An enumeration of 1-perfect binary codes

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Abstract

We enumerate the extended perfect 1–error correcting binary codes of length 16 which can be constructed by the concatenation or doubling construction. In the process, we establish some properties of these codes and consider algorithms that effectively establish the nonequivalence of these codes.

1 Introduction

Perfect 1-error correcting binary codes (briefly 1-perfect codes) are of length $n = 2^r - 1$, minimum distance 3, and have 2^{n-r} codewords. Hamming codes are the linear 1-perfect codes but for $n \geq 15$ there also exist nonlinear 1-perfect codes. Various authors have established an asymptotic lower bound of $2^{2^{.5n(1-o(n))}}$ on the number of nonequivalent 1-perfect codes of length n but no nontrivial upper bound is known (i.e. better than $2^{2^{n(1-o(n))}}$ [1], cf. [3],[9]). In fact, little is even known about the number of nonequivalent 1-perfect codes of length 15. In this paper, we establish a reasonable lower bound on this number by enumerating all nonequivalent *extended* 1-perfect codes of length 16 that can be constructed by *doubling*, or *concatenation* as it is called by some (cf. [10],[6]).

Two codes of length n, C, D, are said to be equivalent if there is a codeword $a \in K^n$, ($K = GF(2)$), and an $n \times n$ permutation matrix P such that $C = \{(a+x)P \mid x \in D\}$. If $C = D$ then the mapping is said to be an automorphism of C. $AutC$ will denote the group of all automorphisms of C. If the codes are linear then the mappings are just permutations of the coordinates, that is, $a = \vec{0}$. If $P = I_n$ then the mappings are *translations*. Assuming that $\vec{0} \in C$, then the set of translations leaving C invariant is a linear subspace of C called the *kernel* of C [7],[2].

Extended 1-perfect codes are formed from 1-perfect codes by adding an overall parity check bit. The reverse process, puncturing the code, deletes one specified coordinate from all codewords. If two 1-perfect codes are equivalent then the corresponding extended codes are equivalent. However, the converse is false : puncturing an extended 1-perfect code in different coordinates can give rise to nonequivalent 1-perfect codes. Because of the nature of the construction method and the potentially large number of 1-perfect codes of length 15, we choose to enumerate the number of extended 1-perfect codes of length 16 constructed by the concatenation method, finding 963 such codes. This implies that the number of 1-perfect codes of length 15 constructed by this method is at least 963 and at most 15,408.

2 Partitions

Any partition of (binary) n-space, K^n, into 1-perfect codes of length n is said to be a 1-perfect partition (of length n). Each 1-perfect code in a 1-perfect partition can be extended, forming a partition of the even weight words of K^{n+1} into extended 1-perfect codes (of length $n+1$). Such a partition is said to be an *extended* 1-perfect partition. Associated with each 1-perfect code C of length n is the natural partition of K^n, into translates (or cosets if C is linear) $C+e_i$, $i=1,\ldots,n$, where the support of e_i is $\{i\}$. However, many other partitions exist [6]. Extended 1-perfect partitions are the key element in the doubling construction.

Construction 1 ([6],[10]) *Given two extended 1-perfect partitions of length $n+1$, $C_0, C_1, \ldots, C_n$ and $D_0, D_1, \ldots, D_n$ and a permutation α of the index set $\{0,1,\ldots,n\}$ form the extended 1-perfect code of length $2n+2$,*

$$C_\alpha = \bigcup_{i=0}^{n} C_i | D_{\alpha(i)}$$

where $C_i|D_{\alpha(i)} = \{(x,y) \mid x \in C_i,\ y \in D_{\alpha(i)}\}$.

Obviously for two given (extended) 1-partitions, the above construction can produce $(n+1)!$ different codes. It is natural to ask, how many of these codes are equivalent? This is a difficult question but we can say something.

The automorphism group of a partition of the n-cube, K^n, is just the subgroup of the automorphism group of K^n that acts imprimitively on the partition, mapping parts to parts. The action of the automorphism group on the partition induces a permutation group on the index set $\{0,1,\ldots,n\}$.

