Winquist and the Atkin–Swinnerton–Dyer partition congruences for modulus 11

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Abstract

We derive the Atkin–Swinnerton–Dyer partition congruences for the modulus 11 by use of Winquist's identity.

1. Introduction

In 1969 Winquist [5] published a new identity and made one dramatic application of it — he gave a new and simple proof of Ramanujan's partition congruence for the modulus 11, namely

$$p(11n+6) \equiv 0 \pmod{11}.$$

($p(n)$ denotes the number of partitions of n.)
Proofs of this congruence had previously been given by Ramanujan [4] and Atkin and Swinnerton–Dyer [1]. For a more recent, completely different, elementary proof, see [3].

In their paper, Atkin and Swinnerton–Dyer proved the startling fact that for the three values of m, $m = 5,\ 7,\ 11$ and every value of r, $r = 0,\ 1,\ \cdots, m-1$ the generating function

$$\sum_{n\geq 0} p(mn+r)q^n$$

is congruent modulo m to a simple infinite product.

The object of this note is to make use of Winquist's identity to prove the full set of eleven Atkin–Swinnerton–Dyer congruences for the modulus 11.

If we employ the standard notation

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ \cdots,$$

$$(a_1, a_2, \ \cdots\ a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \ \cdots\ (a_n;q)_\infty,$$

$$(q)_\infty = (q;q)_\infty,$$

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we can write the eleven congruences we are going to prove as one, namely, modulo 11,

$$\begin{aligned}\sum_{n\geq 0} p(n)q^n \equiv\ & (q^{22},q^{33},q^{44},q^{55},q^{66},q^{77},q^{88},q^{99},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + q(q^{11},q^{44},q^{55},q^{55},q^{66},q^{66},q^{77},q^{110},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + 2q^2(q^{22},q^{33},q^{33},q^{55},q^{66},q^{88},q^{88},q^{99},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + 3q^3(q^{22},q^{22},q^{44},q^{55},q^{66},q^{77},q^{99},q^{99},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + 5q^4(q^{11},q^{33},q^{44},q^{55},q^{66},q^{77},q^{88},q^{110},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + 7q^5(q^{11},q^{33},q^{44},q^{44},q^{77},q^{77},q^{88},q^{110},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + 4q^7(q^{11},q^{22},q^{44},q^{55},q^{66},q^{77},q^{99},q^{110},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + 6q^{19}(q^{11},q^{11},q^{22},q^{33},q^{88},q^{99},q^{110},q^{110},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + 8q^9(q^{11},q^{22},q^{33},q^{55},q^{66},q^{88},q^{99},q^{110},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty \\ & + 9q^{10}(q^{11},q^{22},q^{33},q^{44},q^{77},q^{88},q^{99},q^{110},q^{121},q^{121};q^{121})_\infty/(q^{11})_\infty.\end{aligned} \tag{1}$$

(Note that there are no powers of q congruent to 6 modulo 11 on the right of (1), which proves Ramanujan's congruence.)

2. Proof of the congruence (1)

We shall require Winquist's identity, in the form [2,5]

$$\begin{aligned}&(a,a^{-1}q,b,b^{-1}q,ab,a^{-1}b^{-1}q,ab^{-1},a^{-1}bq,q,q;q)_\infty \\ &\quad = (a^3,a^{-3}q^3,q^3;q^3)_\infty\{(b^3q,b^{-3}q^2,q^3;q^3)_\infty - b(b^3q^2,b^{-3}q,q^3;q^3)_\infty\} \\ &\quad\quad - ab^{-1}(b^3,b^{-3}q^3,q^3;q^3)_\infty\{(a^3q,a^{-3}q^2,q^3;q^3)_\infty - a(a^3q^2,a^{-3}q,q^3;q^3)_\infty\}\end{aligned} \tag{2}$$

as well as the identity which can be deduced from (2) [2,5]

$$\begin{aligned}48(q)_\infty^{10} = \sum_{m,n=-\infty}^{\infty} (-1)^{m+n}\big((6m+3)^3(6n+1) \\ - (6m+3)(6n+1)^3\big)\, q^{(3m^2+3m+3n^2+n)/2}.\end{aligned} \tag{3}$$

Modulo 11, (3) becomes

$$\begin{aligned}(q)_\infty^{10} \equiv 3\Bigg(&\sum_{-\infty}^{\infty}(-1)^n(6n+3)^3q^{(3n^2+3n)/2}\sum_{-\infty}^{\infty}(-1)^n(6n+1)q^{(3n^2+n)/2} \\ &-\sum_{-\infty}^{\infty}(-1)^n(6n+3)q^{(3n^2+3n)/2}\sum_{-\infty}^{\infty}(-1)^n(6n+1)^3q^{(3n^2+n)/2}\Bigg).\end{aligned} \tag{4}$$

