A non-planar version of Tutte's Wheels Theorem

Talmage James Reid

Department of Mathematics
The University of Mississippi
University, MS 38677

Haidong Wu

Department of Mathematics
Southern University
Baton Rouge, LA 70813

Abstract

Tutte's Wheels Theorem states that a minimally 3-connected non-wheel graph G with at least four vertices contains at least one edge e such that the contraction of e from G produces a graph which is both 3-connected and simple. The edge e is said to be *non-essential.* We show that a minimally 3-connected graph which is non-planar contains at least six non-essential edges.

The wheel graphs are the fundamental building blocks of graphs [1]. Tutte's Wheels Theorem [7] characterizes the wheels as being the minimally 3-connected graphs with no non-essential edges. Hence a minimally 3-connected graph G that is not a wheel contains at least *one* non-essential edge. Such edges can be used as an important induction tool in the study of graph structure (Tutte [7]). Therefore, it is interesting to investigate the distributions of non-essential edges in minimally 3-connected graphs (see, for example, [6]). Our main result, Theorem 1, is related to Tutte's Wheels Theorem by replacing the condition that G is not a wheel in the Wheels Theorem by the condition that G is non-planar. The lower bound on the number of non-essential edges in a minimally 3-connected non-planar graph given in this theorem is best possible.

Theorem 1 *A minimally 3-connected non-planar graph contains at least 6 non-essential edges.*

The graph given in Figure 1 is a minimally 3-connected non-planar graph with only the 6 edges not appearing in triangles being non-essential.

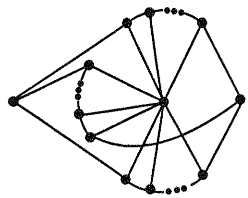

Figure 1

Oxley and Wu [4] characterized all minimally 3-connected graphs with fewer than 5 non-essential edges. They showed that all such graphs are planar. In order to complete the proof of Theorem 1, we characterize in Theorem 2 all minimally 3-connected graphs with exactly 5 non-essential edges as being planar graphs which are contained in 13 families of graphs.

In [6] it is shown that each longest cycle in a minimally 3-connected graph has at least 2 non-essential edges. Moreover, if there is a longest cycle containing exactly 2 such edges, then the graph has at most 5 non-essential edges. This provides further evidence that it is natural to investigate the case of graphs containing exactly 5 non-essential edges, besides the application of Theorem 2 provides in proving the non-planar version of the Wheels Theorem given in Theorem 1. Furthermore, the proof of Theorem 2 indicates that it is likely to be very messy to extend our results to the case of 6 or 7 non-essential edges.

Throughout this paper G is a minimally 3-connected graph which is not a wheel. The vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. The minimum degree of G is denoted by δ_G. Since G is minimally 3-connected, $\delta_G \geq 3$. Let e be an edge of G. Then G/e denotes the contraction of e from G. The edge e is *non-essential* if and only if G/e is both 3-connected and simple. The set of non-essential edges of G is denoted by $\mathcal{C}$.

A *triad* of G is a set of three edges of G which meet a vertex of degree three. Suppose $k \geq 1$ is odd and $F = \{a_1, a_2, ..., a_{k+2}\}$ is a set of distinct edges of G. Then F is a *fan* of G if and only if F is maximal with respect to the property that $\{a_i, a_{i+1}, a_{i+2}\}$ is a triad when i is odd, and a triangle when i is even. If $k = 1$ and F consists of a single triad, then F is called a *trivial* fan. The edges a_1 and a_{k+2} are called *ends* of F. We name a fan by its ends. Thus F is called an $a_1a_{k+2}-$fan.

