Packing and covering the complete graph with cubes*

Peter Adams and Darryn E. Bryant

Department of Mathematics
The University of Queensland
Queensland 4072, Australia

Saad I. El-Zanati

Department of Mathematics
Illinois State University
Normal, Illinois 61790-4520 U.S.A.

Abstract

A decomposition of $K_n \setminus L$, the complete graph of order n with a subgraph L (called the leave) removed, into edge disjoint copies of a graph G is called a maximum packing of K_n with G if L contains as few edges as possible. A decomposition of $K_n \cup P$, the complete graph union a graph P (called the padding), into edge disjoint copies of a graph G is called a minimum covering of K_n with G if P contains as few edges as possible. We construct maximum packings and minimum coverings of K_n with the 3-cube for all n.

1 Introduction

A subgraph will be called a *G-subgraph* if it is isomorphic to a given graph G. A *(partial) G-design* of a graph X is a collection of edge disjoint G-subgraphs of X such that each edge of X is in (at most) exactly one G-subgraph. The *spectrum* for G-designs is the set of integers n such that there is a G-design of K_n. When n is not in the spectrum of G-designs two natural questions arise:

(1) For a given graph G, what is the maximum possible number of G-subgraphs of K_n such that no edge of K_n occurs in more than one G-subgraph?

(2) For a given graph G, what is the minimum possible number of G-subgraphs of K_n such that every edge of K_n occurs in at least one G-subgraph?

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These questions are respectively called the maximum packing and minimum covering problem for G-designs. A partial G-design of K_n is called a *maximum packing* with *leave* L if there is no partial G-design of K_n having a leave with fewer edges (equivalently, containing more G-subgraphs). A *G-covering* of K_n with *padding* P is a collection of G-subgraphs of K_n such that every edge of $K_n \cup P$ is in exactly one G-subgraph. A G-covering of K_n is a *minimum covering* if there is no G-covering of K_n having a padding with fewer edges (equivalently, no G-covering of K_n containing fewer .... G-subgraphs). The minimum covering problem usually involves considering the case where the padding is required to be a simple graph and the case where multiple edges are permitted in the padding.

Maximum packings and minimum coverings of graphs have been and continue to be popular topics of research. In particular, maximum packings and minimum coverings of K_n have attracted the most attention. For K_3-designs, the maximum packing problem was solved by Schönheim [25] (see also Spencer [27]) and the minimum covering problem was solved by Fort and Hedlund [10]. For K_4-designs, Brouwer [4] solved the maximum packing problem and Mills [21, 22] solved the minimum covering problem. Various results on maximum packings of K_n with cycles can be found in [9], [11], [12] and [14]. Maximum packings of K_n with various other specific graphs have also been considered (see [2], [13], [23] and [24] for examples). Maximum packings with triples of graphs other than K_n have also been considered (see [3] and [6]). Similarly, minimum coverings of K_n with various graphs have been investigated (see [15], [17], [18], [19], [23], [24] and [26] for examples). It is also worth mentioning the recent asymptotic results [7, 8] which settle the maximum packing and minimum covering problems for very large values of n.

The *d-cube* is the graph Q_d whose vertex set is the set of all binary d-tuples, and whose edge set consists of all pairs of vertices which differ in exactly one coordinate. Note that the 1-cube is simply K_2, the 2-cube is isomorphic to the 4-cycle and the 3-cube is the graph of the well known regular solid that we all know as the cube.

We note that the packings and coverings of K_n with 1-cubes are trivial. Maximum packings and minimum coverings of K_n with 2-cubes (i.e. 4-cycles) are constructed for all values of n in [26]. Kotzig [16] was the first to investigate the spectrum problem for cubes in general. In [20], Maheo showed that the spectrum for 3-cubes is precisely the set of all $n \equiv 1$ or $16 \pmod{24}$. Necessary and sufficient conditions for the existence of Q_3-designs of $K_{m,n}$ and $K_n \setminus K_m$ are found in [5] and in [1], respectively. We note further that the spectrum for Q_d-designs is still not completely settled when $d \geq 5$ is odd (see [5]).

In this paper we construct for every positive integer n, a maximum packing (see Section 2) and a minimum covering (see Section 3) of K_n with 3-cubes. Since in many cases the number of possible leaves and paddings increases without bound as n approaches infinity, we present only one possible leave for each maximum packing and one possible padding for each minimum covering. We construct minimum coverings having paddings with multiple edges in all cases where such paddings contain fewer edges than any possible simple padding.

We will use the following notation. The graph consisting of t vertex disjoint copies of a graph G will be denoted by tG. For H a subgraph of a graph G, define $G \setminus H$ to be the graph with vertex set $V(G\backslash H) = V(G)$ and edge set $E(G\backslash H) = E(G)\backslash E(H)$.

For two graphs G and H, $G \cup H$ will denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let the 8-tuple (a, b, c, d, e, f, g, h) denote the graph of the 3-cube (henceforth denoted by C) as shown in Figure 1.

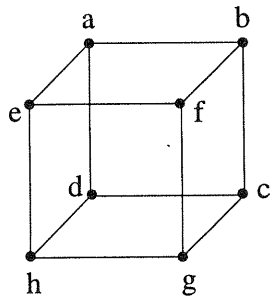


Figure 1: The 3-cube which we denote by C

2 The packing problem

Lemma 2.1 *Let $|E(K_n)| \equiv r$ (mod 12), let $n-1 \equiv d$ (mod 3) and let L be a spanning subgraph (possibly containing isolated vertices) of K_n with $\deg(v) \equiv d$ (mod 3) for each $v \in L$. If $|E(L)| = \min\{\lambda \geq \lceil dn/2 \rceil \mid \lambda \equiv r \text{ (mod 12)}\}$, then a partial C-design of K_n with leave L is a maximum packing.*

Proof: Suppose there is a partial C-design D' of K_n with leave L' which contains more cubes than a partial C-design of K_n which satisfies the conditions of the lemma. Then $|E(L')| \leq |E(L)| - 12 \leq \lceil dn/2 \rceil - 1$. Since $\deg(v) \equiv d$ (mod 3) for each $v \in L'$, we have a contradiction. □

Lemma 2.2 *Suppose there exists a partial C-design of K_n with leave L. Then there exists a partial C-design of K_{n+24} with leave L^* where $L^* = L$ if $n \equiv 1$ (mod 3), $L^* = L + 12K_2$ if $n \equiv 2$ (mod 3) and $L^* = L + 6C_4$ if $n \equiv 0$ (mod 3).*

Proof: Note that the C-designs referred to below can be found in [5] or in the Appendix.

If $n \equiv 0$ (mod 3) then we construct a partial C-design of K_{n+24} by taking the union of the partial C-design of K_n, a C-design of $K_{n,24}$ and a maximum packing of K_{24} with leave $6C_4$.

If $n \equiv 2$ (mod 3) then we construct a partial C-design of K_{n+24} by taking the union of the partial C-design of K_n, a C-design of $K_{n-2,24}$ and a maximum packing of K_{26} with leave $13K_2$. Note that $V(K_{26})$ has two vertices in common with K_n and that these are joined by an edge which is in both L and $13K_2$.