Lemma 1 *Let P_0 and P_1 be two (extended 1-perfect) partitions (of length $n+1$) and let H_0 and H_1 be the permutation groups on the index set induced by the automorphism groups of P_0 and P_1 respectively. The codes C_α , C_β formed by the above construction using partitions P_0, P_1 are equivalent if α and β are in the same double coset $H_1 x H_0$.*

Proof: Let γ_0 and γ_1 be automorphisms of the partitions and $h \in H_0$ and $k \in H_1$ be the corresponding induced permutations. Applying these automorphisms to the code C_α gives the code

$$C_\beta = \bigcup_{i=0}^{n} C_{h(i)} | D_{k(\alpha(i))}$$

where $C_{h(i)}|D_{k(\alpha(i))} = \{(\gamma_0(x), \gamma_1(y)) \mid x \in C_i,\ y \in D_{\alpha(i)}\}$ and $\beta = k \circ \alpha \circ h^{-1}$. ∎

In order to enumerate extended 1-perfect codes of length 16, we need to find all nonequivalent partitions of K^7 into 1-perfect codes of length 7. (Two partitions are said to be equivalent if there is an automorphism of K^n which maps one partition to the other.) There are 30 different (but isomorphic) linear 1-perfect codes of length 7 corresponding to the 30 different Steiner triple systems of order 7. Each linear code has 8 cosets. Any 1-perfect partition will either have at least 2 cosets of these linear codes or will have at most one coset of any linear code. In the first case, we can choose one linear code and its coset and enumerate all solutions. We found 192 different partitions but only 10 nonequivalent partitions. In the second case we found only one nonequivalent solution. The 11 nonequivalent partitions are listed in the appendix as a partition of the vertices $\{1,2,\ldots,128\}$ of the 7-cube where vertex i corresponds to the binary codeword representing $i-1$. Also listed for each partition P_i is the order of its automorphism group, the order of the induced permutation group L_i on the index set, and a list of generators for that induced group L_i.

Of these 11 partitions, six had previously been found by the author [6].

Construction 1 uses extended 1-perfect partitions. As with 1-perfect codes and extended 1-perfect codes, if two partitions are equivalent then the corresponding extended partitions will also be equivalent; however, the converse is again false. Puncturing extended partitions in different coordinates can result in nonequivalent (1-perfect) partitions. In fact, although there are 11 nonequivalent 1-perfect partitions of K^7, there are only 10 nonequivalent extended 1-perfect partitions of length 8. (Partitions 2 and 7 have equivalent extensions). The second fact to consider is that the extended code or extended partition can have a larger automorphism group than the code or partition itself. Our computations on 1-perfect partitions still apply as every automorphism of a 1-perfect partition can be uniquely extended to an automorphism of the extended 1-perfect partition. However, in computing double coset representatives of these subgroups, equivalent codes may be generated.

We remark that graph isomorphism program *NAUTY* [5] was used to compute the automorphism group and to test equivalence of these partitions and extended partitions. The program *MAGMA* was used to compute the induced permutation group of the partitions of K^7 as well as the corresponding double coset representatives. *MAGMA* was not able to handle the computations for the extended 1-perfect partitions in K^8. Instead of writing a separate routine, the computations on K^7 were used since equivalent codes would be eliminated by the isomorphism checking routines. For the 55 pairs of these 11 partitions *MAGMA*, found 2278 double coset representatives. Table 1 below lists the number of double coset representatives for each pair P_i, P_j of partitions. Since extended 1-perfect partitions 2 and 7 are equivalent, all codes constructed using extended 1-perfect partition 7 were eliminated, leaving 1656 codes to consider. Table 2 below lists the number of nonequivalent 1-perfect codes constructed for each pair of (extended) partitions.

partitions	0	1	2	3	4	5	6	7	8	9	10
0	16	13	20	32	51	3	8	29	7	18	21
1		32	27	30	45	8	11	54	20	36	22
2			39	51	72	5	11	57	15	36	32
3				82	126	5	14	72	16	43	50
4					219	6	24	117	24	66	81
5						4	3	10	6	8	4
6							7	19	7	12	11
7								114	30	72	48
8									14	20	13
9										48	28
10											34

Table 1: Number of double coset representatives

partitions	0	1	2	3	4	5	6	7*	8	9	10
0	14	9	20	32	34	3	7		3	13	21
1		13	17	19	15	5	7		6	12	13
2			29	51	45	5	11		5	24	32
3				57	80	5	13		5	28	50
4					53	4	12		4	27	49
5						3	3		4	3	4
6							5		3	8	9
7											
8									5	4	4
9										10	19
10											27

Table 2: Number of nonequivalent codes by partitions

3 Equivalence of 1-perfect codes

Determining the equivalence of codes can be a difficult and time-comsuming problem. On the one hand, if the codes are not equivalent then the algorithm should produce a certificate of this. If they are equivalent then the algorithm must find the mapping.