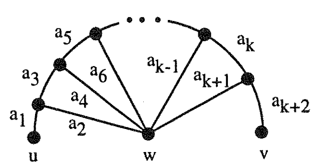

Figure 2

Let $\mathcal{S}$ be the union of the thirteen families of graphs given in Figure 3 subject to the following rules. If $G \in \mathcal{S}\backslash(\mathbf{B}_3 \cup \mathbf{C}_4)$, then the only fan of G which may be trivial is one labelled with an F. If $G \in \mathbf{B}_3$, then at most one of the fans labelled by E and F may be trivial. If $G \in \mathbf{C}_4$, then one or both of the fans labelled by E and F may be trivial.

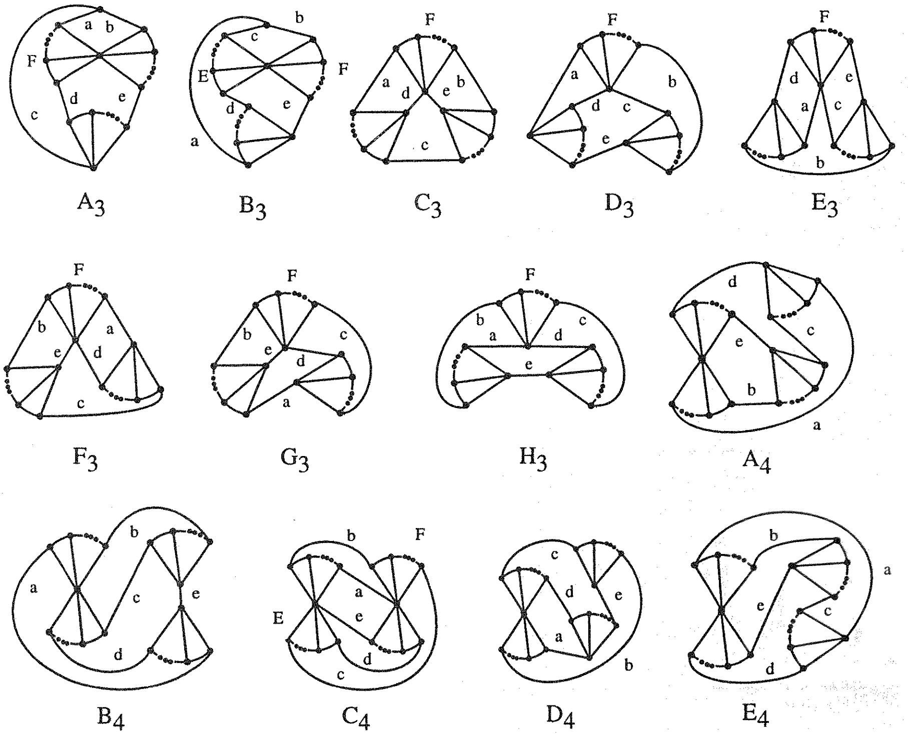

Figure 3

The second main result of the paper is given next.

Theorem 2 *A graph G is minimally 3-connected with exactly 5 non-essential edges if and only if G is a member of $\mathcal{S}$.*

Note that Theorem 1 follows from Theorem 2 as each graph in $\mathcal{S}$ is planar. The following result on the structure of 3-connected graphs of Oxley and Wu [3] is a key part of the proof of Theorem 2.

Theorem 3 *Let G be a minimally 3-connected graph which is not a wheel. If e is an edge of G which is essential, then e is a member of a fan which contains two non-essential ends. Moreover, e is in a unique fan unless e is in exactly two fans* *which are triads as shown in Figure 4(a), or in exactly three fans formed by mutually joining three vertices of degree three as in Figure 4(b).* □

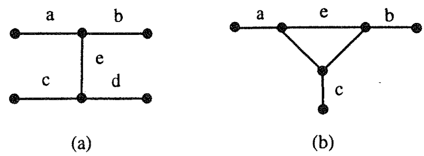

Figure 4

Let two edges of G which are essential be *related* if and only if there exists a fan of G containing both. This is an equivalence relation on the edges of G which are essential. Let $\mathcal{F}$ be a subset of the fans of G whose members consist of an equivalence class of edges which are essential together with two fixed ends of a fan containing them. For example, only one fan of the $ab-$ and $cd-$fans of Figure 4(a) would be a member of $\mathcal{F}$. Likewise, only one fan of the $ab-$, $ac-$, and $bc-$fans of Figure 4(b) would be a member of $\mathcal{F}$.

