A few more incomplete self-orthogonal Latin squares and related designs*

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Abstract

An incomplete self-orthogonal Latin square of order v with an empty subarray of order n, an ISOLS(v, n), can exist only if $v \geq 3n + 1$. This necessary condition is known to be sufficient apart from 2 known exceptions $(v, n) = (6, 1)$ and $(8, 2)$ plus 14 possible exceptions (v, n) with $v = 3n + 2$. In this paper, we construct eleven new ISOLS$(3n + 2, n)$ reducing unknown n to $6, 8, 10$ only. This result is then used to improve the existence of HSOLS of type $3^n u^1$. To do this, two newly found unipotent SOLSSOMs, SOLSSOM(66) and SOLSSOM(70) are also useful.

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1 Introduction

Let S be a set and $\mathcal{H} = \{S_1, S_2, \cdots, S_n\}$ be a set of disjoint subsets of S. A *holey Latin square* having *hole set* $\mathcal{H}$ is an $|S| \times |S|$ array L, indexed by S, satisfying the following properties:

(1) every cell of L either contains an element of S or is empty,

(2) every element of S occurs at most once in any row or column of L,

(3) the subarrays indexed by $S_i \times S_i$ are empty for $1 \le i \le n$ (these subarrays are referred to as *holes*),

(4) element $s \in S$ occurs in row or column t if and only if $(s,t) \in (S \times S) \setminus \bigcup_{1 \le i \le n}(S_i \times S_i)$.

The *order* of L is $|S|$. Two holey Latin squares on symbol set S and hole set $\mathcal{H}$, say L_1 and L_2, are said to be *orthogonal* if their superposition yields every ordered pair in $(S \times S) \setminus \bigcup_{1 \le i \le n}(S_i \times S_i)$. We shall use the notation IMOLS$(s; s_1, \cdots, s_n)$ to denote a pair of orthogonal holey Latin squares on symbol set S and hole set $\mathcal{H} = \{S_1, S_2, \cdots, S_n\}$, where $s = |S|$ and $s_i = |S_i|$ for $1 \le i \le n$. If $\mathcal{H} = \emptyset$, we obtain a MOLS(s). If $\mathcal{H} = \{S_1\}$, we simply write IMOLS(s, s_1) for the orthogonal pair of holey Latin squares.

If $\mathcal{H} = \{S_1, S_2, \cdots, S_n\}$ is a partition of S, then a holey Latin square is called a *partitioned incomplete Latin square* , denoted by PILS. The *type* of the PILS is defined to be the multiset $\{|S_i| : 1 \le i \le n\}$. We shall use an "exponential" notation to describe types: so type $t_1^{u_1} \cdots t_k^{u_k}$ denotes u_i occurrences of $t_i, 1 \le i \le k$, in the multiset. Two orthogonal PILSs of type T will be denoted by HMOLS(T).

A holey Latin square is called *self-orthogonal* if it is orthogonal to its transpose. For self-orthogonal holey Latin squares we use the notations SOLS(s), ISOLS(s, s_1) and HSOLS(T) for the cases of $\mathcal{H} = \emptyset$, $\{S_1\}$ and a partition $\{S_1, S_2, \cdots, S_n\}$, respectively. An HSOLS(T) is also denoted *frame* SOLS or FSOLS of type T. If $\mathcal{H} = \{S_1, S_2, \cdots, S_n, H\}$, where $\{S_1, S_2, \cdots, S_n\}$ is a partition of S, then the holey SOLS is called *incomplete frame* SOLS or an *I-frame* SOLS. The type of the I-frame SOLS is defined to be the multiset $\{(|S_i|, |S_i \cap H|) : 1 \le i \le n\}$. We may also use an "exponential" notation to describe types of I-frame SOLS.

If A, B, C are 3 MOLS such that $B = A^T$ and $C = C^T$, then A is a *self-orthogonal Latin square* (SOLS) with C a *symmetric orthogonal mate* (SOM). The set of A, B, C is denoted SOLSSOM. If C has constant diagonal, the SOLSSOM is called *unipotent.*

The existence problem for ISOLS(v, n) has been almost completely solved. The following is known (see [5],[6],[10]).