If $n \equiv 1$ (mod 3) then we construct a partial C-design of K_{n+24} by taking the union of the partial C-design of K_n, a C-design of $K_{n-1,24}$ and a C-design of K_{25}. Note that $V(K_{25})$ has exactly one vertex in common with K_n. □

We are now ready to construct maximum packings of K_n for all positive integers n. For $n \leq 7$, a maximum packing of K_n contains zero copies of C and has leave K_n. For $8 \leq n \leq 31$ and $n = 34$, a maximum packing of K_n is given in the Appendix and the corresponding leave L_n is shown in Table 1. Note that the case $n = 10$ is special. It can be shown by exhaustive computer search that there do not exist three edge disjoint copies of C in K_{10} and so the leave in a maximum packing of K_{10} must contain at least 21 edges. However, there exists a maximum packing of K_{34} having just nine edges in its leave, so we start the recursive construction for the case $n \equiv 10 \pmod{24}$ at $n = 34$. For $n = 32, 33$ and for $n \geq 35$, we construct a maximum packing of K_n by repeated application of Lemma 2.2 to the maximum packings given for K_n with $n \in \{8, 9, 11, 12, 13, \ldots, 31, 34\}$. It is straight forward to verify, using Lemma 2.1 that the partial C-designs obtained in this manner are maximum packings. We now describe the leaves of the constructed maximum packings. Let $t \in \{8, 9, 11, 12, 13, \ldots, 31, 34\}$, let L_t denote the leave of a maximum packing of K_t as shown in Table 1, and let $x \geq 0$. If $t \equiv 0 \pmod{3}$, then the leave of the constructed maximum packing of K_{24x+t} is the vertex disjoint union of L_t with $6xC_4$. If $t \equiv 1 \pmod{3}$, then the leave of the constructed maximum packing of K_{24x+t} is L_t. If $t \equiv 2 \pmod{3}$, then the leave of the constructed maximum packing of K_{24x+t} is the vertex disjoint union of L_t with a 1-factor on the remaining $24x$ vertices of K_{24x+t}.

Combining the above results, we have the following theorem:

Theorem 2.3 *A maximum packing of K_n has leave of size:*

21	for $n = 7$ and 10	n	for $n \equiv 0, 3 \pmod{24}$
0	for $n \equiv 1, 16 \pmod{24}$	$n+3$	for $n \equiv 9, 18 \pmod{24}$
6	for $n \equiv 4, 13 \pmod{24}$	$n+6$	for $n \equiv 12, 15 \pmod{24}$
9	for $n \equiv 7, 10 \pmod{24}$, $n \geq 31$	$n+9$	for $n \equiv 6, 21 \pmod{24}$
15	for $n \equiv 19, 22 \pmod{24}$	$\frac{n}{2}$	for $n \equiv 2, 8, 14, 20 \pmod{24}$
$\frac{n+3}{2}$	for $n \equiv 11, 23 \pmod{24}$	$\frac{n+15}{2}$	for $n \equiv 5, 17 \pmod{24}$

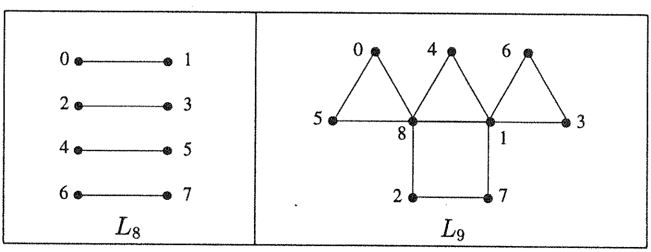


Table 1: The leave L_n for a maximum packing of K_n with 3-cubes in the cases $8 \leq n \leq 31$ and $n = 34$ (continued on subsequent pages).

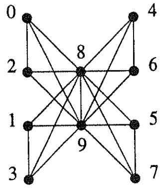
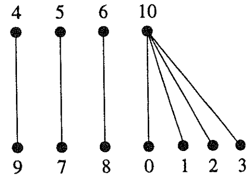
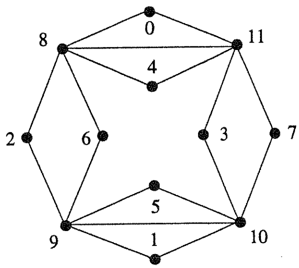
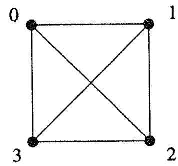
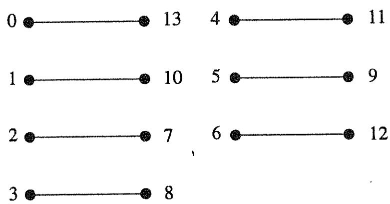
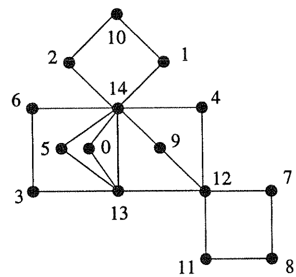
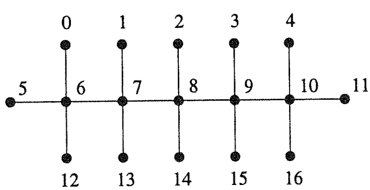
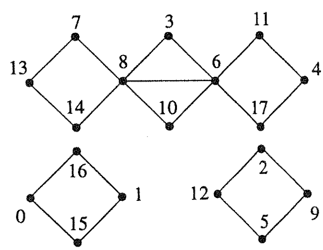
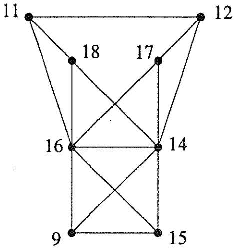
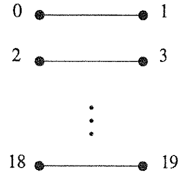
 L_{10}	 L_{11}
 L_{12}	 L_{13}
 L_{14}	 L_{15}
 L_{17}	 L_{18}

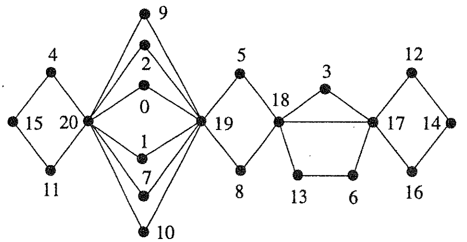
Table 1. (continued)



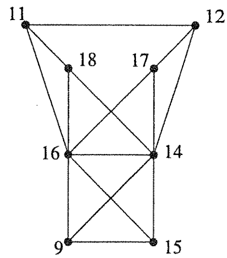
L_{19}



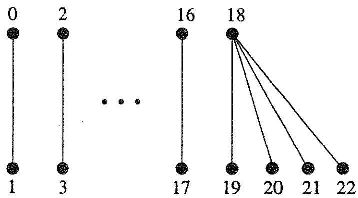
L_{20}



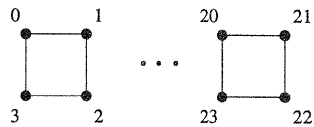
L_{21}



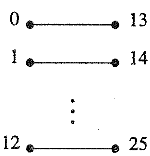
L_{22}



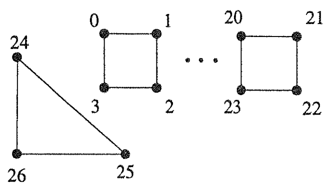
L_{23}



L_{24}

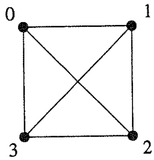


L_{26}

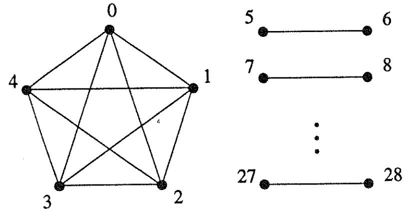


L_{27}

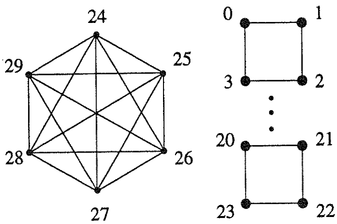
Table 1. (continued)



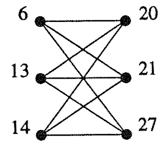
L_{28}



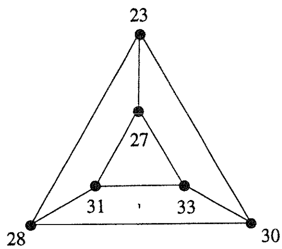
L_{29}



L_{30}



L_{31}



L_{34}

Table 1. (continued)