Suppose that F is a fan as given in Figure 2. Vertices u and v are called *vertex-ends* of F. Vertex w is called the *hub* of F. The two vertices meeting edges $\{a_1, a_2, a_3\}$ and $\{a_k, a_{k+1}, a_{k+2}\}$ are called the *rim-vertices* of F. If F is trivial, then it has a unique rim-vertex which meets all three of its edges.

Several observations which are used in the proof of Theorem 2 are given next. The first of these follows from the fact that an end of a fan of $\mathcal{F}$ is non-essential and hence is not in a triangle. The second of these follows from the definition of $\mathcal{F}$.

Lemma 4 *Distinct fans of $\mathcal{F}$ which share an end have distinct hubs.* □

Lemma 5 *An edge of C is an end of at most two fans of $\mathcal{F}$.* □

Lemma 6 *Each hub of a fan F of $\mathcal{F}$ either meets an edge of C or is the common hub of at least two fans of $\mathcal{F}$.*

Proof. The vertex-ends of F are not a vertex-cut of G. Thus there exists an edge e of G meeting the hub of F which is not a member of F. Suppose that $e \notin C$. Then e is essential and by Theorem 3 is a member of a fan E of $\mathcal{F}$ that is distinct from F. Evidently, the hubs of E and F agree. □

Lemma 7 *The vertex-ends of a fan of G are distinct.*

Proof. Suppose not. It follows from G being simple that F is non-trivial. Since G is not a wheel, $V(G) \neq V(F)$. Thus the hub and unique vertex-end of F form a vertex-cut of G. This contradicts that G is 3-connected. □

Lemma 8 *If G has more than three non-essential edges, then distinct fans F_1 and F_2 of $\mathcal{F}$ do not share both ends.*

Proof. Suppose that F_1 and F_2 share both ends. The set of hubs of F_1 and F_2 is not a vertex-cut of G. Thus $V(G) = V(F_1) \cup V(F_2)$. Lemma 4 implies that the hubs of F_1 and F_2 are distinct. Hence $E(G)$ consists of the edges of F_1 and F_2 together with an edge x joining the hubs of F_1 and F_2 because G is 3-connected. Then $\delta_G \geq 3$ implies that F_1 and F_2 are non-trivial. Thus the two common ends of F_1 and F_2 and x are the only non-essential edges of G. This contradicts that G has more than three non-essential edges. $\square$

Form a graph $G_{\mathcal{F}}$ with vertex set $\mathcal{C}$ as follows. If e and f are distinct members of $\mathcal{C}$, then join e and f by an edge in $G_{\mathcal{F}}$ if and only if e and f are the ends of a fan F in $\mathcal{F}$. For example, if $G \in \mathbf{C}_3$, then $\mathcal{F}$ has three fans and so $G_{\mathcal{F}}$ has three edges. It consists of the cycle a, b, c together with isolated vertices d and e.

Lemma 9 $\mid \mathcal{F} \mid = \frac{1}{2} \sum_{v \in \mathcal{C}} d_{G_{\mathcal{F}}}(v) \leq \mid \mathcal{C} \mid$.