Theorem 1.1 *There exists an ISOLS(v,n) for all values of v and n satisfying $v \ge 3n+1$, except for $(v,n) = (6,1), (8,2)$ and possibly for $v = 3n+2$, $n \in \{4, 6, 8, 10, 14, 16, 18, 20, 22, 26, 28, 32, 34, 46\}$.*

In Section 2, we shall first give a direct construction for ISOLS$(14, 4)$ and then use it to solve ten other cases. This leaves only $n = 6, 8, 10$ undecided. To do this some known results on HSOLS of certain types are useful. We collect them in the following.

Theorem 1.2 ([2], [7], [8]) *For $h \geq 1$, there exists an HSOLS(h^n) if and only if $n \geq 4, (h,n) \neq (1,6)$.*

Theorem 1.3 ([9]) *There exists an HSOLS($3^n u^1$) if and only if $n \geq 4$ and $n \geq 1 + \frac{2u}{3}$, with seventeen possible exceptions $(n,u) = (5,1)$ and $(n,u) = (n, \frac{3n}{2} - 2)$ for $n \in \{6, 10, 14, 18, 22, 30, 34, 38, 42, 46, 54, 58, 62, 66, 70, 94\}$.*

In Section 4, we use the updated Theorem 1.1 to further improve Theorem 1.3. We solve twelve of the seventeen cases leaving five numbers between 6 and 22 undecided. To do this, two unipotent SOLSSOMs, SOLSSOM(66) and SOLSSOM(70) are useful; they are newly constructed in Section 3.

2 New ISOLS

The following ISOLS(14, 4) is found by an exhaustive search using a computer.

0	6	w	7	z	3	8	x	9	y	5	4	2	1
x	1	7	z	5	y	2	4	w	3	9	6	8	0
8	y	2	9	7	w	x	6	z	1	4	5	0	3
w	7	x	3	6	4	9	1	y	z	8	0	5	2
9	z	0	y	4	6	3	2	x	w	7	8	1	5
6	8	1	x	w	5	z	y	7	2	0	3	9	4
z	9	8	w	y	0	6	5	3	x	2	1	4	7
5	w	z	8	x	2	y	7	4	0	1	9	3	6
y	5	3	0	1	x	w	z	8	4	6	2	7	9
4	x	y	1	2	z	0	w	5	9	3	7	6	8
7	0	6	2	9	8	4	3	1	5				
1	2	9	4	3	7	5	8	0	6				
3	4	5	6	0	1	7	9	2	8				
2	3	4	5	8	9	1	0	6	7				

Lemma 2.1 *There exists an ISOLS(14, 4).*

We shall use this ISOLS and some recursive constructions to solve other ten cases listed unknown in Theorem 1.1. We describe some known constructions below (see [10]).

Construction 2.2 *Suppose a frame SOLS of type $t_1^{u_1} \cdots t_k^{u_k}$, and an IMOLS($m+a, a$) both exist. Then there exists an I-frame SOLS of type $\Pi_{1 \leq i \leq k}(t_i(m+a), t_i a)^{u_i}$.*

Construction 2.3 *Suppose there exists a frame SOLS of type $\{s_i : 1 \leq i \leq n\}$, and let $a \geq 0$ be an integer. For $1 \leq i \leq n-1$, suppose an ISOLS($s_i + a, a$) exists. Then there exists an ISOLS($s + a, s_n + a$), where $s = \sum_{1 \leq i \leq n} s_i$.*

Construction 2.4 *Suppose an I-frame SOLS of type $\{(s_i, t_i) : 1 \leq i \leq n\}$ exists, and let $a \geq 0$ be an integer. For $1 \leq i \leq n-1$, suppose there exists an ISOLS$(s_i + a; t_i, a)$. Also, suppose an ISOLS$(s_n + a, t_n)$ exists. Then there exists an ISOLS$(s + a, t)$, where $s = \sum s_i$ and $t = \sum t_i$.*

The following known result is also needed.