3 The covering problem

Lemma 3.1 *A C-covering of K_n which contains $\lceil n\lceil (n-1)/3\rceil /8\rceil$ copies of C is a minimum covering.*

Proof: Each vertex of K_n must occur in at least $\lceil (n-1)/3\rceil$ copies of C and so there must be at least $\lceil n\lceil (n-1)/3\rceil /8\rceil$ copies of C. □

Lemma 3.2 *Suppose there exists a C-covering of K_n with padding P. Then there exists a C-covering of K_{n+24} with padding P^* where $P^* = P$ if $n \equiv 1 \pmod 3$, $P^* = P + 6C_4$ if $n \equiv 2 \pmod 3$ and $P^* = P + 12K_2$ if $n \equiv 0 \pmod 3$.*

Proof: Note that the required C-designs can be found in [5] or in the Appendix. If $n \equiv 1 \pmod 3$, we construct a C-covering of K_{n+24} by taking the union of the C-covering of K_n, a C-design of $K_{n-1,24}$ and a C-design of K_{25}. Note that $V(K_{25})$ has exactly one vertex in common with K_n. If $n \equiv 2 \pmod 3$, we construct a C-covering of K_{n+24} by taking the union of the C-covering of K_n, a C-design of $K_{n-2,24}$, a maximum packing of K_{26} with leave $13K_2$ (one of the edges in this leave has the two vertices which are not in the C-design of $K_{n-2,24}$ as its endpoints) and a C-covering of $12K_2$. If $n \equiv 0 \pmod 3$, we construct a C-covering of K_{n+24} by taking the union of the C-covering of K_n, a C-design of $K_{n,24}$ and a minimum C-covering of K_{24}. □

We are now ready to construct minimum coverings of K_n for all positive integers n. For $2 \leq n \leq 7$, no covering of K_n exists. For $8 \leq n \leq 31$ and $n = 34$, minimum coverings of K_n are given in the Appendix and the corresponding paddings are shown in Table 2. Note that the cases $n \equiv 7$ and $10 \pmod{24}$ are special: the number of edges in K_n is $9 \pmod{12}$ and hence it is possible that a minimum covering of K_n will have a padding that consists of just three edges. However, such a padding must have all vertices of degree $0 \pmod 3$ and so the only such padding is the multigraph consisting of a triple edge on two vertices. If we require that the padding be a simple graph, then it must contain at least 15 edges. It can be shown by exhaustive computer search that it is not possible to cover the edges of K_{10} with four or fewer copies of C (even if non-simple paddings are permitted) and so the minimum covering of K_{10} has a padding consisting of at least 15 edges. A covering of K_{10} and a covering of K_{31} each with simple 15 edge paddings is given in the Appendix. Minimum coverings of K_{31} and K_{34} with paddings consisting of a triple edge are also given in the Appendix. Hence, by repeated application of Lemma 3.2 to our minimum coverings of K_{10}, K_{31} and K_{34}, we have minimum coverings of K_{24x+7} and K_{24x+10} (for all $x \geq 0$) in both the case when simple paddings are required and also in the case where they are not. If non-simple paddings are permitted, then the padding for these minimum coverings is always a triple edge. If simple paddings are required, then for the case $n \equiv 10 \pmod{24}$, the padding is isomorphic to the 15 edge padding of K_{10} shown in Table 2 and for the case $n \equiv 7 \pmod{24}$, the padding is isomorphic to the 15 edge padding of K_{31} shown in Table 2. The cases $n \equiv 7, 10 \pmod{24}$ are the only ones for which it is possible to obtain a smaller minimum covering if non-simple paddings are permitted.

For the remaining congruence classes of n, with $n \geq 32$, we construct a minimum covering of K_n by repeated application of Lemma 3.2 to the minimum coverings given for K_n with $n \in \{8, 9, 11, 12, \ldots, 30\}$. It is straight forward to verify, using Lemma 3.1 that the C-coverings obtained in this manner are minimum coverings. We now describe the paddings of the constructed minimum coverings. Let $t \in \{8, 9, 11, 12, \ldots, 31\}$, let P_t denote the padding of a minimum covering of K_t as shown in Table 2, and let $x \geq 0$. If $t \equiv 0 \pmod 3$, then the padding of the constructed minimum covering of K_{24x+t} is the vertex disjoint union of P_t with a 1-factor on the remaining $24x$ vertices of K_{24x+t}. If $t \equiv 1 \pmod 3$, then the padding of the constructed minimum covering of K_{24x+t} is P_t. If $t \equiv 2 \pmod 3$, then the padding of the constructed minimum covering of K_{24x+t} is the vertex disjoint union of P_t with $6xC_4$.

Combining the above results, we have the following theorem:

Theorem 3.3 *For $2 \leq n \leq 7$, no covering of K_n exists. For the remaining values of n, a minimum covering of K_n has padding of size:*

15	for $n = 10$	0	for $n \equiv 1, 16 \pmod{24}$
3	for $n \equiv 7, 10 \pmod{24}$, $n \geq 31$ (non-simple paddings allowed)	15	for $n \equiv 7, 10 \pmod{24}$, $n \geq 31$ (if simple paddings required)
6	for $n \equiv 4, 13 \pmod{24}$	9	for $n \equiv 19, 22 \pmod{24}$
n	for $n \equiv 8, 23 \pmod{24}$	$n+3$	for $n \equiv 14, 17 \pmod{24}$
$n+6$	for $n \equiv 11, 20 \pmod{24}$	$n+9$	for $n \equiv 2, 5 \pmod{24}$
$\frac{n+12}{2}$	for $n \equiv 6, 18 \pmod{24}$	$\frac{n}{2}$	for $n \equiv 0, 12 \pmod{24}$
$\frac{n+15}{2}$	for $n \equiv 3, 9, 15, 21 \pmod{24}$		

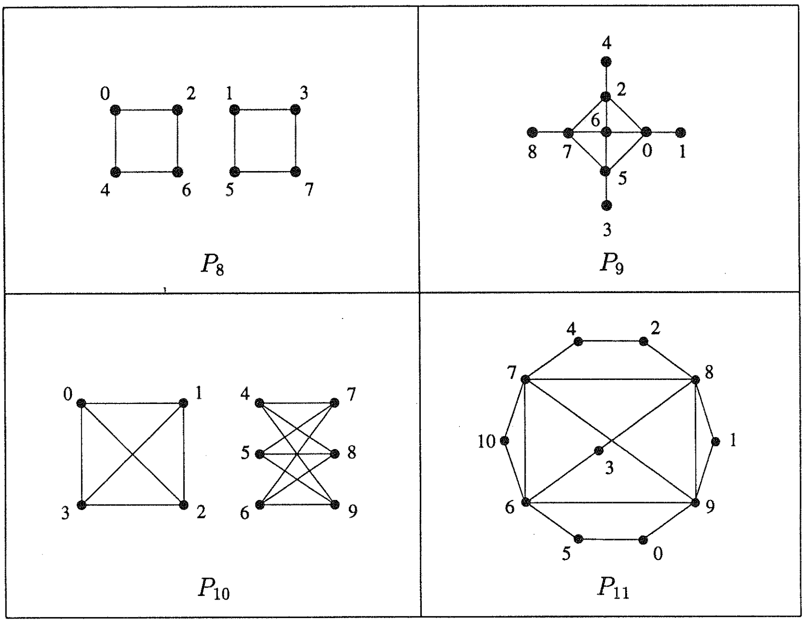
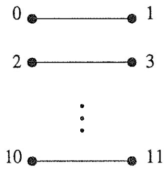
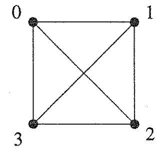


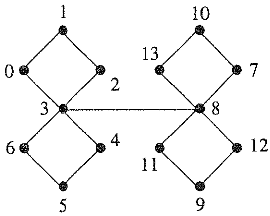
Table 2: The padding P_n for a minimum covering of K_n with 3-cubes in the cases $8 \leq n \leq 31$ and $n = 34$. Note that for $n = 31$ and $n = 24$, the minimum covering is smaller if multiple edges are permitted in the padding. In these cases, the multigraph paddings are denoted by P^*_{31} and P^*_{34} respectively (continued on subsequent pages).