Proof. By the handshaking lemma, $\sum_{v \in \mathcal{C}} d_{G_{\mathcal{F}}}(v) = 2 \mid E(G_{\mathcal{F}}) \mid = 2 \mid \mathcal{F} \mid$. By Lemma 5, the maximum degree of $G_{\mathcal{F}}$ is at most two. Hence $\sum_{v \in \mathcal{C}} d_{G_{\mathcal{F}}}(v) \leq 2 \mid \mathcal{C} \mid$. $\square$

The proof of Theorem 2. Suppose that $G \in \mathcal{S}$. It is straightforward to check that G is minimally 3-connected. It can also be checked that if $G \in \mathcal{S} \backslash (\mathbf{A}_3 \cup \mathbf{B}_3)$, $G \in \mathbf{A}_3$ and F is non-trivial, or $G \in \mathbf{B}_3$ and E and F are non-trivial, then a, b, c, d, and e are the edges of G whose contraction is simple and 3-connected. If $G \in \mathbf{A}_3$ and F is trivial, then b, c, d, e and the unique edge of $F \backslash \{a, d\}$ are the non-essential edges of G. If $G \in \mathbf{B}_3$ and E is trivial, then a, b, d, e, and the unique edge of $E \backslash \{c, d\}$ are the non-essential edges of G. If $G \in \mathbf{B}_3$ and F is trivial, then a, c, d, e, and the unique edge of $F \backslash \{b, e\}$ are the non-essential edges of G. Hence if $G \in \mathcal{S}$, then G has exactly five non-essential edges.

Suppose that G has exactly five non-essential edges $\mathcal{C} = \{a, b, c, d, e\}$ and that G is not a member of $\mathcal{S}$. Suppose $\mid \mathcal{F} \mid = 1$ and F is the unique fan of G. Then $E(G)$ consists of the edges of F and three non-essential edges of G which are not in F. The vertex-ends of F are not joined to its hub. Thus there exists a vertex v in $V(G) \backslash V(F)$. Hence $\delta_G \geq 3$ implies that v meets all three edges of $E(G) \backslash E(F)$. Thus the vertex-ends of F have degree at most two; a contradiction. It follows from Lemma 9 that $2 \leq \mid \mathcal{F} \mid \leq 5$.

Suppose that $\mid \mathcal{F} \mid = 2$. Let F_1 and F_2 be the distinct fans of G. By Lemma 8, F_1 and F_2 do not share both ends. Suppose they share exactly one end. It follows from Theorem 3 that $E(G) \backslash \{E(F_1) \cup E(F_2)\}$ consists of two non-essential edges. Hence $\delta_G \geq 3$ implies that $V(G) = V(F_1) \cup V(F_2)$. By Lemma 4, the hubs of F_1 and F_2 are distinct. Let u and v be the vertex-ends of F_1 and F_2, respectively, not meeting the common end of F_1 and F_2. If u is the hub of F_2, then one of the two edges of $E(G) \backslash \{E(F_1) \cup E(F_2)\}$ would join the hubs of F_1 and F_2. Thus F_1 would have a non-essential end which is in a triangle; a contradiction. Thus u, and likewise v, are distinct from the hubs of F_1 and F_2. If $u = v$, then u is joined to neither of the hubs of F_1 and F_2. Thus $d(u) = 2$; a contradiction. Thus $u \neq v$. Then $\mid E(G) \backslash \{E(F_1) \cup E(F_2)\} \mid = 2$ implies that either the degree of u or v is at most two; a contradiction. Thus F_1 and F_2 have distinct ends. It follows that $E(G)\backslash\{E(F_1) \cup E(F_2)\}$ consists of one non-essential edge f.

Suppose that F_1 and F_2 share two vertex-ends. The 3-connectivity of G implies that F_1 and F_2 share a hub. The remaining non-essential edge f of G connects the vertex-ends of F_1.

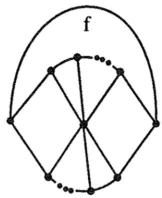

Figure 5

Thus G is as given in Figure 5. Then G/f is not 3-connected; a contradiction. Hence F_1 and F_2 share at most one vertex-end. If F_1 and F_2 share a hub, then $\delta_G \geq 3$ implies that these fans share two vertex-ends; a contradiction. Hence F_1 and F_2 have distinct hubs. The 3-connectivity of M implies that the the hubs of each of F_1 and F_2 are identical with a vertex-end of the other fan. Hence F_1 and F_2 share a vertex-end z. Either the fifth non-essential edge is incident with z and G has a vertex of degree one or it is not and z has degree two in G; a contradiction. Thus $3 \leq\mid \mathcal{F} \mid \leq 5$.