Theorem 2.5 ([4]) *There exists an IMOLS(v, n) for all values of v and n satisfying $v \geq 3n$, except for $(v, n) = (6, 1)$.*

Lemma 2.6 *There exists an ISOLS$(44, 14)$.*

Proof. Start with an HSOLS(10^4) which exists from Theorem 1.2. Applying Construction 2.3 with $a = 4$ gives an ISOLS$(44, 14)$. The required ISOLS$(14, 4)$ comes from Lemma 2.1. □

The remaining nine cases can be treated uniformly.

Lemma 2.7 *There exists an ISOLS$(3u + 2; u, 2)$ for $u = 3, 4$.*

Proof. An ISOLS$(11; 3, 2)$ is known, see [10, Table 4.1]. An ISOLS$(14; 4, 2)$ is shown below. The square is based on $\{0, 1, \cdots, 9, a, b, x, y\}$ with two holes based on $\{8, 9, a, b\}$ and $\{x, y\}$.

1	8	0	b	6	9	4	a	x	7	3	y	2	5
x	2	b	7	9	3	y	5	1	0	6	4	8	a
9	6	4	a	1	8	0	b	3	x	y	7	5	2
3	9	x	5	y	2	b	7	6	4	1	0	a	8
a	5	3	9	7	b	2	8	4	y	x	1	0	6
4	a	y	6	x	0	8	1	5	3	7	2	b	9
7	b	8	2	a	5	3	9	y	1	4	x	6	0
y	0	1	8	4	a	x	6	7	2	5	3	9	b
2	x	7	0	3	6	5	y					1	4
6	3	5	x	2	y	7	0					4	1
5	4	6	y	0	x	1	2					7	3
0	y	2	1	5	4	6	x					3	7
8	1	a	4	b	7	9	3	2	5	0	6		
b	7	9	3	8	1	a	4	0	6	2	5		

□

Lemma 2.8 *If there is an HSOLS$(3^b 4^c h^1)$ for $h = 1, 3, 4, 5$, then there exists an ISOLS$(3u + 2, u)$ for $u = 3b + 4c + h$.*

Proof. Start with an HSOLS$(3^b 4^c h^1)$ and apply Construction 2.2 with $m = 2$ and $a = 1$ to obtain an I-frame SOLS of type $(9, 3)^b (12, 4)^c (3h, h)^1$. The required IMOLS$(3, 1)$ comes from Theorem 2.5. Further apply Construction 2.4 with $a = 2$. The required ISOLS$(11; 3, 2)$ and ISOLS$(14; 4, 2)$ come from Lemma 2.7; also ISOLS$(3h+2, h)$ for $h = 1, 3, 4, 5$ come from Theorem 1.1 and Lemma 2.1. This gives an ISOLS$(3u+2, u)$ for $u = 3b + 4c + h$. □

Lemma 2.9 *There is an ISOLS$(3u+2, u)$ for $u \in \{16, 18, 20, 22, 26, 28, 32, 34, 46\}$.*

Proof. For each u, we have an HSOLS$(3^b 4^c h^1)$ such that $u = 3b+4c+h$. The HSOLS comes from either Theorem 1.2 or Theorem 1.3. The parameters are listed in Table 2.1. Then the conclusion follows from Lemma 2.8. □

u	b	c	h	HSOLS
16	0	3	4	HSOLS$(4^3 4^1)$
18	5	0	3	HSOLS$(3^5 3^1)$
20	0	4	4	HSOLS$(4^4 4^1)$
22	6	0	4	HSOLS$(3^6 4^1)$
26	7	0	5	HSOLS$(3^7 5^1)$
28	8	0	4	HSOLS$(3^8 4^1)$
32	0	7	4	HSOLS$(4^7 4^1)$
34	10	0	4	HSOLS$(3^{10} 4^1)$
46	15	0	1	HSOLS$(3^{15} 1^1)$

Table 2.1

We can now update Theorem 1.1 as follows.