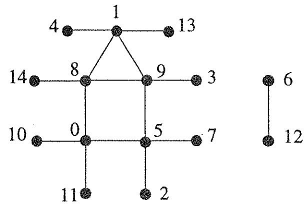
P_{12}



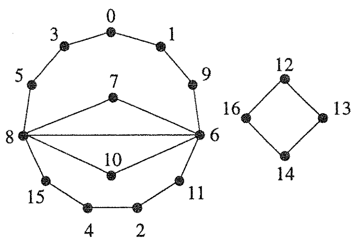
P_{13}



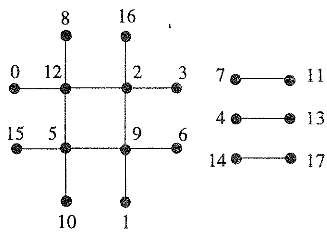
P_{14}



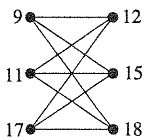
P_{15}



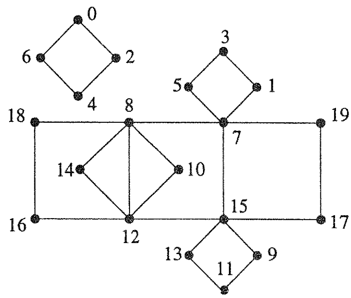
P_{17}



P_{18}

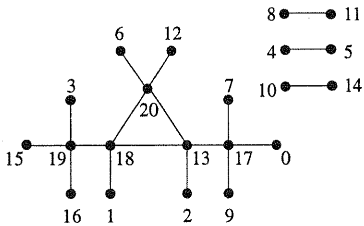


P_{19}

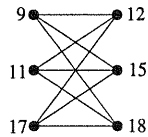


P_{20}

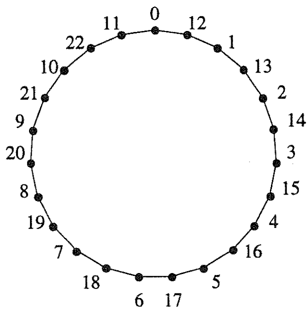
Table 2. (continued)



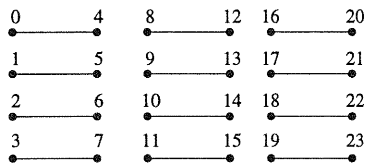
P_{21}



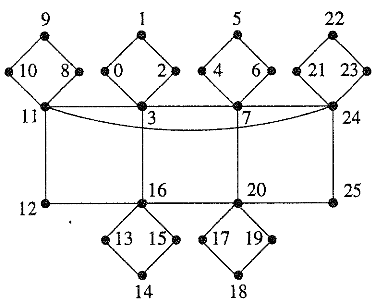
P_{22}



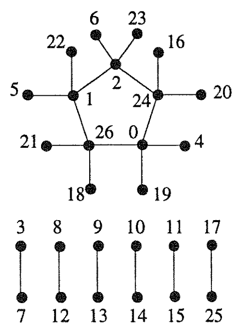
P_{23}



P_{24}

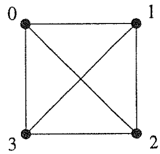


P_{26}

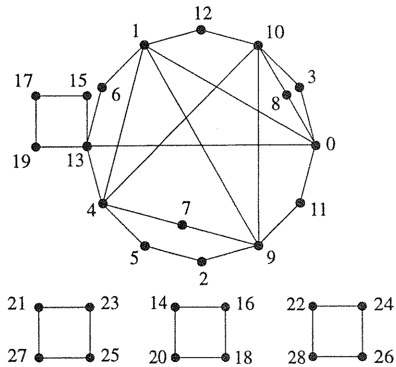


P_{27}

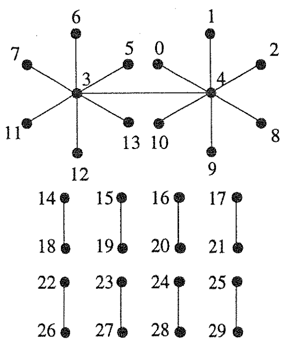
Table 2. (continued)



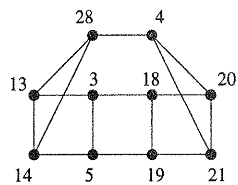
P_{28}



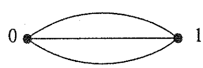
P_{29}



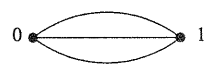
P_{30}



P_{31}



P_{31}^{*}



P_{34}^{*}

Table 2. (continued)

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4 Appendix

Within this Appendix, the graph C of the 3-cube with vertices and edges as shown in Figure 1 will be denoted by $(v_1, v_2, \ldots, v_8)$.

$\boxed{K_8}$

Let the vertex set of K_8 be $\{0, 1, \ldots, 7\}$. A maximum packing of K_8 with 3-cubes and leave as shown in Table 1 is given by:

$$\boxed{(0,1,2,3,4,5,6,7), \quad (0,6,3,5,7,1,4,2).}$$

A covering of K_8 with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cube:

$$\boxed{(0,2,4,6,1,3,5,7).}$$

$\boxed{K_9}$

Let the vertex set of K_9 be $\{0,1,\ldots,8\}$. A maximum packing of K_9 with 3-cubes and leave as shown in Table 1 is given by:

$$\boxed{(0,1,2,3,4,5,6,7),\quad (0,2,4,6,7,5,3,8).}$$

A covering of K_9 with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$$\boxed{(0,2,8,5,6,7,1,3),\quad (0,8,4,1,5,7,2,6).}$$

$\boxed{K_{10}}$

Let the vertex set of K_{10} be $\{0,1,\ldots,9\}$. A maximum packing of K_{10} with 3-cubes and leave as shown in Table 1 is given by:

$$\boxed{(0,5,3,6,7,2,4,1),\quad (0,1,2,3,4,5,6,7).}$$

A covering of K_{10} with 3-cubes and padding as shown in Table 2 is given by:

$$\boxed{\begin{array}{lll}(0,1,2,3,4,5,6,7), & (0,1,3,2,5,7,8,9), & (0,2,1,3,6,8,4,9),\\ (0,7,5,8,9,4,3,6), & (1,6,4,8,9,7,2,5). & \end{array}}$$

$\boxed{K_{11}}$

Let the vertex set of K_{11} be $\{0,1,\ldots,10\}$. A maximum packing of K_{11} with 3-cubes and leave as shown in Table 1 is given by:

$$\boxed{(0,1,2,3,4,5,6,7),\quad (0,2,4,6,8,9,1,3),\quad (0,5,8,7,9,3,4,10),\quad (1,6,9,7,8,10,5,2).}$$

A covering of K_{11} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$$\boxed{(0,10,1,9,8,6,5,7),\quad (9,6,7,4,8,3,10,2).}$$

$\boxed{K_{12}}$

Let the vertex set of K_{12} be $\{0,1,\ldots,11\}$. A maximum packing of K_{12} with 3-cubes and leave as shown in Table 1 is given by:

$$\boxed{(0,1,2,3,4,5,6,7),\quad (0,2,4,6,7,5,3,1),\quad (0,5,8,9,10,11,1,4),\quad (2,7,8,10,11,9,3,6).}$$

A covering of K_{12} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$$(0,8,4,11,1,9,5,10), \quad (8,2,9,6,11,3,10,7).$$

K_{13}

A maximum packing of K_{13} with 3-cubes and leave as shown in Table 1 can be found in [1]. A covering of K_{13} with 3-cubes and padding as shown in Table 2 is given by:

$(0,1,2,3,4,5,6,7)$,	$(0,1,2,3,5,7,8,9)$,	$(0,2,4,6,7,9,1,10)$,	$(0,2,5,10,11,7,12,4)$,
$(0,8,6,9,12,3,1,11)$,	$(1,3,5,8,12,6,11,10)$,	$(2,10,3,11,12,9,4,8)$.	