Lemma 10 *Each vertex v of G is contained in some fan of $\mathcal{F}$ as a vertex which is not a vertex-end of that fan.*

Proof. Suppose that v meets an edge of G which is essential. It follows from Theorem 3 that this edge which is essential is in a fan of $\mathcal{F}$ and hence the result holds. Suppose that v meets only the non-essential edges of $\mathcal{C}$. Then $d(v) \in \{3,4,5\}$.

Suppose that $d(v) = 5$. Then each edge of $\mathcal{C}$ meets v. Let F be a fan of G. Then both ends of F are in $\mathcal{C}$ and hence meet v. This contradicts Lemma 7. Hence $d(v) < 5$.

Suppose that $d(v) = 4$. Let f be the unique edge of $\mathcal{C}$ not meeting v. Then $\mid \mathcal{F} \mid \geq 3$ and Lemma 5 imply that there exists a fan F of $\mathcal{F}$ not using f as an end. Thus F uses two edges of $\mathcal{C}$ meeting v as end-edges. This contradicts Lemma 7. Hence $d(v) = 3$.

Suppose that the set of edges of G incident with v is $\{a, b, c\}$ without loss of generality. Vertex v does not meet a an edge which is essential and in a fan of $\mathcal{F}$. Thus each edge of $\{a, b, c\}$ is an end of at most one fan of $\mathcal{F}$. Hence $\sum_{w \in C} d_{G_{\mathcal{F}}}(w) \leq 3 \cdot 1 + 2 \cdot 2 = 7$. It follows Lemma 9 that $\mid \mathcal{F} \mid= 3$. It follows from using symmetry and the facts that each of a, b, and c are in at most one fan of $\mathcal{F}$, d and e are in at most two fans of $\mathcal{F}$, and $\mid \mathcal{F} \mid\geq 3$, that we may assume that there exists an $ad-$fan F_1 and a $be-$fan F_2. The remaining fan F_3 of G is a $cd-$, $ce-$, or $de-$fan. By the symmetry induced by interchanging a and b, and d and e, we may assume that F_3 is a $cd-$ or $de-$fan. Suppose the latter holds. Lemma 4 implies that the hub of F_3 is distinct from the hubs of F_1 and F_2. The vertex-ends of F_3 do not form a vertex-cut of G. Thus edge c joins v to the hub of F_3. The vertex-ends of F_1 do not form a vertex-cut of G. Thus the hubs of F_1 and F_2 are identical. Then $\delta_G \geq 3$ implies that F_3 is non-trivial. Moreover, at least one of F_1 and F_2 is non-trivial. Hence $G \in \mathbf{A}_3$; a contradiction. Thus F_3 is a $cd-$fan.

Fans F_1 and F_3 have distinct hubs by Lemma 4. Suppose that f is an edge of G which is not in F_2 and is incident with e. Then $f \notin \mathcal{C}$. Hence f is an edge of G which is essential and is in F_1 or F_3. Thus e meets either the hub of F_1 or the hub of F_3. By the symmetry induced by interchanging edges a and c and appropriately interchanging the edges of F_1 and F_3 which are essential, we may assume that e meets the hub of F_1. The vertex-ends of F_2 are not a vertex-cut of G. Thus the hubs of F_2 and F_3 agree. Then $\delta_G \geq 3$ implies that F_1 is non-trivial. Moreover, at least one of F_2 and F_3 is non-trivial.Thus $G \in \mathbf{B}_3$; a contradiction. □

The following immediate corollary of Lemma 10 is used throughout the remainder of the paper.