The easiest approach is to represent the code by a graph and use the graph isomorphism package *NAUTY*. For linear codes, one can form a bipartite graph as one would for any incidence structure. For nonlinear codes of length $n+1$, one first forms a subgraph having vertices $i_j, i = 0, 1 \ldots, n, j = 0, 1$ and edges $[i_0, i_1]$, $i = 0, 1 \ldots, n$. The remaining vertices of the graph will be the codewords with a codeword x adjacent to vertex i_j if and only if coordinate $x_i = j$. It is easy to see that two codes will be equivalent if and only if the corresponding graphs are isomorphic. Unfortunately, the graph for an extended 1-perfect code of length 16 would have 2080 vertices and, without further information, *NAUTY* is rather slow on such graphs.

The codes were first classified according to several invariants. The *rank* of a code is just the dimension of the space spanned by the codewords. The *kernel* of a code C is just the set of translations which leave C invariant. Assuming that the code contains the zero vector, then we have:

$$kernel(C) = \{a \in C \mid C = C + a\}.$$

The kernel is the key to developing a faster algorithm for code equivalence. We summarize and extend some known results on kernels of codes.

Proposition 2 *Let $C_K \subset C$ denote the kernel of a code C (assuming $\vec{0} \in C$), then:*

- *C_K is the intersection of all maximal linear subcodes of C;*
- *C_K is the maximum linear subcode such that C is the union of cosets of C_K;*
- *C_K is a normal subgroup in the automorphism group of C;*
- *if $x \to (x+a)P$ is an automorphsim (isomorphism) of C then P is an automorphism (isomorphism) of the kernel, i.e. $C_K = C_K P$.*

Proof: We establish the last two assertions; see [2], [7] for the other arguments.

Let $\gamma(x) = (x+a)P$ and $\beta(x) = x+b$ for any $b \in C_K$, then $\gamma^{-1}(y) = (y+aP)P^{-1}$. Hence

$$\gamma^{-1}\beta\gamma(x) = x + bP^{-1}$$

is an automorphism of C, which means $bP^{-1} \in C_K$. Thus P is an automorphism of C_K and the translations by C_K form a normal subgroup in the automorphism group of C. ∎

The rank and kernel of a code can be computed relatively quickly ([8]) and give an initial classification of the extended 1-perfect codes of length 16. Since a code is the union of cosets of the kernel, one can represent the code C by its kernel C_K and a (canonical) set of coset representatives C_H. Two codes with the same kernel

rank\kernel	11	9	8	7	6	5	4	3	2
11	1								
12		2	2	3					
13			7	11	38	34	20		
14			1	4	48	210	374	172	36

Table 3: Number of nonequivalent codes by rank and dimension of kernel

are equivalent if there is an isomorphism $\gamma(x) = (a + x)P$ for some $P \in AutC_K$ and $a \in C_H$. *NAUTY* can then be applied to the subcode $C_K \cup C_H$ to find a canonical isomorphic representation for the code C.

Associated with each codeword in an extended 1-perfect code of length $n+1$ is a Steiner quadruple system $SQS(x) = \{x + y \mid d(x,y) = 4, y \in C\}$. One can classify codewords by the isomorphism class of the $SQS(x)$. There is a simple, quickly computable isomorphism invariant for $SQS(x)$: A triangle in an SQS is a configuration of 3 quadruples on 6 points with any two quadruples intersecting in a pair. For each codeword $x \in C_H$, we computed the number of triangles in the associated $SQS(x)$. Besides partitioning the codewords (in C_H) this isomorphism invariant for SQS also gives a certificate of nonequivalence. Along with the rank and kernel, this certificate divided the 1656 codes into 721 classes of codes.