Theorem 2.10 *There exists an ISOLS(v, n) for all values of v and n satisfying $v \geq 3n+1$, except for $(v, n) = (6, 1), (8, 2)$ and possibly for $v = 3n+2$, $n \in \{6, 8, 10\}$.*

3 New SOLSSOMs

It is known that a unipotent SOLSSOM of order n exists if and only if n is even and $n \geq 4$ with one exception of $n = 6$ and four possible exceptions of $n = 10, 14, 66, 70$ (see [3], [1]). In this section, we shall construct two new SOLSSOMs of orders 66 and 70, which are also unipotent. They are useful in the next section.

The following direct construction is based on difference methods, which is a modification of Lemma 2.1 in [8].

Lemma 3.1 *Let $G = Z_g$ with g even, and let X be any set disjoint from G, $|X| = h$ is even. Suppose there exists a set of 5-tuples $\mathcal{B} \subseteq (G \cup X)^5$ which satisfies the following properties:*

1. for each $i, 1 \leq i \leq 5$, and each $x \in X$, there is a unique $B \in \mathcal{B}$ with $b_i = x$ (b_i denotes the i-th co-ordinate of B);

2. no $B \in \mathcal{B}$ has two co-ordinates in X;

3. for each i, j $(1 \leq i < j \leq 5)$ and each $d \in G$, there is a unique $B \in \mathcal{B}$ with $b_i, b_j \in G$ and $b_i - b_j = d \pmod{g}$;

4. for $b_5 \in G$, $(b_1, b_2, b_3, b_4, b_5) \in \mathcal{B}$ if and only if $(b_2, b_1, b_4, b_3, b_5) \in \mathcal{B}$;

5. the differences $b_1 - b_2$, $b_3 - b_4 \pmod{g}$ are both odd if (b_1, b_2, b_3, b_4, x) and (b_2, b_1, b_4, b_3, y) are both in $\mathcal{B}$ for any $x, y \in X, x \neq y$;

6. $(0,0,0,0,x) \in \mathcal{B}$ *for some* $x \in X$ *and* $(0, \frac{g}{2}, a, a+\frac{g}{2}, y) \in \mathcal{B}$ *for some* $a \in G$ *and some* $y \in X$.
Then there exists an ISOLSSOM$(g+h, h)$*. If further a unipotent SOLSSOM*(h)*, exists, then so does a unipotent SOLSSOM*$(g+h)$*.*

Proof. From the first four properties, we can obtain an ISOLS$(g+h, h)$ with an orthogonal mate. In fact, by developing $\mathcal{B}$ through G and using the first two co-ordinates as row and column indices, the squares from the last three co-ordinates are orthogonal to each other. The first square has the second square as its transpose and therefore is an ISOLS.

The third square is almost symmetric except when cell (b_1, b_2) contains $x \in X$ while cell (b_2, b_1) contains $y \in X, x \neq y$. Since the difference $b_1 - b_2$ is odd, we can make the following adjustment to obtain a symmetric square: replace x by y for cells (b_1+t, b_2+t) when t is odd; and also replace y by x for cells (b_2+t, b_1+t) when t is even. Since the difference $b_3 - b_4$ is also odd, such adjustment will not damage the orthogonality between the first and the third squares.

If we construct a unipotent SOLSSOM(h) based on X and fill it in the size h hole, we obtain a SOLSSOM$(g+h)$. Since $\mathcal{B}$ contains $(0,0,0,0,x)$ for some $x \in X$, we may suppose that the third square of SOLSSOM(h) has constant diagonal x so that the third square of SOLSSOM$(g+h)$ also has constant diagonal x. □

In what follows, we take $h = 16$. As mentioned above, there exists a unipotent SOLSSOM(16).