K_{14}

Let the vertex set of K_{14} be $\{0,1,\ldots,13\}$. A maximum packing of K_{14} with 3-cubes and leave as shown in Table 1 is given by:

$(0,2,4,6,7,5,3,1)$,	$(1,9,2,12,13,3,10,5)$,	$(0,5,8,9,10,11,1,4)$,	$(0,8,2,11,12,4,13,7)$,
$(0,1,2,3,4,5,6,7)$,	$(3,6,8,11,12,13,10,9)$,	$(6,9,7,10,11,13,8,12)$.	

A covering of K_{14} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$$(0,1,2,3,13,10,7,8), \quad (3,4,5,6,7,11,9,12).$$

K_{15}

Let the vertex set of K_{15} be $\{0,1,\ldots,14\}$. A maximum packing of K_{15} with 3-cubes and leave as shown in Table 1 is given by:

$(0,1,2,3,4,5,6,7)$,	$(1,9,5,12,13,7,10,8)$,	$(0,5,8,9,10,11,1,4)$,	$(0,8,2,11,12,3,9,6)$,
$(0,2,4,6,7,5,3,1)$,	$(2,7,14,12,13,11,3,10)$,	$(4,8,14,11,13,6,10,9)$.	

A covering of K_{15} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$$(0,13,5,14,11,12,7,8), \quad (3,6,14,13,9,12,4,1), \quad (0,5,9,8,10,2,14,1).$$

K_{16}

It is well-known that there exists a decomposition of K_{16} into 3-cubes (see [5]).

K_{17}

Let the vertex set of K_{17} be $\{0,1,\ldots,16\}$. A maximum packing of K_{17} with 3-cubes and leave as shown in Table 1 is given by:

$(0,1,2,3,4,5,7,11)$,	$(0,2,4,7,8,5,3,10)$,	$(0,5,9,11,12,10,1,8)$,
$(0,9,2,10,13,4,6,14)$,	$(0,14,1,15,16,2,11,13)$,	$(1,3,8,6,12,7,16,11)$,
$(1,4,8,13,16,14,15,12)$,	$(2,12,3,13,15,4,16,6)$,	$(3,6,9,14,15,10,13,5)$,
$(5,11,14,12,16,15,7,9)$.		

A covering of K_{17} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(11,10,4,2,6,9,15,8)$,	$(5,6,0,3,8,7,1,9)$,	$(6,7,8,10,12,13,14,16)$.

$\boxed{K_{18}}$

Let the vertex set of K_{18} be $\{0,1,\ldots,17\}$. A maximum packing of K_{18} with 3-cubes and leave as shown in Table 1 is given by:

$(10,15,13,16,17,14,12,11)$,	$(8,11,10,12,17,9,14,16)$,	$(6,12,17,13,16,7,15,8)$,
$(0,1,2,3,4,5,6,7)$,	$(0,2,4,6,7,5,3,1)$,	$(0,5,8,9,10,13,1,4)$,
$(0,8,2,11,12,4,16,3)$,	$(0,13,3,14,17,2,10,7)$,	$(1,9,6,14,17,3,15,5)$,
$(1,10,5,11,12,9,16,15)$,	$(2,7,11,14,15,9,13,4)$.	

A covering of K_{18} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(8,7,13,14,6,11,4,17)$,	$(3,6,10,8,2,9,5,12)$,	$(16,1,15,0,2,9,5,12)$.

$\boxed{K_{19}}$

Let the vertex set of K_{19} be $\{0,1,\ldots,18\}$. A maximum packing of K_{19} with 3-cubes and leave as shown in Table 1 is given by:

$(0,1,2,3,4,5,6,7)$,	$(0,2,4,6,7,5,3,1)$,	$(0,5,8,9,10,11,1,4)$,
$(0,8,2,11,12,3,9,6)$,	$(0,13,1,14,15,2,10,3)$,	$(0,16,1,17,18,2,12,4)$,
$(1,9,5,15,18,7,10,6)$,	$(2,7,8,14,17,11,4,13)$,	$(3,6,8,11,13,16,10,9)$,
$(3,16,5,17,18,8,12,9)$,	$(4,14,11,15,16,7,13,12)$,	$(5,13,6,14,18,15,17,10)$,
$(7,12,10,15,17,18,13,8)$.		

A covering of K_{19} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(12,17,15,14,11,16,9,18)$,	$(14,9,15,16,17,12,11,18)$.

$\boxed{K_{20}}$

Let the vertex set of K_{20} be $\{0,1,\ldots,19\}$. A maximum packing of K_{20} with 3-cubes and leave as shown in Table 1 is given by:

$(0,2,1,3,4,6,5,7)$,	$(0,5,2,7,8,3,4,1)$,	$(0,6,1,9,10,3,11,2)$,
$(0,11,4,12,13,5,8,2)$,	$(0,14,1,15,16,2,17,3)$,	$(0,17,4,18,19,5,9,3)$,
$(1,10,4,13,16,5,14,6)$,	$(1,12,5,18,19,6,15,2)$,	$(3,12,7,13,14,8,10,9)$,
$(4,15,7,16,19,8,11,9)$,	$(6,8,7,9,10,13,14,12)$,	$(6,11,12,17,18,13,15,9)$,
$(7,17,8,18,19,10,16,12)$,	$(10,14,11,15,18,16,19,17)$,	$(11,16,13,17,18,15,19,14)$.

A covering of K_{20} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(0,2,4,6,1,3,5,7)$,	$(8,10,12,14,9,11,13,15)$,	$(16,18,8,12,17,19,7,15)$.

$\boxed{K_{21}}$

Let the vertex set of K_{21} be $\{0, 1, \ldots, 20\}$. A maximum packing of K_{21} with 3-cubes and leave as shown in Table 1 is given by:

$(0,1,2,3,4,5,6,7)$,	$(0,2,4,6,7,5,3,1)$,	$(0,5,8,9,10,11,1,4)$,
$(0,8,2,11,12,3,9,6)$,	$(0,13,1,14,15,2,10,3)$,	$(0,16,1,17,18,2,12,4)$,
$(1,9,5,15,18,7,10,6)$,	$(2,7,8,14,17,11,4,13)$,	$(3,6,8,11,13,16,10,9)$,
$(3,16,4,19,20,5,14,6)$,	$(5,12,7,13,17,8,15,10)$,	$(7,14,9,16,17,15,12,20)$,
$(8,13,12,16,20,15,18,19)$,	$(9,15,19,17,18,16,11,14)$,	$(10,12,19,14,18,11,13,20)$.

A covering of K_{21} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(5,15,8,19,4,20,11,15)$,	$(9,19,10,20,17,16,14,12)$,	$(2,19,7,20,13,6,17,18)$,
$(0,19,1,20,17,3,18,13)$.		