A stronger invariant was needed. Several were tried. Finally, *NAUTY* was used to test for isomorphism classes of the associated $SQS(x)$. *NAUTY* computes a complete invariant but it is much slower and the invariant (a canonically labeled graph) is large. We used several hash functions of the canonically labeled graph to derive a short certificate of nonisomorphism of the SQS. This set of certificates was then used to derive a certificate of nonequivalence for the code. This set of certificates was also used to put the code into a canonical order and establish equivalence between codes when applicable.

4 Summary

Appendix A contains a listing of the 11 nonequivalent partitions of the 7-cube. Text files containing all 11 extended partitions, and a listing of the 963 nonequivalent codes is available at the author's home page

http://www.dms.auburn.edu/~phelpkt

in the zipped file codes16.gz. A compact listing is also available in postscript format in the file appendixB.ps.

More precisely, each line in the listing of the 963 nonequivalent code looks like:

$$xxxyz \ rank \ kernal \ C \ D \ \alpha()$$

where *xxxyz* is the identification number, followed by the rank and the dimension of the kernel. The code is formed by using (extended) partitions C, D and the permutation $\alpha()$ as in Construction 1 (doubling). In the identification number, *xxx* is the initial equivalence class (1-721); y indicates codes in the same initial equivalence class that had nonisomorphic kernels; z further subdivides the initial partition into the nonequivalent codes.

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A Appendix : 1-perfect partitions of the 7-cube

Order of automorphism group of 7-cube = 645120
Partition 0:

$$\begin{gathered}
\{1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128\}, \\
\{2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127\}, \\
\{3, 6, 28, 29, 42, 47, 49, 56, 73, 80, 82, 87, 100, 101, 123, 126\}, \\
\{4, 5, 27, 30, 41, 48, 50, 55, 74, 79, 81, 88, 99, 102, 124, 125\}, \\
\{9, 16, 18, 23, 36, 37, 59, 62, 67, 70, 92, 93, 106, 111, 113, 120\}, \\
\{10, 15, 17, 24, 35, 38, 60, 61, 68, 69, 91, 94, 105, 112, 114, 119\}, \\
\{11, 14, 20, 21, 33, 40, 58, 63, 66, 71, 89, 96, 108, 109, 115, 118\}, \\
\{12, 13, 19, 22, 34, 39, 57, 64, 65, 72, 90, 95, 107, 110, 116, 117\}
\end{gathered}$$

Order of automorphism group of partition 0 =768
Order of automorphism group of extended partition 0 = 6144
Permutation group L_0 acting on a set of cardinality 8,
Order = 96 $=2^5 \cdot 3$, generators:

$$\begin{gathered}
(5, 6)(7, 8) \\
(3, 5)(4, 6) \\
(1, 3)(2, 4)(5, 6) \\
(1, 2)(3, 4)(5, 6)(7, 8)
\end{gathered}$$

Partition 1:

$$\begin{gathered}
\{1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128\}, \\
\{2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127\}, \\
\{3, 6, 28, 29, 42, 47, 49, 56, 73, 80, 82, 87, 100, 101, 123, 126\}, \\
\{4, 5, 27, 30, 41, 48, 50, 55, 74, 79, 81, 88, 99, 102, 124, 125\}, \\
\{9, 16, 19, 22, 34, 39, 60, 61, 68, 69, 90, 95, 107, 110, 113, 120\}, \\
\{11, 14, 17, 24, 36, 37, 58, 63, 66, 71, 92, 93, 105, 112, 115, 118\}, \\
\{10, 15, 20, 21, 33, 40, 59, 62, 67, 70, 89, 96, 108, 109, 114, 119\}, \\
\{12, 13, 18, 23, 35, 38, 57, 64, 65, 72, 91, 94, 106, 111, 116, 117\}
\end{gathered}$$

Order of automorphism group of partition 1 =1536
Order of automorphism group of extended partition 1 = 12288
Permutation group L_1 acting on a set of cardinality 8
Order = 96 = $2^5 \cdot 3$, generators:

$$\begin{gathered}
(2, 3, 4)(6, 8, 7) \\
(5, 7)(6, 8) \\
(5, 8)(6, 7) \\
(1, 5)(2, 6)(3, 7)(4, 8) \\
(1, 2)(3, 4)(5, 7)(6, 8)
\end{gathered}$$

