On digraphs with unique walks of closed lengths between vertices

J. Gimbert

Departament de Matemàtica, Universitat de Lleida
Avda. Jaume II, 69, Campus Cappont
25001 Lleida, Spain
email: joangim@eup.udl.es

Abstract

It is known that regular digraphs of degree d, diameter k and unique walks of length not smaller than h and not greater than k between all pairs of vertices ($[h, k]$-digraphs), exist only for $h = k$ and $h = k - 1$, if $d \geq 2$. This paper deals with the problem of the enumeration of $[k-1, k]$-digraphs in the case of diameter $k = 2$ or degree $d = 2$. It is shown, using algebraic techniques, that the line digraph $L K_{d+1}$ of the complete digraph K_{d+1} is the only $[1, 2]$-digraph of degree d, that is to say the only digraph —up to isomorphisms— whose adjacency matrix A fulfills the equation $A + A^2 = J$, where J denotes the all-one matrix. As a consequence, we deduce that there does not exist any other almost-Moore digraph of diameter $k = 2$ with all selfrepeat vertices apart from Kautz digraph. In addition, the cycle structure of a $[k-1, k]$-digraph is studied. Thus, a formula that provides the number of short cycles (cycles of length $\leq k$) of such a digraph is obtained. From this formula, using graphical arguments, the enumeration of $[k-1, k]$-digraphs of degree 2 and diameter not greater than 4 is concluded.

1 Introduction

Digraphs with unique walks of length in a fixed interval $[h, k]$ between all pairs of vertices, were introduced by Plesník and Znám [19] and, since then, they have been extensively studied, because they are suitable models for dense interconnection networks. Some problems related to such a class of digraphs, denoted by $[h, k]$-digraphs, have been solved by means of algebraic (spectral) techniques applied to the equation

$$A^h + A^{h+1} + \cdots + A^k = J \tag{1}$$

satisfied by their corresponding adjacency matrix A, where J denotes the all-one matrix. Thus, for instance, the regularity of a $[h,k]$-digraph and the computation of its order $n = d^h + d^{h+1} + \cdots + d^k$, where d is its degree, are easily derived from (1) (see Hoffman and McAndrew [12].) Furthermore, the problem of their existence was completely solved by Bosák [4], who proved that, for degree $d \geq 2$, there only exist $[k,k]$- and $[k-1,k]$-digraphs, where k is precisely their diameter. Among such digraphs there are Good-De Bruijn (see [16]) and Kautz (proposed in [13, 14]), which, in fact, can be seen as iterated line digraphs of the complete digraph of degree d with loops and without loops, respectively (see Fiol et al. [7, 8].) With regard to their structure and enumeration, several results concerning the class of $[k,k]$-digraphs[1] have already been obtained (see [9], [15], [16].) Thus, it is known that, in general, the family of Good-De Bruijn digraphs is not the only one whose adjacency matrix A fulfills the equation $A^k = J$. In particular, Mendelsohn [16] proved that for diameter $k = 2$ and degree $d = 3$ there are exactly six non-isomorphic digraphs of that type. Furthermore, Fiol et al. [9] described two direct methods of constructing digraphs of this kind by adequately modifying Good-De Bruijn digraphs.

In this paper, we focus our attention on the study of $[k-1,k]$-digraphs and, in particular, we face the problem of their enumeration in the case of diameter $k = 2$ or degree $d = 2$. First, in Section 2, we introduce the concepts of (l,m)-reachable digraph and reachable digraph with a maximum delay m (an extension of the notions of l-reachable and equi-reachable digraph, respectively, introduced in [9]), we give a characterization of the class of reachable digraphs with a maximum delay m and we meet the class of $[k-1,k]$-digraphs when we consider $(k,1)$-reachable digraphs with maximum order. In Section 3, we prove, by means of algebraic methods, that there is only one $[1,2]$-digraph of degree d, namely the line digraph $L\,K_{d+1}$ of the complete digraph K_{d+1} and, as a consequence, we deduce that there does not exist any other almost-Moore digraph of diameter $k = 2$ with all selfrepeat vertices apart from Kautz digraph (this solves a question formulated by Baskoro et al. in [1].)

Some results about the cycle structure of a $[k-1,k]$-digraph are presented in Section 4. Thus, using the spectrum of a $[k-1,k]$-digraph and the property that any of its closed walks of length $l \leq k$ is either a cycle or a repeated cycle, we obtain a formula for the computation of its short cycles, that is to say cycles of length not greater than k. Such a formula, which almost coincides with that given in [16] for $[k,k]$-digraphs, allow us to know the number of cycles of length a divisor of $k-1$ or k, which constitute a partition of the set of vertices of a $[k-1,k]$-digraph. Finally, in Section 5 we make a first attempt to study the enumeration of $[k-1,k]$-digraphs of degree $d = 2$ and diameter $k \geq 3$. Using the previous results about their cycle structure, together with other graphical arguments, we prove that for $k = 3, 4$ there is only one digraph of degree $d = 2$ of that type.