$\boxed{K_{22}}$

Let the vertex set of K_{22} be $\{0, 1, \ldots, 21\}$. A maximum packing of K_{22} with 3-cubes and leave as shown in Table 1 is given by:

$(0,1,2,3,4,5,6,7)$,	$(0,2,4,6,7,5,3,1)$,	$(0,5,8,9,10,11,1,4)$,
$(0,8,2,11,12,3,9,6)$,	$(0,13,1,14,15,2,10,3)$,	$(0,16,1,17,18,2,12,4)$,
$(0,19,1,20,21,3,18,5)$,	$(1,9,5,15,21,7,10,6)$,	$(2,7,8,14,17,11,4,13)$,
$(2,19,4,20,21,8,15,10)$,	$(3,6,8,11,13,16,10,9)$,	$(3,16,5,17,20,7,12,8)$,
$(4,14,19,16,21,11,13,12)$,	$(5,13,6,14,19,18,17,10)$,	$(6,18,12,19,20,21,15,11)$,
$(7,13,8,18,19,21,16,20)$,	$(7,14,20,15,17,21,9,19)$,	$(9,12,20,17,18,10,13,15)$.

A covering of K_{22} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(12,17,15,14,11,16,9,18)$,	$(14,9,15,16,17,12,11,18)$.

$\boxed{K_{23}}$

Let the vertex set of K_{23} be $\{0, 1, \ldots, 22\}$. A maximum packing of K_{23} with 3-cubes and leave as shown in Table 1 is given by:

$(0,1,2,3,4,5,6,7)$,	$(0,2,4,6,7,5,3,1)$,	$(0,5,8,9,10,11,1,4)$,
$(0,8,2,11,12,3,9,6)$,	$(0,13,1,14,15,2,10,3)$,	$(0,16,1,17,18,2,12,4)$,
$(0,19,1,20,21,2,22,3)$,	$(1,9,5,15,18,7,10,6)$,	$(2,7,8,14,17,11,4,13)$,
$(3,6,8,11,13,16,10,9)$,	$(3,16,5,17,18,8,12,9)$,	$(4,14,5,19,20,6,13,7)$,
$(4,15,7,16,21,8,17,12)$,	$(5,18,10,20,21,11,12,13)$,	$(6,17,10,19,21,14,15,9)$,
$(7,12,15,21,22,14,18,17)$,	$(8,13,18,20,22,15,19,12)$,	$(9,14,19,16,22,20,21,18)$,
$(10,13,11,14,22,19,20,16)$.		

A covering of K_{23} with 3-cubes and padding as shown in Table 2 may be obtained by rotating the 3-cube $(0, 10, 1, 12, 5, 3, 7, 4)$ modulo 23.

$\boxed{K_{24}}$

Let the vertex set of K_{24} be $\{0,1,\ldots,23\}$. A maximum packing of K_{24} with 3-cubes and leave as shown in Table 1 is given by:

$(0,2,4,6,7,5,1,3)$,	$(0,4,3,5,8,10,12,14)$,	$(0,9,1,10,11,2,6,13)$,
$(0,12,1,13,14,2,7,9)$,	$(0,15,1,16,17,2,8,3)$,	$(0,18,1,19,20,2,21,3)$,
$(0,21,4,22,23,5,8,6)$,	$(1,11,3,14,17,4,9,6)$,	$(1,20,5,22,23,4,12,7)$,
$(2,10,3,13,16,5,15,4)$,	$(2,19,8,22,23,9,12,11)$,	$(3,18,9,22,23,8,15,10)$,
$(4,14,7,18,19,10,16,6)$,	$(5,9,11,17,18,16,14,20)$,	$(5,11,7,13,19,15,17,22)$,
$(6,10,21,11,15,7,8,20)$,	$(6,12,16,20,21,18,11,19)$,	$(7,19,12,20,21,14,17,9)$,
$(8,13,15,16,17,19,23,21)$,	$(10,17,23,18,20,13,16,22)$,	$(12,21,15,22,23,13,18,14)$.

A covering of K_{24} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(0,1,2,3,4,5,6,7)$,	$(8,9,10,11,12,13,14,15)$,	$(16,17,18,19,20,21,22,23)$.

$\boxed{K_{25}}$

It is well-known that there exists a decomposition of K_{25} into 3-cubes (see [5]).

$\boxed{K_{26}}$

Let the vertex set of K_{26} be $\{0,1,\ldots,25\}$. A maximum packing of K_{26} with 3-cubes and leave as shown in Table 1 may be obtained by rotating the 3-cube $(0,10,1,12,5,3,7,4)$ modulo 26.
A covering of K_{26} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(0,1,2,3,13,14,15,16)$,	$(4,5,6,7,17,18,19,20)$,	$(8,9,10,11,21,22,23,24)$,
$(11,12,7,3,24,25,20,16)$.		

$\boxed{K_{27}}$

Let the vertex set of K_{27} be $\{0,1,\ldots,26\}$. A maximum packing of K_{27} with 3-cubes and leave as shown in Table 1 is given by:

$(0,2,4,6,7,5,1,3)$,	$(0,4,3,5,8,10,12,14)$,	$(0,9,1,10,11,2,6,13)$,
$(0,12,1,13,14,2,7,9)$,	$(0,15,1,16,17,2,8,3)$,	$(0,18,1,19,20,2,21,3)$,
$(0,21,4,22,23,5,8,6)$,	$(0,24,1,25,26,2,22,3)$,	$(1,11,3,14,17,4,9,6)$,
$(1,20,4,23,26,5,12,7)$,	$(2,10,3,13,16,5,15,4)$,	$(2,19,8,23,25,4,18,9)$,
$(3,18,10,23,24,5,17,11)$,	$(4,14,7,24,26,10,15,6)$,	$(5,9,11,19,22,12,6,10)$,
$(5,11,7,13,25,12,8,15)$,	$(6,16,7,18,19,9,17,12)$,	$(6,20,7,21,25,8,22,11)$,
$(7,10,20,19,25,16,11,14)$,	$(8,13,16,21,24,17,14,18)$,	$(8,16,15,17,26,12,20,22)$,
$(9,15,26,20,22,18,16,24)$,	$(9,21,10,24,26,17,25,23)$,	$(11,15,23,18,26,19,21,13)$,
$(12,21,14,23,24,15,22,19)$,	$(13,19,25,22,23,17,20,16)$,	$(13,20,14,24,25,18,26,21)$.

A covering of K_{27} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(0,1,2,3,4,5,6,7)$,	$(8,9,10,11,12,13,14,15)$,	$(16,17,18,19,20,21,22,23)$,
$(16,17,18,19,24,25,26,0)$,	$(20,21,22,23,24,26,1,2)$.	

$\boxed{K_{28}}$

Let the vertex set of K_{28} be $\{0, 1, \dots, 27\}$. A 3-cube decomposition of $K_{28} \setminus K_4$ is given in [1]. This gives a maximum packing of K_{28} with 3-cubes and leave as shown in Table 1.
A covering of K_{28} with 3-cubes and leave as shown in Table 2 is given by:

(0, 1, 2, 3, 4, 5, 6, 7),	(0, 1, 2, 3, 5, 7, 8, 9),	(0, 2, 4, 6, 7, 9, 1, 10),
(0, 2, 5, 8, 9, 10, 3, 4),	(0, 10, 4, 11, 12, 5, 13, 1),	(0, 13, 2, 14, 15, 3, 11, 5),
(0, 16, 1, 17, 18, 2, 15, 4),	(0, 19, 1, 20, 21, 2, 22, 3),	(0, 22, 4, 23, 24, 5, 16, 3),
(0, 25, 1, 26, 27, 2, 23, 5),	(1, 3, 6, 8, 14, 12, 9, 11),	(1, 3, 8, 18, 21, 14, 10, 11),
(1, 6, 11, 24, 27, 12, 7, 13),	(2, 7, 14, 17, 20, 15, 6, 13),	(2, 12, 4, 24, 26, 8, 14, 9),
(3, 17, 5, 18, 19, 6, 20, 7),	(3, 25, 4, 26, 27, 6, 21, 7),	(4, 19, 8, 20, 27, 9, 13, 10),
(5, 19, 10, 21, 25, 11, 12, 13),	(6, 16, 7, 22, 23, 8, 17, 9),	(6, 18, 10, 24, 26, 12, 15, 14),
(7, 23, 12, 24, 25, 10, 16, 15),	(8, 15, 9, 21, 22, 11, 16, 17),	(8, 24, 16, 25, 27, 17, 18, 14),
(9, 18, 19, 20, 25, 21, 12, 17),	(10, 17, 15, 22, 26, 11, 13, 16),	(11, 20, 14, 23, 27, 16, 19, 24),
(12, 20, 23, 22, 25, 18, 26, 24),	(13, 14, 22, 18, 23, 16, 21, 15),	(13, 19, 25, 22, 26, 17, 23, 27),
(15, 19, 22, 26, 27, 21, 20, 25),	(18, 23, 21, 24, 27, 19, 26, 20).	