Corollary 11 *Let $x \in \mathcal{C}$.*

(a) *x joins the hubs of distinct fans of $\mathcal{F}$* in G *if and only if x has degree zero in $G_{\mathcal{F}}$.*

(b) *x joins a rim-vertex of a unique fan of $\mathcal{F}$ to the common hub of possibly several fans of $\mathcal{F}$ if and only if x has degree one in $G_{\mathcal{F}}$.*

(c) *x is an end of two distinct fans of $\mathcal{F}$ in G if and only if x has degree two in $G_{\mathcal{F}}$.* □

Suppose $\mid \mathcal{F} \mid = 5$. Then equality holds throughout in the statement of Lemma 9. Thus $G_{\mathcal{F}}$ is a regular graph of degree two with five vertices and five edges. Hence $G_{\mathcal{F}}$ is a cycle. Suppose the vertices of this 5-cycle are listed consecutively in alphabetic order without loss of generality. Then each edge of $\mathcal{C}$ does not meet a hub of a fan of $\mathcal{F}$ by Corollary 11(c). It follows from Lemma 6 that each of the hubs of the five fans $ab-$, $bc-$, $cd-$, $de-$, and $ae-$ of $\mathcal{F}$ is the common hub of at least two fans of $\mathcal{F}$. Hence there exist two distinct fans of $\mathcal{F}$ which share an end and a hub contradicting Lemma 4. Thus $\mid \mathcal{F} \mid \in \{3, 4\}$. Thus $G_{\mathcal{F}}$ is a graph with three or four edges, five vertices, and maximum degree two. Hence $G_{\mathcal{F}}$ is isomorphic to one of the six graphs given in Figure 6.

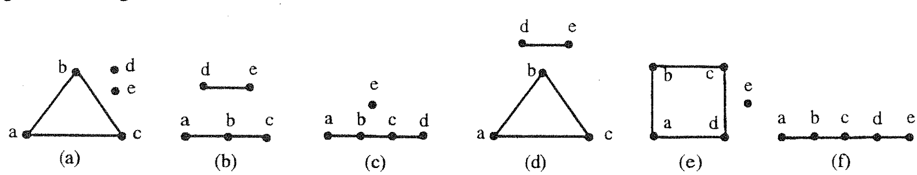

Figure 6

Suppose that $G_{\mathcal{F}}$ is as given in Figure 6(a). The hubs of the $ab-$, $bc-$, and $ac-$fans of $\mathcal{F}$ are distinct by Lemma 4. By Corollary 11(a) and symmetry, we may assume that d joins the hubs of the $ab-$ and $ac-$fans and e joins the hubs of the $ab-$ and $bc-$fans. Then $\delta_G \geq 3$ implies that the $ac-$ and $bc-$fans are non-trivial. Thus $G \in \mathbf{C}_3$; a contradiction.

Suppose that $G_{\mathcal{F}}$ is as given in Figure 6(b). The hubs of the $ab-$ and $bc-$fans of $\mathcal{F}$ are distinct by Lemma 4. By Corollary 11(b), each edge of $\{a, c, d, e\}$ meets a hub of the three fans of $\mathcal{F}$. Suppose that the $de-$fan shares a hub with another fan of $\mathcal{F}$. By symmetry, we may assume that the $ab-$ and $de-$fans share a hub. Then edges a, d, and e all meet the hub of the $bc-$fan. Edge c meets the common hub of the $ab-$ and $de-$fans. Thus the two hubs of the $ab-$ and $bc-$fans form a vertex-cut of G; a contradiction. Hence the hubs of the three fans of $\mathcal{F}$ are pairwise distinct. Edges d and e meet distinct hubs of $\mathcal{F}$ by Lemma 7. We may assume that edges d and e meet the hubs of the $ab-$ and $bc-$fans of G, respectively. Edge a or c meets the hub of the $de-$fan by Lemma 6. Suppose the former holds without loss of generality. Edge c meets either the hub of the $ab-$ or $de-$fan. In the former case, $\delta_G \geq 3$ implies that the $bc-$ and $de-$fans are non-trivial. Thus $G \in \mathbf{D}_3$; a contradiction. Hence c meets the hub of the $de-$fan. The $ab-$ and $bc-$fans are non-trivial because their hubs have degree at least three. Thus $G \in \mathbf{E}_3$; a contradiction.