We conclude this introduction fixing the terminology used throughout the paper. Thus, a *digraph* G consists of a finite non-empty set $V(G)$ of objects called *vertices*

[1]The class of $[k,k]$-digraphs has been studied by Mendelsohn in [16] as "UPP digraphs" (digraphs with the unique path property of order k) and by Conway and Guy in [5] as "tight precisely k-steps digraphs".

and a set $E(G)$ of ordered pairs of vertices called *arcs*. The *order* n of G is the cardinality of $V(G)$, $n = |V(G)|$. If (u, v) is an arc, it is said that v is *adjacent from* u [u is *adjacent to* v] and also that a is *incident from* u *to* v [a is an *out-arc* of u and an *in-arc* of v]. The set of vertices which are adjacent from [to] a given vertex v, also called *successors* of v, is denoted by $\Gamma^+(v)$ [$\Gamma^-(v)$] and its cardinality is the *out-degree* of v, $d^+(v) = |\Gamma^+(v)|$ [*in-degree* of v, $d^-(v) = |\Gamma^-(v)|$]. A vertex v is *isolated* if $d^+(v) = d^-(v) = 0$. A digraph is *regular* of degree d (d-*regular*) if, for any vertex v, $d^+(v) = d^-(v) = d$. A *walk* of length h from a vertex u to a vertex v ($u \to v$ *walk*) is a sequence of vertices $u = u_0, u_1, \ldots, u_{h-1}, u_h = v$ such that (u_{i-1}, u_i) is an arc. A *circuit* is a closed walk with all its arcs distinct. A *cycle* of length $h > 0$ (h-*cycle*) is a closed walk with h distinct vertices. A *repeated cycle* is a closed walk originated by the repetition of a cycle. The length of a shortest $u \to v$ walk is the *distance from* u *to* v. Its maximum value over all pairs of vertices is the *diameter* k of the digraph. A p-*generalized cycle* is a digraph G such that $V(G)$ can be partitioned into p-parts V_i such that all adjacent vertices from vertices of V_i belong to V_{i+1}, for i modulo $p \geq 1$. The reader is referred to Chartrand and Lesniak [6] for additional graph concepts.

2 Reachable digraphs with a fixed maximum delay

Let G be a strongly connected digraph with adjacency matrix A and let m be a non-negative integer. Then, we say that G is (l, m)-*reachable* if $l \geq m$ is the smallest integer such that for each pair of vertices $u, v \in V(G)$, there is at least one $u \to v$ walk of length in the interval $[l - m, l]$, that is to say $A^{l-m} + \cdots + A^l \geq J$ and $A^{l'-m} + \cdots + A^{l'} \not\geq J$, if $l' < l$. Moreover, if m is the smallest non-negative integer such that G is (l, m)-reachable for some $l \geq m$, then we say that G is *reachable with a maximum delay* m. Notice that in the case $m = 0$ the notions of l-*reachable* and *equi-reachable* digraph, presented in [9], are recovered. We also remark that if G is a $[k-1, k]$-digraph of degree $d \geq 2$, then G is $(k, 1)$- and $(k+1, 0)$-reachable, since $A^{k-1} + A^k = J$ and $A^{k+1} = AJ - A^k \geq (d-1)J$, and, consequently, G is reachable with a maximum delay 0 (equi-reachable).

Now, we state a characterization of reachable digraphs with a maximum delay m, which is in fact a natural extension of the one given in [9] for the case $m = 0$, taking into account that any digraph G can be seen as a p-generalized cycle with $p = 1$.

Proposition 1. *A strongly connected digraph G is reachable with a maximum delay m iff $p = m + 1$ is the greatest integer such that G is a p-generalized cycle.*

Proof. We will prove that a strongly connected digraph G is reachable with a maximum delay $\leq m$ unless G is a p-generalized cycle with $p \geq m + 2$. Since any reachable digraph G with a maximum delay $\leq m$ has walks of lengths that belong to a set of cardinal $m + 1$ between all its pairs of vertices, the condition of not being a p-generalized cycle with a number of parts $p \geq m + 2$ is clearly necessary. The sufficiency of such a condition can be derived using the same reasoning detailed in [9] for $m = 0$. □

From now on, we focus our attention on the study of largest (l,m)-reachable digraphs. Thus, if G is (l,m)-reachable with a maximum out-degree d, then its order n satisfies the following inequality

$$n \leq d^{l-m} + \cdots + d^{l},$$

since this bound is the maximum number of distinct walks of length not smaller than $l-m$ and not greater than l from any vertex of G. To attain this bound there must be exactly one walk of length in the interval $[l-m,l]$ between each pair of vertices. So, any digraph with such a property is an $[l-m,l]$-digraph and, consequently, it only exists for $m=0,1$, if $d \geq 2$.

3 Enumeration of $[1,2]$-digraphs

In this section we present the main result of this paper. We prove that Kautz digraphs of diameter 2 are the unique $[1,2]$-digraphs —up to isomorphisms. This means that all $n \times n$ binary matrices that are solution of the equation $A + A^2 = J$ are of the form PAP^t, where A is the adjacency matrix of the line digraph $L\,K_{d+1}$, where $n = d + d^2$, and P is a permutation matrix.