$\boxed{K_{29}}$

Let the vertex set of K_{29} be $\{0, 1, \dots, 28\}$. A maximum packing of K_{29} with 3-cubes and leave as shown in Table 1 is given by:

(0, 5, 1, 6, 7, 2, 8, 3),	(0, 8, 4, 9, 10, 5, 7, 1),	(0, 11, 1, 12, 13, 2, 14, 3),
(0, 14, 4, 15, 16, 5, 11, 3),	(0, 17, 1, 18, 19, 2, 15, 5),	(0, 20, 1, 21, 22, 2, 16, 4),
(0, 23, 1, 24, 25, 2, 26, 3),	(0, 26, 4, 27, 28, 5, 12, 2),	(1, 13, 4, 19, 22, 5, 17, 3),
(1, 25, 5, 27, 28, 4, 20, 3),	(2, 6, 4, 10, 18, 7, 23, 3),	(2, 9, 6, 21, 24, 7, 10, 8),
(3, 5, 23, 9, 21, 24, 6, 11),	(4, 5, 21, 18, 24, 9, 12, 10),	(6, 8, 9, 13, 14, 11, 15, 7),
(6, 12, 7, 16, 17, 8, 19, 9),	(6, 15, 8, 18, 19, 10, 13, 11),	(6, 20, 7, 22, 25, 8, 26, 9),
(6, 26, 10, 27, 28, 11, 16, 8),	(7, 11, 10, 17, 21, 20, 14, 15),	(7, 25, 11, 27, 28, 10, 22, 12),
(8, 14, 16, 22, 23, 12, 13, 15),	(9, 14, 17, 20, 21, 19, 12, 16),	(9, 18, 13, 27, 28, 14, 21, 17),
(10, 20, 25, 21, 23, 13, 17, 26),	(11, 17, 16, 23, 24, 19, 18, 20),	(12, 15, 18, 24, 25, 19, 26, 27),
(12, 18, 22, 20, 26, 28, 24, 15),	(13, 19, 27, 22, 25, 28, 23, 14),	(13, 24, 14, 26, 28, 16, 27, 20),
(15, 25, 18, 27, 28, 22, 23, 21),	(16, 19, 23, 25, 26, 22, 17, 24).	

A covering of K_{29} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

(0, 1, 2, 3, 4, 5, 6, 13),	(0, 4, 1, 3, 8, 7, 9, 10),	(0, 2, 4, 1, 11, 9, 10, 12),
(13, 15, 17, 19, 14, 16, 18, 20),	(21, 23, 25, 27, 22, 24, 26, 28).	

$\boxed{K_{30}}$

Let the vertex set of K_{30} be $\{0, 1, \dots, 29\}$. A maximum packing of K_{30} with 3-cubes and leave as shown in Table 1 is given by:

(0, 2, 4, 8, 9, 10, 1, 3),	(0, 4, 3, 10, 11, 6, 12, 5),	(0, 12, 1, 13, 14, 2, 8, 5),
(0, 15, 1, 16, 17, 2, 9, 4),	(0, 18, 1, 19, 20, 2, 11, 3),	(0, 21, 1, 22, 23, 2, 24, 3),
(0, 24, 4, 25, 26, 5, 15, 3),	(0, 27, 1, 28, 29, 2, 25, 5),	(1, 14, 3, 17, 20, 4, 13, 6),
(1, 23, 4, 26, 29, 6, 10, 7),	(2, 13, 7, 16, 19, 8, 9, 5),	(2, 22, 6, 26, 28, 4, 18, 8),
(3, 5, 7, 18, 21, 17, 8, 10),	(3, 6, 8, 16, 27, 9, 14, 10),	(3, 7, 11, 28, 29, 12, 4, 19),
(4, 21, 5, 27, 29, 8, 20, 11),	(5, 18, 9, 22, 23, 11, 12, 13),	(6, 14, 7, 15, 16, 11, 17, 9),
(6, 19, 7, 21, 24, 9, 20, 12),	(6, 25, 7, 27, 28, 9, 23, 12),	(7, 22, 8, 24, 28, 10, 15, 13),
(8, 23, 10, 25, 27, 14, 19, 13),	(9, 21, 11, 26, 29, 14, 13, 16),	(10, 11, 15, 17, 20, 22, 12, 18),
(10, 24, 14, 26, 29, 15, 18, 13),	(11, 19, 16, 24, 25, 12, 14, 20),	(12, 16, 20, 17, 26, 18, 23, 19),
(13, 17, 26, 20, 21, 22, 24, 27),	(14, 22, 16, 25, 28, 26, 23, 17),	(15, 19, 21, 26, 27, 22, 28, 25),
(15, 20, 28, 23, 25, 29, 24, 21),	(15, 21, 29, 22, 28, 16, 27, 18),	(17, 24, 19, 27, 29, 18, 25, 23).

A covering of K_{30} with 3-cubes and padding as shown in Table 2 is obtained by using the additional 3-cubes:

$(0,1,2,3,7,6,5,4)$,	$(1,4,2,7,5,0,6,3)$,	$(4,8,11,9,10,12,3,13)$,
$(14,15,16,17,18,19,20,21)$,	$(22,23,24,25,26,27,28,29)$.	

$\boxed{K_{31}}$

Let the vertex set of K_{31} be $\{0,1,\ldots,30\}$. A maximum packing of K_{31} with 3-cubes and leave as shown in Table 1 is given by:

$(0,1,2,6,7,8,9,3)$,	$(0,2,7,9,10,8,4,5)$,	$(0,8,3,11,12,5,13,1)$,
$(0,13,2,14,15,4,10,1)$,	$(0,16,1,17,18,2,19,3)$,	$(0,19,4,20,21,5,3,10)$,
$(0,22,1,23,24,2,20,3)$,	$(0,25,1,26,27,2,21,3)$,	$(0,28,1,29,30,4,6,5)$,
$(1,7,5,18,24,6,11,4)$,	$(1,9,4,27,30,6,12,7)$,	$(2,11,7,15,17,8,13,6)$,
$(2,12,3,28,29,8,14,6)$,	$(2,23,4,26,30,8,16,9)$,	$(3,15,5,16,22,8,20,6)$,
$(3,25,4,29,30,10,14,11)$,	$(4,17,7,21,22,5,14,9)$,	$(5,23,6,25,26,7,10,11)$,
$(5,24,8,27,28,7,18,9)$,	$(6,8,19,18,21,25,7,16)$,	$(6,19,10,26,27,11,9,12)$,
$(7,20,11,22,29,9,13,10)$,	$(8,21,13,26,28,11,12,14)$,	$(9,15,10,17,19,12,16,13)$,
$(9,23,10,24,25,12,18,13)$,	$(10,12,17,27,28,20,14,13)$,	$(11,15,13,23,24,14,22,16)$,
$(11,16,15,17,18,14,19,20)$,	$(12,21,14,29,30,15,23,17)$,	$(12,22,15,24,28,17,18,21)$,
$(13,20,15,29,30,16,25,18)$,	$(14,25,19,27,30,20,21,22)$,	$(15,26,16,27,28,19,17,24)$,
$(16,19,22,28,29,23,20,26)$,	$(17,21,30,25,26,23,24,22)$,	$(18,22,29,24,28,23,27,25)$,
$(18,23,25,26,27,30,29,21)$,	$(19,24,20,29,30,26,27,28)$.	

