Antimagic valuations of generalized Petersen graphs

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Abstract

A connected graph G is said to be (a,d)-antimagic, for some positive integers a and d, if its edges admit a labeling by the integers $1, 2, \ldots, |E(G)|$ such that the induced vertex labels consist of an arithmetic progression with the first term a and the common difference d. In this paper we prove that the generalized Petersen graph $P(n, 2)$ is $(\frac{3n+6}{2}, 3)$-antimagic for $n \equiv 0 \pmod 4$, $n \geq 8$.

1. Introduction and Definitions

The graphs considered here will be finite, undirected and simple. The vertex (edge) set of a graph G will be denoted by $V(G)$ $(E(G))$, respectively. The weight $w(v)$ of a vertex $v \in V(G)$ under an edge labeling f is the sum of values $f(e)$ assigned to all edges incident to a given vertex v.

A connected graph $G = (V(G), E(G))$ is said to be (a,d)-antimagic if there exist positive integers a, d and a bijection $f : E(G) \to \{1, 2, ..., |E(G)|\}$ such that the induced mapping $g_f : V(G) \to W$ is also a bijection, where

$$W = \{w(v) : v \in V(G)\} = \{a, a+d, a+2d, ..., a+(|V(G)|-1)d\}$$

is the set of the weights of vertices.

If $G = (V, E)$ is (a,d)-antimagic and $f : E(G) \to \{1, 2, ..., |E(G)|\}$ is a corresponding bijective mapping of G then f is said to be an (a,d)-antimagic labeling of G.

Hartsfield and Ringel [12] introduced the concept of an antimagic graph. An antimagic graph G is a graph whose edges can be labeled with the integers $1, 2, 3, \ldots,$ $|E(G)|$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices receive the same weight.

Hartsfield and Ringel conjecture that every tree other than K_2 is antimagic and, more strongly, that every connected graph other than K_2 is antimagic.

Bodendiek and Walther [6] defined the concept of an (a,d)-antimagic graph as a special case of an antimagic graph. They showed [7] that the theory of linear Diophantine equations and other concepts of number theory can be applied to determine the set of all connected (a,d)-antimagic graphs. For special graphs called parachutes, (a,d)-antimagic labelings are described in [8,9].

The generalized Petersen graphs $P(n,k)$, $1 \le k < \frac{n}{2}$, consist of an outer n-cycle $y_1, y_2, \dots, y_n$, a set of n spokes $y_i x_i$, $(1 \le i \le n)$, and n inner edges $x_i x_{i+k}$, $1 \le i \le n$, with indices taken modulo n. The standard Petersen graph is the instance $P(5,2)$. Generalized Petersen graphs were first defined by Watkins [15]. The classification of the Hamiltonicity of $P(n,k)$ was begun in [15], continued by Bondy [10] and Bannai [5], and completed by Alspach [1]. Schwenk [14] determined the precise number of Hamiltonian cycles in $P(n,2)$. Upper and lower bounds for the toughness of the $P(n,k)$ are established by Ferland in [11]. The irregularity strength of the $P(n,k)$ was determined by Jendroľ and Žoldák in [13]. Bodendiek and Walther [7] conjecture that $P(n,1)$, $n \equiv 0 \pmod 2$ is $(\frac{7n+4}{2},1)$-antimagic and $P(n,1)$, $n \equiv 1 \pmod 2$, is $(\frac{5n+5}{2},2)$-antimagic. In [3] are given the proofs of the conjectures of Bodendiek and Walther and it is shown that $P(n,1)$, $n \equiv 0 \pmod 2$, is $(\frac{3n+6}{2},3)$-antimagic. In [4] it is proved that the generalized Petersen graph $P(n,k)$ is $(a,1)$-antimagic if and only if n is even, $n \ge 4$, $k \le \frac{n}{2} - 1$ and $a = \frac{7n+4}{2}$.

A connected graph $G = (V(G), E(G))$ with p vertices and q edges is said to be (a,b)-consecutive, where b is a positive divisor of p with $t = \frac{p}{b} \ge 2$, if there exist an integer $a \ge 0$ and a bijection δ from $E(G)$ to $\{1, 2, \dots, q\}$ such that the induced mapping $\delta^* : V(G) \to W$ is also a bijection, where the set of weights of vertices W can be partitioned into t intervals

$$W_j = [w_{min} + (j-1)b + (j-1)a, w_{min} + jb + (j-1)a - 1],\ 1 \le j \le t \text{ and}$$

$$w_{min} = min\{\delta^*(v) : v \in V(G)\}.$$

In [2] is shown that (i) if $n \ge 3$ and $1 \le k < \frac{n}{2}$, then the generalized Petersen graph $P(n,k)$ is (n,n)-consecutive and $(3n,n)$-consecutive and (ii) if n is even, $n \ge 4$, $1 \le k \le \frac{n}{2} - 1$, then $P(n,k)$ is $(2n,n)$-consecutive.

Now, we will concentrate on the (a,d)-antimagicness of $P(n,k)$ and we show that if $n \equiv 0 \pmod 4$ then $P(n,2)$ is $(\frac{3n+6}{2},3)$-antimagic.