We use, as an auxiliary result, the following characterization of regular line digraphs, which is based on Heuchenne's condition (see [11]). Such a condition says that a digraph G is a line digraph iff every pair of vertices u, v of G satisfy that their corresponding sets of successors $\Gamma^+(u), \Gamma^+(v)$ are either disjoint or equal. In matrix terms, a $(0,1)$-matrix A represents the adjacency matrix of a line digraph G iff the rows of A are either mutually ortogonal or identical.

Lemma 1. *A regular digraph G of degree $d \geq 1$ and order n is a line digraph iff the rank of its adjacency matrix A is equal to $\frac{n}{d}$.*

Proof. Let us suppose that G is a regular digraph of degree d, order $n = dn'$ and such that the rank of its adjacency matrix A is n'. Then, we take n' row vectors of A that constitute a basis of the row-space of A. Since the total number of entries of these $(0,1)$-vectors that are equal to 1 is exactly n, we deduce that these vectors must be mutually ortogonal. Otherwise, A would have a null column, which is impossible since $JA = dJ$ (G is d-regular.) Moreover, since each row has just d elements equal to 1, we deduce that any other row is identical to one of the basis. Hence, G satisfies Heuchenne's condition and, consequently, G is a line digraph.

Conversely, let G be the line digraph of a digraph G' of order n' without isolated vertices. Then, since G is regular, we have that G' is also a regular digraph with the same degree d and, consequently, the order of G is $n = n'd$. Therefore, using Heuchenne's condition, we deduce that the adjacency matrix A of G has a set of n' mutually ortogonal rows such that any other row is equal to one of those. Hence, $\operatorname{rank} A = n' = \frac{n}{d}$. □

From the relation between the rank of a square matrix A and the dimension of its null-space, we can reformulate the previous characterization, saying that a regular digraph G of degree $d \geq 2$ and order n is a line digraph, iff 0 is an eigenvalue of G with geometric multiplicity equal to $n - \frac{n}{d}$.

Besides, from the construction of a line digraph it follows that if the adjaceny matrix A of a digraph G fulfills the equation $P(A) = J$, where $P(x)$ is a polynomial, then the adjacency matrix A_L of LG satisfies the equation $A_L P(A_L) = J$. The converse is also true if we assume that G has no isolated vertices. Taking into account this property, the following result is derived.

Lemma 2. *Let G be a digraph without isolated vertices. Then, LG is a $[k-1,k]$-digraph iff G is a $[k-2,k-1]$-digraph, and both of them have the same degree.*

Now, we can characterize the class of $[1,2]$-digraphs.

Theorem 1. *There is only one $[1,2]$-digraph of degree d, namely $L K_{d+1}$.*

Proof. Let A be the adjacency matrix of a $[1,2]$-digraph G of degree d and order $n = d + d^2$. Then, A fulfills the equation $A + A^2 = J$ and, consequently, $\operatorname{tr} A = 0$. From these relations, it follows that the characteristic polynomial of A is

$$\det(xI - A) = (x-d)x^{n-d-1}(x+1)^d.$$

Now, we will prove, using the previous lemmas, that G is the line digraph $L K_{d+1}$. In fact, for $d = 1$ the result is trivial. So, from here on, we assume $d \geq 2$.

Since 0 is an eigenvalue of A with algebraic multiplicity equal to $n - d - 1$, we have that $\operatorname{rank} A \geq d + 1 = \frac{n}{d}$. Suppose that $\operatorname{rank} A > d+1$. Then, A would have at least $d+2$ column vectors $u_1, \ldots, u_{d+2}$ which are linearly independent. However, the relation $A + A^2 = (A+I)A = J$ implies that $(A+I)u_i = (1, \ldots, 1)^t$, from which we deduce that the vectors $u_i - u_1$ belong to the null-space of $A + I$. Then, since these vectors $u_i - u_1$ $(2 \leq i \leq d+2)$ are also linearly independent, we conclude that 0 must be an eigenvalue of $A + I$ with algebraic multiplicity not smaller than $d+1$. But, this is impossible because the characteristic polynomial of $A + I$ is

$$\det(xI - (A+I)) = \det((x-1)I - A) = (x - (d+1))(x-1)^{n-d-1}x^d.$$

Therefore, since rank of A must be equal to $d+1$, from Lemma 1 it turns out that G is a line digraph.

Let G' be a digraph —without isolated vertices— such that $G = LG'$. Then, from Lemma 2, we have that G' is a $[0,1]$-digraph of degree d. Hence, G' is the complete digraph K_{d+1}. □