It is easy to show by considering $n \equiv -5, -4, \cdots, 5 \pmod{11}$ that

$$\sum_{-\infty}^{\infty}(-1)^n(6n+3)q^{(3n^2+3n)/2} \equiv -5P_3 + 4q^3P_9 - 3q^9P_{15} + 2q^{18}P_{21} - q^{30}P_{27}, \tag{5}$$

$$\sum_{-\infty}^{\infty}(-1)^n(6n+3)^3q^{(3n^2+3n)/2} \equiv -P_3 + 5q^3P_9 - 4q^9P_{15} + 2q^{18}P_{21} - 3q^{30}P_{27}, \tag{6}$$

$$\sum_{-\infty}^{\infty}(-1)^n(6n+1)q^{(3n^2+n)/2} \equiv P_1 + 5qP_5 + 4q^2P_7 + 2q^7P_{13} - 5q^{12}P_{17} + 3q^{15}P_{19} - q^{22}P_{23} + 3q^{26}P_{25} - 4q^{35}P_{29} + 2q^{40}P_{31} \tag{7}$$

and

$$\sum_{-\infty}^{\infty}(-1)^n(6n+1)^3q^{(3n^2+n)/2} \equiv P_1 + 4qP_5 - 2q^2P_7 - 3q^7P_{13} - 4q^{12}P_{17} + 5q^{15}P_{19} - q^{22}P_{23} + 5q^{26}P_{25} + 2q^{35}P_{29} - 3q^{40}P_{31} \tag{8}$$

where

$$P_k = \sum_{-\infty}^{\infty}(-1)^n q^{(363n^2+11kn)/2} = (q^{(363-11k)/2}, q^{(363+11k)/2}, q^{363}; q^{363})_\infty.$$

If we substitute (5), (6), (7) and (8) into (4) and rearrange, we obtain

$$\begin{aligned}(q)_\infty^{10} \equiv\ & (P_1P_3 - q^{22}P_3P_{23} - q^{33}P_{19}P_{21} - q^{44}P_{21}P_{25}) \\ & + q(P_3P_5 - q^{11}P_3P_{17} - q^{44}P_{19}P_{27} - q^{55}P_{25}P_{27}) \\ & + 2q^2(P_3P_7 - q^{22}P_{15}P_{19} - q^{33}P_{15}P_{25} - q^{33}P_3P_{29}) \\ & + 3q^3(P_1P_9 - q^{22}P_9P_{23} - q^{22}P_{13}P_{21} - q^{55}P_{21}P_{31}) \\ & + 5q^4(P_5P_9 - q^{11}P_9P_{17} - q^{33}P_{13}P_{27} - q^{66}P_{27}P_{31}) \\ & + 7q^5(P_7P_9 - q^{11}P_{13}P_{15} - q^{33}P_9P_{29} - q^{44}P_{15}P_{31}) \\ & + 4q^7(P_3P_{13} - q^{11}P_9P_{19} - q^{22}P_9P_{25} + q^{33}P_3P_{31}) \\ & + 6q^{19}(P_5P_{21} - q^{11}P_1P_{27} - q^{11}P_{17}P_{21} + q^{33}P_{23}P_{27}) \\ & + 8q^9(P_1P_{15} - q^{11}P_7P_{21} - q^{22}P_{15}P_{23} + q^{44}P_{21}P_{29}) \\ & + 9q^{10}(P_5P_{15} - q^{11}P_{15}P_{17} - q^{22}P_7P_{27} + q^{55}P_{27}P_{29}).\end{aligned} \tag{9}$$

Every bracketed term on the right of (9) can, remarkably, be expressed as a single product. For example, the first term is

$$
\begin{aligned}
&P_1P_3 - q^{22}P_3P_{23} - q^{33}P_{19}P_{21} - q^{44}P_{21}P_{25} \\
&\quad = P_3(P_1 - q^{22}P_{23}) - q^{33}P_{21}(P_{19} + q^{11}P_{25}) \\
&\quad = (q^{165}, q^{198}, q^{363}; q^{363})_\infty \\
&\qquad\qquad \times \left((q^{176}, q^{187}, q^{363}; q^{363})_\infty - q^{22}(q^{55}, q^{308}, q^{363}; q^{363})_\infty\right) \\
&\qquad - q^{33}(q^{66}, q^{297}, q^{363}; q^{363})_\infty \\
&\qquad\qquad \times \left((q^{77}, q^{286}, q^{363}; q^{363})_\infty + q^{11}(q^{44}, q^{319}, q^{363}; q^{363})_\infty\right) \\
&\quad = (q^{22}, q^{33}, q^{44}, q^{55}, q^{66}, q^{77}, q^{88}, q^{99}, q^{121}, q^{121}; q^{121})_\infty,
\end{aligned}
$$

where we have used (2) with q replaced by q^{121} and $a = q^{55}$, $b = q^{22}$.
For the remaining terms, the relevant substitutions in (2) (after replacing q by q^{121}) are respectively $(a,b) = (q^{55}, q^{11})$, (q^{55}, q^{33}), (q^{44}, q^{22}), (q^{44}, q^{11}), (q^{44}, q^{33}), (q^{55}, q^{44}), (q^{22}, q^{11}), (q^{33}, q^{22}), (q^{33}, q^{11}).

(1) then follows from the fact that

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q)_\infty} = \frac{(q)_\infty^{10}}{(q)_\infty^{11}} \equiv \frac{(q)_\infty^{10}}{(q^{11})_\infty}. \quad \blacksquare$$

References

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