Suppose that $G_{\mathcal{F}}$ is as given in Figure 6(c). Then the hub of the $bc-$fan is distinct from the hubs of the $ab-$ and $cd-$fans. Suppose that the hubs of the $ab-$ and $cd-$fans agree. Then edge e joins the two distinct hubs of fans of $\mathcal{F}$ by Corollary 11(a). Edges a and d meet the hub of the $bc-$fan by Corollary 11(b). Hence e is a non-essential edge of G which is in a triangle; a contradiction. Thus the hubs of the 3 fans of $\mathcal{F}$ are pairwise distinct. It follows from Corollary 11(a) and symmetry that we may assume that edge e joins the hubs of the $ab-$ and $bc-$fans or e joins the hubs of the $ab-$ and $cd-$ fans. Suppose the former holds. By Lemma 6, edge a meets the hub of the $cd-$fan. Edge d meets the hub of the $ab-$ or $bc-$fan. In the former case, $\delta_G \geq 3$ implies that the $bc-$ and $cd-$fans are non-trivial. Thus $G \in \mathbf{F}_3$; a contradiction. Hence d meets the hub of the $bc-$fan. The $ab-$ and $cd-$fans are non-trivial as $\delta_G \geq 3$. Hence $G \in \mathbf{G}_3$; a contradiction. Thus e joins the hubs of the $ab-$ and $cd-$fans. Edge a does not meet the hub of the $cd-$fan as it is in no triangle. Thus edge a meets the hub of the $bc-$fan. By symmetry, d meets the hub of the $bc-$fan. The $ab-$ and $cd-$fans are non-trivial because $\delta_G \geq 3$. Thus $G \in \mathbf{H}_3$; a contradiction.

Suppose that $G_{\mathcal{F}}$ is as given in Figure 6(d). The hubs of the $ab-$, $bc-$, and $ac-$fans are pairwise distinct. By Lemma 6, the hub of the $de-$fan agrees with the hub of one of the three other fans of $\mathcal{F}$. By symmetry, suppose that the hubs of the $ab-$ and $de-$fans agree. By Lemma 6, each of the hubs of the $ac-$ and $bc-$fans meets edge d or e. We may assume that edge d meets the hub of the $ac-$fan and edge e meets the hub of the $bc-$fan. The $ac-$ and $bc-$fans are non-trivial because $\delta_G \geq 3$. Likewise, either the $ab-$ or $de-$fan is non-trivial. If exactly one of these two fans is trivial, then the contraction of its non-end is 3-connected and simple. The contraction of a, b, c, d, or e is also 3-connected and simple. Thus G has six non-essential edges; a contradiction. Hence each fan of G is non-trivial. Thus $G \in \mathbf{A}_4$; a contradiction.

Suppose that G is as given in Figure 6(e). Then only the hubs of the fans $ab-$ and $cd-$, or $bc-$ and $ad-$ may be identical. Suppose that all four hubs of fans of $\mathcal{F}$ are pairwise distinct. Then e joins two of these hubs. Then the hubs of the remaining two fans of $\mathcal{F}$ do not meet a member of $\mathcal{C}$ contradicting Lemma 6. Hence we may assume that the hubs of the fans $ab-$ and $cd-$ are identical. Suppose that the hubs of the $bc-$ and $ad-$fans are distinct. By Lemma 6, edge e joins the hubs of these two fans. Then $\delta_G \geq 3$ implies that fans $ad-$ and $bc-$ are non-trivial. As in the previous paragraph, the fans $ab-$ and $cd-$ are non-trivial. Hence $G \in \mathbf{B}_4$; a contradiction. Thus the hubs of the $ad-$ and $bc-$fans are identical. Hence e joins the two distinct hubs of fans of $\mathcal{F}$. Since G is minimally 3-connected, $G \backslash e$ is not 3-connected. Thus two of the fans of $\mathcal{F}$ sharing a hub are trivial. Suppose the $ab-$ and $cd-$fans are trivial without loss of generality. Then a,b, c, d, e, and the non-end of the $ab-$fan are six non-essential edges of G; a contradiction. It follows that $G_{\mathcal{F}}$ is as given in Figure 6(f).