The previous result answers a question formulated by Baskoro et al. in [1] about the enumeration of almost-Moore digraphs (also called (d,k)-digraphs) of diameter $k = 2$ with all selfrepeat vertices. A *(d,k)-digraph* is a regular directed graph of degree $d > 1$, diameter $k > 1$ and order one less than the (unattainable) *Moore* bound. Every (d,k)-digraph G has the property that for each vertex $v \in V(G)$ there exists only one vertex, denoted by $r(v)$ and called the *repeat* of v, such that there are exactly two $v \to r(v)$ walks of length less than or equal to k (one of them must be of length k.) If $r(v) = v$, which means that v is contained in exactly one k-cycle, v is called a *selfrepeat* of G. The map r, which assigns the vertex $r(v)$ to each vertex $v \in V(G)$, is an automorphism of G and its associated permutation matrix P is related to the adjacency matrix A of G by means of the following equation $I + A + \cdots + A^k = J + P$ (see, for instance, [2], [3] and [17].) Therefore, the notion of (d,k)-digraph with all selfrepeat vertices ($P = I$) and the concept of $[1,k]$-digraph are equivalent. Hence, (d,k)-digraphs with all selfrepeat vertices do only exist for diameter $k = 2$ (see [2] and [4].) Moreover, in such a case, using Theorem 1, we have that $L\,K_{d+1}$ is the only $(d,2)$-digraph with all selfrepeat vertices. Furthermore, using the results about the structure of almost-Moore digraphs with selfrepeat vertices, presented in [1], together with the necessary conditions in terms of the cycle structure of the permutation r, given in [10], it can be proved that all vertices of any $(d,2)$-digraph of degree d, $3 \leq d \leq 12$, are selfrepeats. Hence, the enumeration of almost-Moore digraphs of diameter 2 and small degree is reduced to the resolution of the equation $A + A^2 = J$ (given in Theorem 1.)

4 Short cycles in a $[k-1,k]$-digraph

Several properties about the cycle structure of a $[k,k]$-digraph are already known. Thus, in [16] it is proved, using matrix techniques, that each closed walk of a $[k,k]$-digraph of length $\leq k$ is either a cycle or a repeated cycle. We will extend such a result, by means of graphical techniques, and, as a consequence, we will deduce that any $[k-1,k]$-digraph shares this same property. From this fact and the knowledge of the spectrum of a $[k-1,k]$-digraph, we will obtain a formula for the number of its short cycles.

Lemma 3. *If G is a digraph such that for each pair of (not necessarily different) vertices u,v there is at most one $u \to v$ walk of length k, then each closed walk of G of length $\leq k$ is either a cycle or a repeated cycle.*

Proof. Let $C : u_0, u_1, \ldots, u_l$ be a closed walk of G of length $l \leq k$ such that is neither a cycle nor a repeated cycle. Let u_i and u_j be the two first repeated vertices of the sequence C, where $0 \leq i < j < l$. Then, $C_1 : u_i, u_{i+1}, \ldots, u_j$ is a cycle of length $j - i$ and $C_2 : u_j, u_{j+1}, \ldots, u_l, u_1, \ldots, u_i$ is a closed walk of length $l + i - j$. Since C is not a repeated cycle, it can be seen that the two concatenation of these sequences, C_1C_2 and C_2C_1, represent different $u_i \to u_i$ closed walks of length l. But, then, there would be two $u_i \to u_{i+r}$ walks of length k, where $k - l \equiv r \pmod{j-i}$, which is impossible. □

Corollary 1. *Every closed walk of a $[k-1,k]$-digraph of length $\leq k$ is either a cycle or a repeated cycle.*

Moreover, since the adjacency matrix A of a $[k-1,k]$-digraph G of degree d fulfills the equation $A^k + A^{k-1} = J$, we can deduce that the characteristic polynomial of A is $(x-d)x^{n-1-d}(x+1)^d$, where $n = d^{k-1} + d^k$. Therefore,

$$\operatorname{tr} A^l = d^l + (-1)^l d. \tag{2}$$

Working with such identities and using the previous corollary, we obtain the following result for the computation of the number of distinct short cycles of a $[k-1,k]$-digraph. We say that two cycles $C_1 : u_0, u_1, \dots, u_l$ and $C_2 : u'_0, u'_1, \dots, u'_l$ are equal iff one can be obtained from the other by means of a rotation, that is to say $C_2 : u_i, u_{i+1}, \dots, u_{i+l}$, for some i modulo l. Although the number of such cycles depends on the degree d of the digraph as well as on the length l, we simply denote it by $c(l)$.

Theorem 2. *Let G be a $[k-1,k]$-digraph of degree d. Then, the number $c(l)$ of distinct cycles of G of length l, $l \le k$, is given by the formula*

$$c(l) = \begin{cases} 0, & \text{if } l = 1, \\ \binom{d+1}{2}, & \text{if } l = 2, \\ \frac{1}{l}\sum_{m|l} \mu(\frac{l}{m}) d^m, & \text{if } 3 \le l \le k, \end{cases} \tag{3}$$

where $\mu(l)$ denotes the Möbius function.

Proof. From Corollary 1 and from identity (2), we have that

$$\operatorname{tr} A^l = d^l + (-1)^l d = \sum_{m|l} m \cdot c(m), \text{ if } l \le k.$$

Therefore, applying Möbius's inversion formula [18], we deduce that

$$c(l) = \frac{1}{l}\sum_{m|l} \mu(\frac{l}{m}) d^m + \frac{1}{l} d \sum_{m|l} \mu(\frac{l}{m})(-1)^m. \tag{4}$$

Now, we compute the auxiliary function

$$f(l) = \sum_{m|l} \mu(\frac{l}{m})(-1)^m = \sum_{m|l} \mu(m)(-1)^{\frac{l}{m}}.$$

If l is odd, then

$$\begin{aligned} f(l) &= -\textstyle\sum_{m|l} \mu(m) = -\sum_{m|l} \operatorname{tr} \Phi_m(x) = -\operatorname{tr} \prod_{m|l} \Phi_m(x) \\ &= -\operatorname{tr}(x^l - 1) = \begin{cases} -1, \text{ if } l = 1, \\ 0, \text{ otherwise,} \end{cases} \end{aligned}$$

where $\Phi_m(x)$ denotes the m-th cyclotomic polynomial and $\operatorname{tr} \Phi_m(x)$ represents the sum of all its complex roots.

