices are in V_1 and V_2, respectively. Then, since F is a spanning subgraph of $\lambda K^*_{m,n}$, we have $pt_1 + qt_2 = m$ and $qt_1 + pt_2 = n$. When $p = q$, we have $pt_1 + pt_2 = m$ and $pt_1 + pt_2 = n$. Therefore, Condition (i) is necessary. When $p \neq q$, we have $t_1 = (pm - qn)/(p^2 - q^2)$ and $t_2 = (pn - qm)/(p^2 - q^2)$. ¿From $0 \leq t_1 \leq m$ and $0 \leq t_2 \leq n$, we must have $pm \geq qn$ and $pn \geq qm$. Condition (ii)-(vi) are, therefore, necessary.
(Sufficiency) When $p = q$, put $m = sp$ and $n = sp$. Then obviously $\lambda K^*_{m,n}$ has a $K_{p,p}$ - factor formed by s $K_{p,p}$'s. When $p \neq q$, for those parameters m and n satisfying (ii)-(vi), let $t_1 = (pm - qn)/(p^2 - q^2)$ and $t_2 = (pn - qm)/(p^2 - q^2)$. Then t_1 and t_2 are integers such that $0 \leq t_1 \leq m$ and $0 \leq t_2 \leq n$. Hence, $pt_1 + qt_2 = m$ and $qt_1 + pt_2 = n$. Using pt_1 vertices in V_1 and qt_1 vertices in V_2, consider t_1 $K_{p,q}$'s whose start-vertices are in V_1. Using the remaining qt_2 vertices in V_1 and the remaining pt_2 vertices in V_2, consider t_2 $K_{p,q}$'s whose start-vertices are in V_2. Then these $t_1 + t_2$ $K_{p,q}$'s are arc-disjoint and they form a $K_{p,q}$ - factor of $\lambda K^*_{m,n}$.

Corollary 2. $\lambda K^*_{n,n}$ has a $K_{p,q}$ - factor if and only if $n \equiv 0$ (mod p) for $p = q$ and $n \equiv 0$ (mod $p + q$) for $p \neq q$.

3. $K_{p,q}$ - factorization of $\lambda K^*_{m,n}$

We use the following notation.

Notation. Given a $K_{p,q}$ - factorization of $\lambda K^*_{m,n}$, let
r be the number of factors
t be the number of components of each factor
b be the total number of components.
Among t components of each factor, let t_1 and t_2 be the numbers of components whose start-vertices are in V_1 and V_2, respectively.

Among r components having vertex x in V_i, let r_{ij} be the numbers of components whose start-vertices are in V_j.

We give the following necessary condition for the existence of a $K_{p,q}$ - factorization of $\lambda K^*_{m,n}$.

Theorem 3. Let $d = (p,q)$, $p' = p/d$, $q' = q/d$, $e = (\lambda, p'q')$. If $\lambda K^*_{m,n}$ has a $K_{p,q}$ - factorization, then (i) $m = n \equiv 0 \pmod{p}$ for $p = q$ and (ii) $m = n \equiv 0$ (mod $d(p'+q')p'q'/e$) for $p \neq q$.

Proof. Suppose that $\lambda K^*_{m,n}$ has a $K_{p,q}$ - factorization. Then $b = 2\lambda mn/pq$, $t = (m+n)/(p+q)$, $r = b/t = 2\lambda mn(p+q)/(m+n)pq$. And $t_1 = (pm-qn)/(p^2-q^2)$, $t_2 = (pn-qm)/(p^2-q^2)$ for $p \neq q$. Moreover, $qr_{11} = \lambda n$, $pr_{12} = \lambda n$, $pr_{21} = \lambda m$, and $qr_{22} = \lambda m$. Thus we have $r = r_{11} + r_{12} = \lambda n(p+q)/pq$ and $r = r_{21} + r_{22} = \lambda m(p+q)/pq$. Therefore, $m = n$ is necessary.
Moreover, when $m = n$ and $p = q$, we have $b = 2\lambda n^2/p^2$, $t = n/p$, $r = 2\lambda n/p$, $r_{11} = r_{12} = r_{21} = r_{22} = \lambda n/p$. Therefore, $n \equiv 0 \pmod{p}$ is also necessary. When $m = n$ and $p \neq q$, we have $b = 2\lambda n^2/d^2p'q'$, $t = 2n/d(p'+q')$, $r = \lambda n(p'+q')/dp'q'$, $r_{11} = r_{22} = \lambda n/dq'$, $r_{12} = r_{21} = \lambda n/dp'$, $t_1 = t_2 = n/d(p'+q')$. Thus we have $n \equiv 0$ (mod $d(p'+q')$) and $\lambda n \equiv 0$ (mod $dp'q'$). Therefore, $n \equiv 0$ (mod $d(p'+q')p'q'/e$) is also necessary.

Lemma 4. Let G, H and K be digraphs. If G has an H - factorization and H has a K - factorization, then G has a K - factorization.

Proof. Let $E(G) = \bigcup_{i=1}^{r} E(F_i)$ be an H - factorization of G. Let $H_j^{(i)}$ $(1 \leq j \leq t)$ be the components of F_i. And let $E(H_j^{(i)}) = \bigcup_{k=1}^{s} E(K_k^{(i,j)})$ be a K - factorization of $H_j^{(i)}$. Then $E(G) = \bigcup_{i=1}^{r} \bigcup_{k=1}^{s} E(\bigcup_{j=1}^{t} K_k^{(i,j)})$ is a K - factorization of G.