Partition 2:

$$\begin{gathered}
\{\ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128\ \},\\
\{\ 2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127\ \},\\
\{\ 3, 6, 28, 29, 41, 48, 50, 55, 74, 79, 81, 88, 100, 101, 123, 126\ \},\\
\{\ 4, 5, 27, 30, 42, 47, 49, 56, 73, 80, 82, 87, 99, 102, 124, 125\ \},\\
\{\ 9, 16, 19, 22, 34, 39, 60, 61, 68, 69, 90, 95, 107, 110, 113, 120\ \},\\
\{\ 11, 14, 17, 24, 36, 37, 58, 63, 66, 71, 92, 93, 105, 112, 115, 118\ \},\\
\{\ 12, 13, 18, 23, 33, 40, 59, 62, 67, 70, 89, 96, 106, 111, 116, 117\ \},\\
\{\ 10, 15, 20, 21, 35, 38, 57, 64, 65, 72, 91, 94, 108, 109, 114, 119\ \}
\end{gathered}$$

Order of automorphism group of partition 2 =256.
Order of automorphism group of extended partition 2 = 1024
Permutation group L_2 acting on a set of cardinality 8,
Order = 64 $=2^6$, generators:

$$\begin{gathered}
(5, 6)(7, 8)\\
(1, 5)(2, 6)(3, 8)(4, 7)\\
(1, 2)(3, 4)(7, 8)\\
(1, 3)(2, 4)(5, 8, 6, 7)
\end{gathered}$$

Partition 3:

$$\begin{gathered}
\{\ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128\ \},\\
\{\ 2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127\ \},\\
\{\ 3, 6, 28, 29, 41, 48, 50, 55, 74, 79, 81, 88, 100, 101, 123, 126\ \},\\
\{\ 5, 10, 24, 27, 36, 47, 49, 62, 67, 80, 82, 93, 102, 105, 119, 124\ \},\\
\{\ 9, 16, 18, 23, 35, 38, 60, 61, 68, 69, 91, 94, 106, 111, 113, 120\ \},\\
\{\ 4, 15, 17, 30, 37, 42, 56, 59, 70, 73, 87, 92, 99, 112, 114, 125\ \},\\
\{\ 11, 14, 20, 21, 33, 40, 58, 63, 66, 71, 89, 96, 108, 109, 115, 118\ \},\\
\{\ 12, 13, 19, 22, 34, 39, 57, 64, 65, 72, 90, 95, 107, 110, 116, 117\ \}
\end{gathered}$$

Order of automorphism group of partition 3 =128.
Order of automorphism group of extended partition 3 = 1024
Permutation group L_3 acting on a set of cardinality 8
Order = 32 = 2^5, generators:

$$\begin{gathered}
(1, 6, 2, 4)(3, 7)(5, 8)\\
(1, 2)(4, 6)(7, 8)\\
(3, 5)(4, 6)
\end{gathered}$$

Partition 4:

$$\begin{gathered}
\{\ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128\ \},\\
\{\ 2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127\ \},\\
\{\ 3, 6, 28, 29, 42, 47, 49, 56, 73, 80, 82, 87, 100, 101, 123, 126\ \},\\
\{\ 4, 5, 27, 30, 41, 48, 50, 55, 74, 79, 81, 88, 99, 102, 124, 125\ \},\\
\{\ 9, 16, 20, 21, 35, 38, 58, 63, 66, 71, 91, 94, 108, 109, 113, 120\ \},
\end{gathered}$$

$\{ 10, 15, 17, 24, 36, 37, 59, 62, 67, 70, 92, 93, 105, 112, 114, 119 \}$,
$\{ 11, 14, 18, 23, 33, 40, 60, 61, 68, 69, 89, 96, 106, 111, 115, 118 \}$,
$\{ 12, 13, 19, 22, 34, 39, 57, 64, 65, 72, 90, 95, 107, 110, 116, 117 \}$

Order of automorphism group of partition 4 =128
Order of automorphism group of extended partition 4 = 1024
Permutation group L_4 acting on a set of cardinality 8
Order = 16 = 2^4, generators:

$$(5, 7)(6, 8)$$
$$(1, 4)(2, 3)(5, 7)(6, 8)$$
$$(1, 3)(2, 4)$$
$$(1, 2)(3, 4)(6, 8)$$
$$(1, 3)(2, 4)(5, 7)(6, 8)$$