Likewise, if $l = 2^e l'$, where $e \ge 1$ and l' is odd, then

$$\begin{aligned} f(l) &= \textstyle\sum_{m|l'} \mu(m)(-1)^{2^e \frac{l'}{m}} + \sum_{m|l'} \mu(2m)(-1)^{2^{e-1} \frac{l'}{m}} \\ &= \textstyle\sum_{m|l'} \mu(m)(1 - (-1)^{2^{e-1}}). \end{aligned}$$

Clearly, $f(l) = 0$, if $l = 2^e l'$ and $e > 1$. Moreover, if $l = 2l'$, then

$$f(l) = 2\sum_{m|l'} \mu(m) = \begin{cases} 2, \text{ if } l = 2 \\ 0, \text{ otherwise.} \end{cases}$$

Hence, $f(1) = -1$, $f(2) = 2$ and $f(l) = 0$ if $l > 2$. The proof is concluded by substituting these values into (4). □

We notice that the number $c(l)$ of distinct cycles of length $l \leq k$ in a $[k-1,k]$-digraph does not depend on k. Moreover, if $l \geq 3$, then $c(l)$ turns out to be equal to the number of distinct cycles of the same length in a $[k,k]$-digraph with equal degree (see [16]). Furthermore, the number of 2-cycles of a $[k-1,k]$-digraph is equal to the number of loops plus the number of 2-cycles of a $[k,k]$-digraph. We also remark that if $d \geq 2$, then $c(l) \geq 1$, which implies that any $[k-1,k]$-digraph of degree $d \geq 2$ has cycles of each length l, $2 \leq l \leq k$. The problem of the existence of longer cycles has been solved in the particular case of Kautz digraphs by Villar, who proved in [21] that any Kautz digraph has cycles of any length, except for 1 and $n-1$, where n is its order.

Taking into account that each vertex of a $[k-1,k]$-digraph is included in exactly one closed walk of length $k-1$ or k, we can also derive some other properties about the cycle structure of such a digraph.

Corollary 2. *If G is a $[k-1,k]$-digraph of degree d, then the following statements hold.*

(i) There exists a partition of the set of vertices of G into $\binom{d+1}{2}$ cycles of length 2 and $\frac{1}{l}\sum_{m|l}\mu(\frac{l}{m})d^m$ cycles of length l, for each $l \geq 3$ a divisor of $(k-1)$ or k. Moreover, the total number $\mathcal{N}(k)$ of these cycles is given by the expression

$$\mathcal{N}(k) = \frac{1}{k}\sum_{l|k}\phi(\frac{k}{l})d^l + \frac{1}{k-1}\sum_{l|(k-1)}\phi(\frac{k-1}{l})d^l - d, \tag{5}$$

where $\phi(i)$ stands for the Euler function.

(ii) Each arc not contained in a cycle of length a divisor of k belongs to a unique closed walk of length $k+1$, which is either a cycle, a repeated cycle or the concatenation of two arc-disjoint cycles.

Proof. From the definition of a $[k-1,k]$-digraph G, the existence of a partition of $V(G)$ into cycles of length a divisor of $k-1$ or k is derived. The number $c(l)$ of such cycles of each length l is given by Theorem 2 and its total number $\mathcal{N}(k)$ can be deduced as follows. Since $\gcd(k-1,k) = 1$ and $c(1) = 0$, we have that $\mathcal{N}(k) = \sum_{l|k} c(l) + \sum_{l|(k-1)} c(l)$. Therefore, using Theorem 2, we obtain that

$$\mathcal{N}(k) = \sum_{l|k}\frac{1}{l}\sum_{m|l}\mu(\frac{l}{m})d^m + \sum_{l|(k-1)}\frac{1}{l}\sum_{m|l}\mu(\frac{l}{m})d^m - d.$$

Then, taking into account that

$$\sum_{l|n}\frac{1}{l}\sum_{m|l}\mu(\frac{l}{m})d^m = \sum_{m|n}d^m\sum_{\substack{m|l\\ l|n}}\frac{1}{l}\mu(\frac{l}{m}) = \frac{1}{n}\sum_{m|n}d^m\sum_{l'|\frac{n}{m}}\frac{\frac{n}{m}}{l'}\mu(l')$$

and using the identity $\sum_{d|n}\frac{n}{d}\mu(d) = \phi(n)$ (see [18]), we can deduce (5).