It follows from Lemma 6 that the hub of each fan of $\mathcal{F}$ either meets a or e or is a hub of at least two fans of $\mathcal{F}$. Thus at least two of the hubs of the fans of $\mathcal{F}$ are identical. By symmetry, we may assume that the hubs of the $ab-$ and $cd-$fans are identical, or the hubs of the $ab-$ and $de-$fans are identical. Suppose the former occurs. Suppose that the hubs of the $bc-$ and $de-$fans are identical. Then a and e meet the hubs of the $bc-$ and $ab-$fans, respectively. The $ab-$ and $de-$fans are non-trivial as otherwise their non end-edge would be a sixth non-essential edge of G. Thus $G \in \mathbf{C}_4$; a contradiction. Hence the hubs of the $bc-$ and $de-$fans are distinct.

Edge a either meets the hub of the $bc-$ or $de-$fan. Suppose the former holds. The hub of the $ab-$fan and the rim-vertex of the $bc-$fan meeting c are not a vertex-cut of G. Thus edge e also meets the hub of the $bc-$ fan. By considering the hub of the $de-$fan we obtain a contradiction of Lemma 6. Thus edge a meets the hub of the $de-$fan. By Lemma 6, edge e meets the hub of the $bc-$fan. From arguing as before, we obtain that each fan of $\mathcal{F}$ is non-trivial. Thus $G \in \mathbf{D}_4$; a contradiction. Hence the hubs of the $ab-$ and $de-$fans are identical. It follows from Lemma 4 that the hubs of the $bc-$ and $cd-$fans are distinct from the common hub of the $ab-$ and $de-$fans. By Lemma 6, the hubs of the $bc-$ and $cd-$fans each meet exactly one of edges a and e. If edge a meets the hub of the $bc-$fan and edge e meets the hub of the $cd-$fan, then the hub of the $ab-$fan and the rim-vertex of the $bc-$fan meeting c is a vertex-cut of G; a contradiction. Thus edge a meets the hub of the $cd-$fan and edge e meets the hub of the $bc-$fan. As before, each fan of $\mathcal{F}$ is non-trivial. Thus $G \in \mathbf{E}_4$; a contradiction. Hence every minimally 3-connected graph with exactly 5 non-essential edges is a member of $\mathcal{S}$. This completes the proof of Theorem 2. □

References

[1] C. R. Coullard and J. G. Oxley, Extensions of Tutte's Wheels- and Whirls-Theorem, *J. Combin. Theory Ser. B* **56** (1992), 130-140.

[2] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[3] J. G. Oxley and H. Wu, On the structure of 3-connected matroids and graphs, submitted.

[4] J. G. Oxley and H. Wu, The 3-connected graphs with exactly three non-essential edges, preprint.

[5] J. G. Oxley and H. Wu, Matroids and graphs with few non-essential elements, submitted.

[6] T. J. Reid and H. Wu, A longest cycle version of Tutte's Wheels Theorem, *J. Combin. Theory Ser. B*, **70**, (1997), 202-215 .

[7] W. T. Tutte, A theory of 3-connected graphs, Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 441-455.

[8] H. Wu, On contractible and vertically contractible elements in 3-connected matroids and graphs, *Discrete Math.*, to appear.

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