Partition 5:

$\{ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128 \}$,
$\{ 2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127 \}$,
$\{ 3, 6, 28, 29, 42, 47, 49, 56, 73, 80, 82, 87, 100, 101, 123, 126 \}$,
$\{ 4, 5, 27, 30, 41, 48, 50, 55, 74, 79, 81, 88, 99, 102, 124, 125 \}$,
$\{ 9, 16, 18, 23, 36, 37, 59, 62, 67, 70, 92, 93, 106, 111, 113, 120 \}$,
$\{ 10, 15, 17, 24, 35, 38, 60, 61, 68, 69, 91, 94, 105, 112, 114, 119 \}$,
$\{ 12, 13, 19, 22, 33, 40, 58, 63, 66, 71, 89, 96, 107, 110, 116, 117 \}$,
$\{ 11, 14, 20, 21, 34, 39, 57, 64, 65, 72, 90, 95, 108, 109, 115, 118 \}$

Order of automorphism group of partition 5 =21504
Order of automorphism group of extended partition 5 = 172032
Permutation group L_5 acting on a set of cardinality 8
Order = 1344 =$2^6 \cdot 3 \cdot 7$, generators:

$$(5, 6)(7, 8)$$
$$(5, 7)(6, 8)$$
$$(3, 5)(4, 6)$$
$$(2, 3)(6, 8)$$
$$(1, 2)(3, 4)(5, 6)(7, 8)$$

Partition 6:

$\{ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128 \}$,
$\{ 2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127 \}$,
$\{ 3, 10, 24, 29, 37, 48, 50, 59, 70, 79, 81, 92, 100, 105, 119, 126 \}$,
$\{ 5, 11, 20, 30, 40, 42, 49, 63, 66, 80, 87, 89, 99, 109, 118, 124 \}$,
$\{ 9, 16, 19, 22, 34, 39, 60, 61, 68, 69, 90, 95, 107, 110, 113, 120 \}$,
$\{ 6, 15, 17, 28, 36, 41, 55, 62, 67, 74, 88, 93, 101, 112, 114, 123 \}$,
$\{ 4, 14, 21, 27, 33, 47, 56, 58, 71, 73, 82, 96, 102, 108, 115, 125 \}$,
$\{ 12, 13, 18, 23, 35, 38, 57, 64, 65, 72, 91, 94, 106, 111, 116, 117 \}$

Order of automorphism group of partition 6 =384
Order of automorphism group of extended partition 6 = 3072
Permutation group L_6 acting on a set of cardinality 8
Order = 192 $=2^6 \cdot 3$, generators:

$$(3, 7, 5)(4, 8, 6)$$
$$(1, 3)(2, 6)(4, 5)(7, 8)$$
$$(1, 2)(3, 7)(4, 6)$$

Partition 7:

{ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128 },
{ 2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127 },
{ 3, 6, 28, 29, 41, 48, 50, 55, 74, 79, 81, 88, 100, 101, 123, 126 },
{ 4, 5, 27, 30, 42, 47, 49, 56, 73, 80, 82, 87, 99, 102, 124, 125 },
{ 9, 16, 19, 22, 34, 39, 60, 61, 68, 69, 90, 95, 107, 110, 113, 120 },
{ 12, 13, 17, 24, 35, 38, 58, 63, 66, 71, 91, 94, 105, 112, 116, 117 },
{ 10, 15, 20, 21, 33, 40, 59, 62, 67, 70, 89, 96, 108, 109, 114, 119 },
{ 11, 14, 18, 23, 36, 37, 57, 64, 65, 72, 92, 93, 106, 111, 115, 118 }

Order of automorphism group of partition 7 =256
Order of automorphism group of extended partition 7 (= 2) = 1024
Permutation group L_7 acting on a set of cardinality 8
Order = 32 = 2^5, generators:

$$(5, 7)(6, 8)$$
$$(1, 6, 2, 8)(3, 5, 4, 7)$$
$$(1, 5)(2, 7)(3, 6)(4, 8)$$
$$(1, 3, 2, 4)(5, 8, 7, 6)$$
$$(1, 2)(3, 4)(5, 7)(6, 8)$$