Besides, given an arc uv of G, and since there exists exactly one $v \to u$ walk of length $k-1$ or k, we have that uv is included in exactly one closed walk of length k or $k+1$. Therefore, using Corollary 1, we deduce that if uv is not included in a cycle of length a divisor of k, then uv belongs to a unique closed walk of length $k+1$. Let $C : u_0, u_1, \ldots, u_{k+1}$ be such a closed walk and let us assume that C is neither a cycle nor a repeated cycle. Then, if u_i and u_j are the first two repeated vertices of the sequence C, we have that $C_1 : u_i, u_{i+1}, \ldots, u_j$ is a cycle and $C_2 : u_j, u_{j+1}, \ldots, u_{k+1}, u_0, \ldots, u_i$ is a cycle or the repetition of a cycle C_2' distinct from C_1. But such a repetition is impossible because, otherwise, we would have more than one walk of length k between u_i and u_{i-1}. Hence, C is equal to the concatenation of the two cycles C_1 and C_2, which are arc-disjoint. □

We point out that the computation of the number of *Kautz necklaces*, that is to say cycles of a Kautz digraph of length a divisor of its diameter k or $k-1$, was previously proved by Tvrdík in [20], by using combinatorial techniques. Here, we have extended such a result (5) to the class of $[k-1,k]$-digraphs.

Thus, if G is a $[k-1,k]$-digraph of degree d and diameter $k \leq 5$, then G has a vertex partition into $c(l)$ distinct l-cycles, where such numbers are shown in Table 1. For $k=2$, since $L\,K_{d+1}$ is the only $[1,2]$-digraph, we have that the number of l-cycles equals the number of circuits of K_{d+1} of the same length, which represents the number of closed sequences of length l of $(d+1)$-ary digits: $0, \ldots, d$ such that two consecutive digits are different and subsequences of length two are all different. It can be verified that these computations turn out to be equal to $c(l)$ for $l \leq 4$ but not for $l=5$, that is to say the expression of $c(l)$, given in Theorem 2 for $l \leq k$, can be extended, in the case $k=2$, for $l = k+1, k+2$. We do not know if such an extension can be generalized, neither do we know if there exists a formula for the computation of long cycles (cycles of length $> k$) of a $[k-1,k]$-digraph.

k	num. 2-cycles	num. 3-cycles	num. 4-cycles	num. 5-cycles
3	$\frac{d^2+d}{2}$	$\frac{d^3-d}{3}$		
4	$\frac{d^2+d}{2}$	$\frac{d^3-d}{3}$	$\frac{d^4-d^2}{4}$	
5	$\frac{d^2+d}{2}$	$\frac{d^3-d}{3}$	$\frac{d^4-d^2}{4}$	$\frac{d^5-d}{5}$

Table 1: Number of short cycles of a $[k-1,k]$-digraph of diameter $k \leq 5$.

5 The case $d=2$

In this section, we will illustrate how the previous results about the cycle structure of a $[k-1,k]$-digraph can be used in order to find the enumeration of such digraphs in the case of degree $d=2$ and small diameter k.

Applying Corollary 2, in the case $k=3$, we derive the following result.

Lemma 4. *If G is a $[2,3]$-digraph of degree d, then the following statements hold.*

(i) There is a partition

$$\mathcal{P} = \{C_1^2, C_2^2, \ldots, C_{\frac{d^2+d}{2}}^2, C_1^3, C_2^3, \ldots, C_{\frac{d^3-d}{3}}^3\}$$

of the set of vertices of G, *where* $C_i^j = \{v_1^i, v_2^i, \ldots, v_j^i\} \subset V(G)$ *is such that* $v_1^i, v_2^i, \ldots, v_j^i, v_1^i$ *represents a* j*-cycle of* G.

(ii) Each arc of G *not contained in a cycle of length* 2 *or* 3 *belongs to a unique cycle of* G *of length* 4.

In particular, every $[2,3]$-digraph G of degree $d = 2$ must contain the subdigraph $G_{\mathcal{P}}$ shown in Figure 1. Moreover, the strongly connected components of the subdigraph $\overline{G}_{\mathcal{P}}$ of G induced by the remaining arcs —one incident arc from each vertex— are cycle digraphs of order 4. The following technical lemma says how the arcs of $\overline{G}_{\mathcal{P}}$ have to be placed. Using this lemma, we will show that there is only one way of constructing (up to isomorphisms) a $[2,3]$-digraph of degree 2. In order to simplify the notation, from now on each part C_i^j will be identified by its associated cycle.

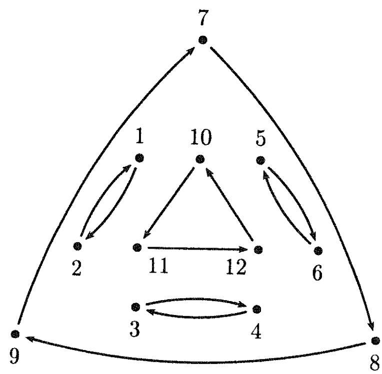

Figure 1: The subdigraph $G_{\mathcal{P}}$.