We prove the following extension theorems, which we use later in this paper.

Theorem 5. If $\lambda K^*_{n,n}$ has a $K_{p,q}$ - factorization, then $s\lambda K^*_{n,n}$ has a $K_{p,q}$ - factorization for every positive integer s.

Proof. Obvious. Construct a $K_{p,q}$ - factorization of $\lambda K^*_{n,n}$ repeatly s times. Then we have a $K_{p,q}$ - factorization of $s\lambda K^*_{n,n}$.

Theorem 6. If $\lambda K^*_{n,n}$ has a $K_{p,q}$ - factorization, then $\lambda K^*_{sn,sn}$ has a $K_{p,q}$ - factorization for every positive integer s.

Proof. Since $\lambda K^*_{n,n}$ has a $K_{p,q}$ - factorization, $\lambda K^*_{sn,sn}$ has a $K_{sp,sq}$ - factorization. Obviously, $K_{sp,sq}$ has a $K_{p,q}$ - factorization. Therefore, by Lemma 4 $\lambda K^*_{sn,sn}$ has a $K_{p,q}$ - factorization.

We use the following notation for a $K_{p,q}$.

Notation. Let $U = \{u_1, u_2, ..., u_p\}$ and $V = \{v_1, v_2, ..., v_q\}$. For a $K_{p,q}$ whose start-vertex set is U and end-vertex set is V, we denote $(u_1, u_2, \dots, u_p; v_1, v_2, ..., v_q)$ or $(U; V)$.

We give the following sufficient conditions for the existence of a $K_{p,q}$ - factorization of $\lambda K^*_{n,n}$.

Theorem 7. When $n \equiv 0 \pmod p$, $\lambda K^*_{n,n}$ has a $K_{p,p}$ - factorization.

Proof. Put $n = sp$. Let $V_1 = \{1, 2, ..., p\}$ and $V_2 = \{1', 2', ..., p'\}$. Then $(V_1; V_2)$ and $(V_2; V_1)$ are $K_{p,p}$ - factors and they comprise a $K_{p,p}$ - factorization of $K^*_{p,p}$. Applying Theorem 5 and Theorem 6, we see that $\lambda K^*_{n,n}$ has a $K_{p,p}$ - factorization.

Theorem 8. For $p \neq q$, let $d = (p, q)$, $p' = p/d$, $q' = q/d$. When $p'q' = x\lambda$ and $n = d(p' + q')p'q'/\lambda$, $\lambda K^*_{n,n}$ has a $K_{p,q}$ - factorization.

Proof. Let $V_1 = \{1, 2, ..., n\}$, $V_2 = \{1', 2', ..., n'\}$. For $i = 1, 2, ..., p' + q'$ and $j = 1, 2, ..., p' + q'$, construct $(p' + q')^2$ $K_{p,q}$ - factors F_{ij} as following:
$F_{ij} = \{\ ((A + 1, ..., A + p);(B + g + 1, ..., B + g + q)')$
$((A + p + 1, ..., A + 2p);(B + g + q + 1, ..., B + g + 2q)')$
...
$((A + (f - 1)p + 1, ..., A + fp);(B + g + (f - 1)q + 1, ..., B + g + fq)')$
$((B + 1, ..., B + p)';(A + g + 1, ..., A + g + q))$
$((B + p + 1, ..., B + 2p)';(A + g + q + 1, ..., A + g + 2q))$
...
$((B + (f - 1)p + 1, ..., B + fp)';(A + g + (f - 1)q + 1, ..., A + g + fq))\ \}$,
where $A = (i - 1)dp'q'$, $B = (j - 1)dp'q'$, $f = p'q'/\lambda$, $g = fp$, and the additions are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a $K_{p,q}$ - factorization of $\lambda K^*_{n,n}$.

Theorem 9. For $p \neq q$, let $d = (p, q)$, $p' = p/d$, $q' = q/d$, $e = (\lambda, p'q')$. When $n \equiv 0 \pmod{d(p' + q')p'q'/e}$, $\lambda K^*_{n,n}$ has a $K_{p,q}$ - factorization.

Proof. Let $x = p'q'/e$ and $y = \lambda/e$. Then $p'q' = xe$ and $\lambda = ye$. Put $n = d(p' + q')p'q's/e$ and $N = d(p' + q')p'q'/e$. By Theorem 8, $eK^*_{N,N}$ has a $K_{p,q}$ - factorization. Applying Theorem 4 and Theorem 5, we see that $\lambda K^*_{n,n}$ has a $K_{p,q}$ - factorization.

We have the following main theorem and its corollary.

Main Theorem. Let $d = (p, q)$, $p' = p/d$, $q' = q/d$, $e = (\lambda, p'q')$. $\lambda K^*_{m,n}$ has a $K_{p,q}$ - factorization if and only if (i) $m = n \equiv 0 \pmod p$ for $p = q$ and (ii) $m = n \equiv 0 \pmod{d(p' + q')p'q'/e}$ for $p \neq q$.

Corollary[13]. Let $d = (p, q)$, $p' = p/d$, $q' = q/d$. $K^*_{m,n}$ has a $K_{p,q}$ - factorization if and only if (i) $m = n \equiv 0 \pmod{p}$ for $p = q$ and (ii) $m = n \equiv 0 \pmod{d(p' + q')p'q'}$ for $p \neq q$.

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