Lemma 3.2 *There is a unipotent SOLSSOM*(66)*.*

Proof. Let $G = Z_{50}$ and let $X = \{x_1, \cdots, x_8, y_1, \cdots, y_8\}$. We use the above direct construction and list about half the members of $\mathcal{B}$ as follows. For $e \in G$, each column (a, b, c, d, e) generates another column (b, a, d, c, e). For $e = x_i, i = 2, \cdots, 8$, each column (a, b, c, d, x_i) generates another column (b, a, d, c, y_i). The last two columns do not generate another one. We thus obtain 82 columns forming the set $\mathcal{B}$. It is readily checked that the conditions in Lemma 3.1 are satisfied; hence a unipotent SOLSSOM(66) exists.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
39	47	37	32	43	36	19	14	11	28	34	9	46	2	4	13
33	39	25	17	45	46	6	5	20	1	49	4	28	16	12	37
36	48	44	38	27	0	22	13	32	21	23	40	34	18	2	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

31	23	6	12	20	30	40	7	48	1	3	18	16	17	10	41
35	29	26	24	44	33	27	38	25	22	45	8	0	15	42	5
11	9	10	7	43	8	31	24	3	41	35	14	29	42	47	26
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

0	0	0	0	0	0	0		21		0	0
17	7	5	11	9	15	1		49		0	25
33	19	27	18	22	12	44		30		0	26
46	18	20	3	17	39	11		19		0	1
x_2	x_3	x_4	x_5	x_6	x_7	x_8		0		x_1	y_1

□

Lemma 3.3 *There is a unipotent SOLSSOM*(70).

Proof. Let $G = Z_{54}$ and let X be as in the previous lemma. We list about half of the members of $\mathcal{B}$ similar to the previous lemma. We obtain 86 columns forming the set $\mathcal{B}$. The conditions in Lemma 3.1 are again satisfied; hence a unipotent SOLSSOM(70) exists.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
23	32	25	0	51	36	37	31	48	19	30	28	38	49	47	3
11	21	1	15	47	20	0	25	4	32	51	46	35	53	12	17
24	19	29	45	41	28	10	37	44	16	33	26	40	2	5	38
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

44	20	4	13	18	39	11	8	10	22	6	29	15	5	27	46
52	40	34	41	21	2	26	33	14	16	50	17	1	43	45	24
14	49	36	34	48	8	31	13	43	6	22	39	9	42	52	3
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

0	0	0	0	0	0	0		53	7	35		0	0
45	19	5	21	11	31	1		12	9	42		0	27
15	47	17	19	36	14	40		7	50	23		0	28
40	32	36	20	45	3	23		30	18	27		0	1
x_2	x_3	x_4	x_5	x_6	x_7	x_8		0	0	0		x_1	y_1

□

Regarding the existence of unipotent SOLSSOMs, we can now update the known results in the following.

Theorem 3.4 *A unipotent SOLSSOM*(n) *exists if and only if* $n \geq 4$ *is even, except for* $n = 6$ *and possibly for* $n = 10, 14$.

4 New HSOLS

In this section, we shall improve Theorem 1.3 by deleting twelve unknown cases. First we give an HSOLS($3^5 1^1$) below, which is found by general computer search. The HSOLS is based on $\{0, \cdots, 14, x\}$ with holes $\{x\}$ and $\{i, 5+i, 10+i\}$ for $0 \le i \le 4$. We replace 10, 11, 12, 13, 14 by a, b, c, d, e, respectively.

Lemma 4.1 *There exists an HSOLS*($3^5 1^1$).

We can treat the remaining eleven cases uniformly. A *transversal* of a holey Latin square of order n is a set of n cells in which the n entries are all distinct. For a transversal T, suppose $(i,j) \in T$ if and only if $(j,i) \in T$, then T is called *symmetric.* For a pair of transversals T and T', suppose $(i,j) \in T$ if and only if $(j,i) \in T'$, then the pair is called *symmetric.* We can now state a known construction, see [9, Construction 2.5].