A covering of K_{31} with 3-cubes and non-simple padding as shown in Table 2 is given by:

$(0,1,2,3,4,5,6,7)$,	$(0,1,3,5,6,4,8,9)$,	$(0,1,6,8,9,7,10,2)$,
$(0,1,8,7,10,9,11,5)$,	$(0,2,4,11,12,5,13,1)$,	$(0,13,2,14,15,3,11,6)$,
$(0,16,1,17,18,2,15,4)$,	$(0,19,1,20,21,2,22,3)$,	$(0,22,4,23,24,5,14,1)$,
$(0,25,1,26,27,2,28,3)$,	$(0,28,4,29,30,5,16,3)$,	$(1,10,3,18,21,4,9,12)$,
$(1,27,5,29,30,4,19,6)$,	$(2,7,11,12,17,13,10,8)$,	$(2,20,4,24,26,5,25,6)$,
$(2,23,7,29,30,8,14,9)$,	$(3,4,12,6,14,26,7,16)$,	$(3,12,10,17,19,13,14,11)$,
$(3,23,9,24,25,10,15,7)$,	$(5,8,13,15,17,16,9,18)$,	$(5,18,6,21,23,11,13,16)$,
$(6,17,7,20,22,9,19,8)$,	$(6,23,12,27,28,13,20,9)$,	$(7,18,8,21,22,10,24,11)$,
$(7,27,8,28,30,10,26,11)$,	$(8,15,11,25,29,12,16,18)$,	$(9,21,13,25,26,15,22,12)$,
$(10,16,15,19,20,22,14,17)$,	$(10,21,14,28,29,17,12,19)$,	$(11,20,14,27,29,15,23,21)$,
$(12,24,15,28,30,13,27,16)$,	$(13,18,19,26,29,14,24,16)$,	$(14,19,16,25,30,21,20,23)$,
$(15,17,22,25,30,24,18,20)$,	$(17,23,18,26,27,22,21,25)$,	$(17,25,29,28,30,19,27,18)$,
$(19,20,28,22,23,24,26,29)$,	$(20,26,23,27,29,30,28,24)$,	$(21,24,22,26,28,25,30,27)$.

$\boxed{K_{34}}$

Let the vertex set of K_{34} be $\{0, 1, \ldots, 33\}$. A maximum packing of K_{34} with 3-cubes and leave as shown in Table 1 is given by:

$(0,1,2,3,4,5,6,7)$,	$(0,1,3,5,6,4,8,9)$,	$(0,1,6,8,9,7,10,2)$,
$(0,1,8,7,10,9,11,5)$,	$(0,2,4,11,12,5,13,1)$,	$(0,13,2,14,15,3,11,6)$,
$(0,16,1,17,18,2,15,4)$,	$(0,19,1,20,21,2,22,3)$,	$(0,22,4,23,24,5,14,1)$,
$(0,25,1,26,27,2,28,3)$,	$(0,28,4,29,30,5,16,3)$,	$(0,31,1,32,33,2,29,5)$,
$(1,10,3,18,21,4,9,12)$,	$(1,27,4,30,33,6,3,12)$,	$(2,7,11,12,17,13,10,8)$,
$(2,20,4,24,26,5,19,6)$,	$(2,23,6,30,32,3,17,7)$,	$(3,14,7,19,24,8,15,9)$,
$(3,25,4,31,33,7,12,10)$,	$(4,26,8,32,33,9,13,11)$,	$(5,8,16,15,17,18,6,12)$,
$(5,18,7,21,23,9,16,10)$,	$(5,25,8,27,31,6,20,7)$,	$(6,13,12,22,28,14,16,11)$,
$(6,21,8,29,32,9,22,10)$,	$(7,22,13,23,24,14,15,11)$,	$(7,26,10,28,29,11,14,9)$,
$(8,19,10,30,31,11,17,9)$,	$(8,23,12,28,33,14,19,13)$,	$(9,20,10,25,27,11,18,13)$,
$(10,15,17,24,27,18,14,12)$,	$(11,21,14,25,30,13,20,15)$,	$(12,20,16,25,26,17,19,18)$,
$(12,29,13,31,32,14,26,15)$,	$(13,16,18,24,32,17,21,19)$,	$(14,27,16,30,31,17,22,18)$,
$(15,19,20,21,22,23,18,28)$,	$(15,23,16,24,27,20,26,21)$,	$(15,28,16,29,33,19,31,20)$,
$(16,21,22,32,33,23,24,25)$,	$(17,23,25,28,29,26,19,27)$,	$(17,25,20,30,33,21,32,24)$,
$(18,29,30,32,33,31,27,26)$,	$(19,22,25,29,30,26,31,21)$,	$(20,22,27,24,28,30,25,26)$,
$(22,29,28,31,33,32,23,30)$,	$(23,27,33,29,31,32,28,24)$.	

A covering of K_{34} with 3-cubes and non-simple padding as shown in Table 2 is given by:

$(0,1,2,3,4,5,6,7)$,	$(0,1,3,5,6,4,8,9)$,	$(0,1,6,8,9,7,10,2)$,
$(0,1,8,7,10,9,11,5)$,	$(0,2,4,11,12,5,13,1)$,	$(0,13,2,14,15,3,11,6)$,
$(0,16,1,17,18,2,15,4)$,	$(0,19,1,20,21,2,22,3)$,	$(0,22,4,23,24,5,14,1)$,
$(0,25,1,26,27,2,28,3)$,	$(0,28,4,29,30,5,16,3)$,	$(0,31,1,32,33,2,29,5)$,
$(1,10,3,18,21,4,9,12)$,	$(1,27,4,30,33,6,3,12)$,	$(2,7,11,12,17,13,10,8)$,
$(2,20,4,24,26,5,19,6)$,	$(2,23,6,30,32,3,17,7)$,	$(3,14,7,19,24,8,15,9)$,
$(3,25,4,31,33,7,12,10)$,	$(4,26,8,32,33,9,13,11)$,	$(5,8,16,15,17,18,6,12)$,
$(5,18,7,21,23,9,16,10)$,	$(5,25,8,27,31,6,20,7)$,	$(6,13,12,22,28,14,16,11)$,
$(6,21,8,29,32,9,22,10)$,	$(7,22,13,23,24,14,15,11)$,	$(7,26,10,28,29,11,14,9)$,
$(8,19,10,30,31,11,17,9)$,	$(8,23,12,28,33,14,19,13)$,	$(9,20,10,25,27,11,18,13)$,
$(10,15,17,24,27,18,14,12)$,	$(11,21,14,25,30,13,20,15)$,	$(12,20,16,25,26,17,19,18)$,
$(12,29,13,31,32,14,26,15)$,	$(13,16,18,24,32,17,21,19)$,	$(14,27,16,30,31,17,22,18)$,
$(15,19,20,21,22,23,18,28)$,	$(15,23,16,24,27,20,26,21)$,	$(15,28,16,29,33,19,31,20)$,
$(16,21,22,32,33,23,24,25)$,	$(17,23,25,28,29,26,19,27)$,	$(17,25,20,30,33,21,32,24)$,
$(18,29,30,32,33,31,27,26)$,	$(19,22,25,29,30,26,31,21)$,	$(20,22,27,24,28,30,25,26)$,
$(22,29,28,31,33,32,23,30)$,	$(23,27,33,29,31,32,28,24)$.	

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