Partition 8:

{ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128 },
{ 2, 11, 23, 30, 37, 48, 52, 57, 72, 77, 81, 92, 99, 106, 118, 127 },
{ 3, 16, 18, 29, 38, 41, 55, 60, 69, 74, 88, 91, 100, 111, 113, 126 },
{ 5, 12, 24, 25, 35, 46, 50, 63, 66, 79, 83, 94, 104, 105, 117, 124 },
{ 6, 9, 19, 32, 36, 47, 53, 58, 71, 76, 82, 93, 97, 110, 120, 123 },
{ 7, 14, 17, 28, 34, 43, 56, 61, 68, 73, 86, 95, 101, 112, 115, 122 },
{ 10, 15, 20, 21, 33, 40, 59, 62, 67, 70, 89, 96, 108, 109, 114, 119 },
{ 4, 13, 22, 27, 39, 42, 49, 64, 65, 80, 87, 90, 102, 107, 116, 125 }

Order of automorphism group of partition 8 =336
Order of automorphism group of extended partition 8 = 2688
Permutation group L_8 acting on a set of cardinality 8
Order = 168 = $2^3 \cdot 3 \cdot 7$, generators:

$$(2, 3, 7)(4, 6, 8)$$
$$(1, 2, 4)(3, 8, 6)$$
$$(1, 2)(3, 6)(4, 5)(7, 8)$$

Partition 9:

{ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128 },
{ 2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127 },
{ 3, 6, 28, 29, 41, 48, 50, 55, 74, 79, 81, 88, 100, 101, 123, 126 },
{ 4, 5, 27, 30, 42, 47, 49, 56, 73, 80, 82, 87, 99, 102, 124, 125 },
{ 9, 16, 18, 23, 36, 37, 59, 62, 67, 70, 92, 93, 106, 111, 113, 120 },
{ 10, 15, 17, 24, 35, 38, 60, 61, 68, 69, 91, 94, 105, 112, 114, 119 },
{ 11, 14, 20, 21, 33, 40, 58, 63, 66, 71, 89, 96, 108, 109, 115, 118 },
{ 12, 13, 19, 22, 34, 39, 57, 64, 65, 72, 90, 95, 107, 110, 116, 117 }

Order of automorphism group of partition 9 =1024
Order of automorphism group of extended partition 9 = 8192
Permutation group L_9 acting on a set of cardinality 8
Order = $64 = 2^6$, generators:

$$(5, 6)(7, 8)$$
$$(3, 7)(4, 8)$$
$$(1, 5)(2, 6)(3, 8)(4, 7)$$
$$(1, 3)(2, 4)(5, 7)(6, 8)$$
$$(1, 2)(3, 4)(5, 6)(7, 8)$$

Partition 10:

{ 1, 8, 26, 31, 44, 45, 51, 54, 75, 78, 84, 85, 98, 103, 121, 128 },
{ 2, 7, 25, 32, 43, 46, 52, 53, 76, 77, 83, 86, 97, 104, 122, 127 },
{ 3, 6, 28, 29, 41, 48, 50, 55, 74, 79, 81, 88, 100, 101, 123, 126 },
{ 5, 10, 24, 27, 36, 47, 49, 62, 67, 80, 82, 93, 102, 105, 119, 124 },
{ 9, 16, 19, 22, 34, 39, 60, 61, 68, 69, 90, 95, 107, 110, 113, 120 },
{ 4, 15, 17, 30, 37, 42, 56, 59, 70, 73, 87, 92, 99, 112, 114, 125 },
{ 11, 14, 20, 21, 33, 40, 58, 63, 66, 71, 89, 96, 108, 109, 115, 118 },
{ 12, 13, 18, 23, 35, 38, 57, 64, 65, 72, 91, 94, 106, 111, 116, 117 }

Order of automorphism group of partition 10 = 96
Order of automorphism group of extended partition 10 = 768
Permutation group L_{10} acting on a set of cardinality 8
Order = $48 = 2^4 \cdot 3$, generators:

$$(1, 6, 2, 4)(3, 7)(5, 8)$$
$$(1, 2)(3, 7)(4, 5)(6, 8)$$

(Received 31/8/99)