Lemma 5. *Let* G *be a* $[2,3]$*-digraph of degree* 2. *Let* $G_{\mathcal{P}}$ *be the subdigraph of* G *induced by its* 2*-cycles* (C_1^2, C_2^2, C_3^2) *and its* 3*-cycles* (C_1^3, C_2^3), *and let* $\overline{G}_{\mathcal{P}}$ *the subdigraph of* G *induced by its remaining arcs. Then, the arcs of* $\overline{G}_{\mathcal{P}}$ *satisfy the following properties:*

(i) Each arc of $\overline{G}_{\mathcal{P}}$ *is incident from a vertex of a* 2*-cycle [*3*-cycle] to a vertex of a* 3*-cycle [*2*-cycle]. Moreover, the arcs of* $\overline{G}_{\mathcal{P}}$ *incident from vertices of* G *included in the same* 2*-cycle [*3*-cycle] are incident to vertices of* G *included in distinct* 3*-cycles [*2*-cycles]. (See Figure 2 (I).)*

(ii) Let u_1, u_2, u_3 *be a walk of* $\overline{G}_{\mathcal{P}}$. *If* u_1 *is included in a* 2*-cycle [*3*-cycle] of* G, *then* u_3 *is contained in a distinct* 2*-cycle [*3*-cycle] of* G. *(See Figure 2 (II).)*

(*iii*) *Let* u_1v_1 [v_1u_1] *be an arc of* $\overline{G}_{\mathcal{P}}$ *such that* u_1 *is included in a 2-cycle* C_i^2 : u_1, u_2, u_1 *and* v_1 *is included in a 3-cycle* C_j^3 : v_1, v_2, v_3, v_1 *of* G. *Then,* v_3u_2 [u_2v_2] *is an arc of* $\overline{G}_{\mathcal{P}}$. *(See Figure 2 (III).)*

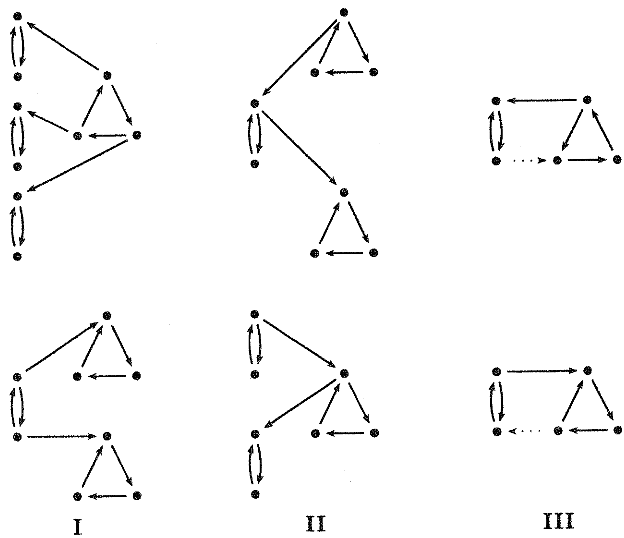

Figure 2: Conditions about the arcs of $\overline{G}_{\mathcal{P}}$.

Proof. Taking into account the definition of a $[k-1,k]$-digraph G, it can be seen that, given a vertex v_0 included in a cycle $C : v_0, v_1, \dots, v_l$ of G of length l a divisor of $k-1$, then each out-arc of v_0, except for v_0v_1, is an in-arc of a vertex included in a cycle of length a divisor of k. Since the converse digraph of G (derived from G by changing the orientation of its arcs) is also a $[k-1,k]$-digraph, every property satisfied by the out-arcs of a vertex v of G is also fulfilled by its in-arcs. In particular, if $k=3$, then each arc of $\overline{G}_{\mathcal{P}}$ that is incident from [to] a vertex included in a 2-cycle must also be incident to [from] a vertex included in a cycle of length 3. Moreover, since in the case $d=2$ there are equal numbers of vertices included in 2-cycles and in 3-cycles, we have that each arc of $\overline{G}_{\mathcal{P}}$ joins two vertices included in cycles of $G_{\mathcal{P}}$ of distinct length. Now, let us suppose that $\overline{G}_{\mathcal{P}}$ has two arcs u_1v_1 and u_2v_2, where $C^2 : u_1, u_2, u_1$ and $C^3 : v_1, v_2, v_3, v_1$ are cycles of $G_{\mathcal{P}}$ of length 2 and 3, respectively. Then, the sequences u_1, v_1, v_2, v_3 and u_1, u_2, v_2, v_3 are two different $u_1 \to v_3$ walks of length 3, which is impossible. The proof of (*ii*) is quite similar and property (*iii*) is a consequence of the first two. □

The application of the previous lemma to the subdigraph shown in Figure 1 allows us to conclude the enumeration of [2,3]-digraphs of degree 2.

Proposition 2. *There is only one* [2,3]*-digraph of degree* 2, *namely* L^2K_3.

Proof. Let G be a $[2,3]$-digraph of degree 2. We will prove, using Heuchenne's condition, that G is a line digraph, from which we will deduce, taking into account Lemma 2 and Theorem 1, that G is the Kautz digraph of diameter $k = 3$ and degree $d = 2$.

Let u_1 and v_1 be two vertices of G such that $\Gamma^+(u_1) \cap \Gamma^+(v_1) \neq \emptyset$. From property *(i)* of Lemma 5, we have that u_1 and v_1 must belong to cycles of $G_{\mathcal{P}}$ of distinct length. Let us assume that u_1 belongs to the cycle $C^2 : u_1, u_2, u_1$ and that v_1 is included in the 3-cycle $C^3 : v_1, v_2, v_3, v_1$. Therefore, since u_1v_2 $[v_1u_2]$ is an arc of $\overline{G}_{\mathcal{P}}$, we deduce, using property *(iii)* of the previous lemma, that $\Gamma^+(u_1) = \Gamma^+(v_1)$ and, consequently, G is a line digraph. Then, from Lemma 2, we have that $G = L\,G'$ where G' is a $[1,2]$-digraph of degree 2. Hence, using Theorem 1, we obtain that G is $L^2\,K_3$. (See Figure 3.)