	3	6	e	b		9	d	2	x		8	4	7	1	c
c		9	4	3	x		a	7	0	8		5	e	2	d
9	5		1	x	b	8		6	d	3	4		a	0	e
7	a	4		6	9	2	e		c	1	5	x		b	0
8	2	5	0		c	7	6	x		d	a	3	b		1
	d	3	c	8		4	x	e	b		9	1	6	7	2
x		e	a	d	7		9	c	5	2		0	4	8	3
6	e		b	5	8	3		0	a	9	x		1	d	4
4	9	a		1	6	0	b		7	x	2	e		c	5
3	7	b	x		2	d	1	a		c	0	8	5		6
	4	1	9	2		e	8	b	6		3	d	c	x	7
2		0	7	c	d		4	9	3	e		a	x	5	8
b	8		6	a	e	x		5	1	4	d		0	3	9
1	0	x		7	4	c	5		2	b	e	9		6	a
d	x	8	5		3	a	0	1		7	c	6	2		b
e	c	d	2	0	1	5	3	4	8	6	7	b	9	a	

Construction 4.2 *Suppose there is an HSOLS(t^n) which has p+$2q$ disjoint transversals, p of them being symmetric and the rest being q symmetric pairs. For $1 \le i \le p$ and $1 \le j \le q$, let $v_i \ge 0$ and $w_j \ge 0$ be integers. Let s and h be positive integers, where $sh \ne 2$ or 6 if $p + 2q < t(n-1)$. Suppose there exist HMOLS($s^h v_i^1$) for $1 \le i \le p$, HSOLS($s^h w_j^1$) for $1 \le j \le q$ and HSOLS($s^{tn} k^1$). Then there exists an HSOLS($(tsh)^n u^1$), where $u = k + \sum v_i + 2\sum w_j$.*

For a unipotent SOLSSOM, each element in the symmetric orthogonal mate determines a symmetric transversal in the SOLS. Especially, the main diagonal is one of such symmetrical transversals. Thus, we have the following.

Lemma 4.3 *If a unipotent SOLSSOM(n) exists, then an HSOLS(1^n) having $n-1$ disjoint symmetric transversals also exists.*

Lemma 4.4 *For even n, if there exist a unipotent SOLSSOM(n) and an ISOLS $(\frac{3n}{2}-1, \frac{n}{2}-1)$, then there exists an HSOLS $(3^n u^1)$, where $u = \frac{3n}{2} - 2$.*

Proof. From a unipotent SOLSSOM(n), we have by Lemma 4.3 an HSOLS(1^n) having $n-1$ disjoint symmetric transversals. Apply Construction 4.2 with $t = 1, p = n-1$ and $q = 0$. Let $s = 1, h = 3$ and $v_i = 1$ for $1 \leq i \leq p$. A unipotent SOLSSOM(4) leads to an HSOLS(1^4), also an HMOLS(1^4). For $k = \frac{n}{2} - 1$, an HSOLS($1^n k^1$) exists from the given ISOLS($n+k, k$). By Construction 4.2, we obtain an HSOLS($3^n u^1$), where $u = k + \sum v_i = \frac{3n}{2} - 2$. □

Lemma 4.5 *There exists an HSOLS $(3^n(\frac{3n}{2}-2)^1)$ for $n \in \{30, 34, 38, 42, 46, 54, 58, 62, 66, 70, 94\}$.*

Proof. For the given n, there exist a unipotent SOLSSOM(n) from Theorem 3.4 and an ISOLS $(\frac{3n}{2}-1, \frac{n}{2}-1)$ from Theorem 2.10. The conclusion follows from Lemma 4.4. □

We can now update Theorem 1.3 as follows.

Theorem 4.6 *There exists an HSOLS($3^n u^1$) if and only if $n \geq 4$ and $n \geq 1 + \frac{2u}{3}$, with five possible exceptions $(n, u) = (n, \frac{3n}{2} - 2)$ for $n \in \{6, 10, 14, 18, 22\}$.*

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