□

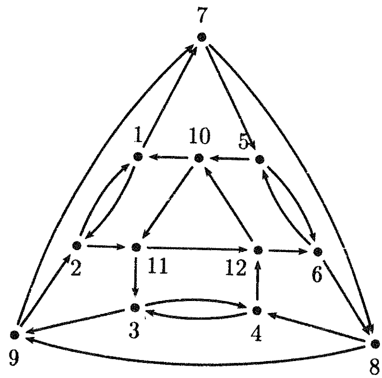

Figure 3: The digraph $L^2\,K_3$.

We notice that while Good-De Bruijn digraph of degree 2 and diameter 3 is one of the three non-isomorphic $[3,3]$-digraphs of such a degree (see [9]), Kautz digraph with equal parameters is the only $[2,3]$-digraph of degree 2. This may stregthen the idea that, as the order becomes closer to the Moore bound, there are fewer digraphs. With a more detailed reasoning it can be seen that $L^3\,K_3$ is the only $[3,4]$-digraph of degree 2. However, because of the ad hoc nature of these techniques, it may be worthwhile to find other approaches to the problem of the enumeration of $[k-1,k]$-digraphs for $k \geq 3$.

Acknowledgments

I thank M.A. Fiol for his encouragement and suggestions.

References

[1] E.T. Baskoro, M. Miller and J. Plesník, Further results on almost Moore digraphs, *Ars Combin.*, to appear.

[2] E.T. Baskoro, M. Miller, J. Plesník and Š. Znám, Digraphs of degree 3 and order close to the Moore bound, *J. Graph Theory* **20** (1995) 339–349.

[3] E.T. Baskoro, M. Miller, J. Plesník and Š. Znám, Regular digraphs of diameter 2 and maximum order, *Australas. J. Combinatorics* **9** (1994) 291–306.

[4] J. Bosák, Directed graphs and matrix equations, *Math. Slovaca* **28** (1978) 189–202.

[5] J.H. Conway and M.J.T. Guy, Message Graphs, *Annals of Discrete Mathematics* **13** (1982) 61–64.

[6] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Chapman & Hall, third edition (1996).

[7] M.A. Fiol, I. Alegre and J.L.A. Yebra, Line digraphs iterations and the (d, k) problem for directed graphs, *Proc. 10th Int. Symp. Comput. Arch.*, Stockholm (1983) 174–177.

[8] M.A. Fiol, J.L.A. Yebra and I. Alegre, Line digraphs iterations and the (d, k) digraph problem, *IEEE Trans. Comput.* **C-32** (1984) 400–403.

[9] M.A. Fiol, I. Alegre, J.L.A. Yebra and J. Fábrega, Digraphs with walks of equal length between vertices, *Graph Theory and its Applications to Algorithms and Computer Science*, Eds Y. Alavi et al. John Wiley & Sons (1985) 702–713.

[10] J. Gimbert, On the existence of (d, k)-digraphs, *Discrete Math.* **197–198** 1–3 (1999) 375–391.

[11] C. Heuchenne, Sur une certaine correspondance entre graphes, *Bull. Soc. Roy. Sci. Liège* **33** (1964) 743–753.

[12] A.J. Hoffman and M.H. McAndrew, The polynomial of a directed graph, *Proc. Amer. Math. Soc.* **16** (1965) 303–309.

[13] W.H. Kautz, Bound on directed (d, k) graphs, Theory of cellular logic networks and machines. AFCRL–68–0668 Final Report (1968) 20–28.

[14] W.H. Kautz, Design of optimal interconnection networks for multiprocessors, *Architecture and Design of Digital Computers.* NATO Advanced Summer Institute (1969) 249–272.

[15] D.E. Knuth, Notes on central groupoids, *J. Combin. Theory* **8** (1970) 376–390.

[16] N.S. Mendelsohn, Directed graphs with the unique path property, *Combinatorial Theory and its Applications II* (Proc. Colloq. Balatonfürer, 1969), North Holland (1970) 783–799.

[17] M. Miller and I. Fris, Maximum order digraphs for diameter 2 or degree 2, *Pullman Volume of Graphs and Matrices, Lecture Notes in Pure and Applied Mathematics* **139** (1992) 269–278.

[18] I. Niven, H.S. Zuckerman and H.L. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley & Sons, fifth edition (1991).

[19] J. Plesník and Š. Znám, Strongly geodetic directed graphs, *Acta F. R. N. Univ. Comen.-Mathematica* **XXIX** (1974) 29–34.

[20] P. Tvrdík, Kautz Necklaces, Research Report 94–08, LIP ENSL, 69364 Lyon, France (1994).

[21] J.L. Villar, Kautz s-ary closed sequences, *Combinatorics 88*, **II**, Research and Lecture Notes in Mathematics, Mediterranean Press (1991) 459–469.

(Received 8/7/98)