2. Necessary conditions

Assume that $P(n,k)$ is (a,d)-antimagic on $|V(P(n,k))| = 2n$ vertices and $|E(P(n,k))| = 3n$ edges. Let $\rho : E(P(n,k)) \to \{1,2,3, \dots, 3n\}$ be edge labeling and $W = \{w(v) : v \in V(P(n,k))\} = \{a, a+d, \dots, a+(2n-1)d\}$ be the set of weights of vertices.

$$\sum_{e\in E(P(n,k))} \rho(e) = \frac{3n(3n+1)}{2} ,$$

$$\sum_{v\in V(P(n,k))} w(v) = 2na + nd(2n-1) .$$

Clearly, the following equations (1), (2) hold

$$(1) \qquad 2 \sum_{e\in E(P(n,k))} \rho(e) = \sum_{v\in V(P(n,k))} w(v) ,$$

$$(2) \qquad 3n(3n+1) = 2na \; + \; nd(2n-1).$$

From the linear Diophantine equation (2) we have

$$d \;=\; \frac{3(3n+1)-2a}{2n-1} .$$

The minimal value of weight which can be assigned to a vertex of degree three is $a = 6$. Thus we get the upper bound on the value d, i.e., $0 < d < \frac{9}{2}$. This implies that:

(3) if $n \equiv 0 \pmod 2$, then d is necessarily odd and the equation (2) has exactly the two different solutions $(a,d) = \left(\frac{7n+4}{2}, 1\right)$ or $(a,d) = \left(\frac{3n}{2}+3, 3\right)$, respectively and

(4) if $n \equiv 1 \pmod 2$, then d is necessarily even and the equation (2) has exactly the two different solutions $(a,d) = \left(\frac{5n+5}{2}, 2\right)$ or $(a,d) = \left(\frac{n+7}{2}, 4\right)$, respectively.

3. Main result

Theorem 1. *For $n \geq 8$, $n \equiv 0 \pmod 4$, the generalized Petersen graph $P(n,2)$ has a $\left(\frac{3n}{2}+3, 3\right)$-antimagic labeling.*

Proof. Define the edge labeling f of $P(n,2)$, $n \equiv 0 \pmod 4$, as follows:

$$f(x_iy_i) = \begin{cases} \frac{6n+1-i}{2} & \text{if} \quad 1 \leq i \leq n-1 \quad \text{is odd and} \quad i \neq 3, \\ = \frac{n}{2}+1 & \text{if} \quad i = 2, \\ = \frac{5n}{2}-1 & \text{if} \quad i = 3, \\ = \frac{3n+4-i}{2} & \text{if} \quad 4 \leq i \leq n \quad \text{is even.} \end{cases}$$

$$f(y_iy_{i+1}) = \begin{cases} \frac{2n+3-i}{2} & \text{if} \quad 3 \le i \le n-1 \quad \text{is odd} \ , \\ = 1 & \text{if} \quad i = 1, \\ = n+1 & \text{if} \quad i = 2, \\ = \frac{n+4-i}{2} & \text{if} \quad 4 \le i \le n \quad \text{is even.} \end{cases}$$

$$f(x_ix_{i+2}) = \begin{cases} 3n-1 & \text{if} \quad i = 1, \\ = \frac{4n+1-i}{2} & \text{if} \quad i \equiv 3 \pmod 4, \quad i \ge 3, \\ = \frac{5n-i-1}{2} & \text{if} \quad i \equiv 1 \pmod 4, \quad i \ge 5, \\ = \frac{3n+2i}{2} & \text{if} \quad i \equiv 0 \pmod 2. \end{cases}$$

It is easy to verify that the labeling f uses each integer $1, 2, \ldots, 3n$ exactly once and this implies that the labeling f is a bijection from the edge set $E(P(n,2))$ to the set $\{1, 2, \ldots, 3n\}$.

Let us denote the weights (under an edge labeling f) of vertices x_i and y_i of $P(n,2)$ by

$w(y_i) = f(y_iy_{i+1}) + f(y_{i-1}y_i) + f(x_iy_i)$ for $1 \le i \le n$,

$w(x_i) = f(x_ix_{i+2}) + f(x_iy_i) + f(x_{n+i-2}x_i)$ for $1 \le i \le n$

with indices taken modulo n.

The weights of vertices of $P(n,2)$ under the edge labeling f constitute the sets

$W_1 = \{w(y_i) : 1 \le i \le n\} = \left\{\frac{3n}{2} + 3i : 1 \le i \le n\right\}$ and

$W_2 = \{w(x_i) : 1 \le i \le n\} = \left\{\frac{9n}{2} + 3i : 1 \le i \le n\right\}$.

We can see that each vertex of $P(n,2)$ receives exactly one label of weight from $W_1 \cup W_2$ and each number from $W_1 \cup W_2$ is used exactly once as a label of weight and further that the set

$W = W_1 \cup W_2 = \{a, a+d, a+2d, \ldots, a+(2n-1)d\}$, where $a = \frac{3n}{2} + 3$ and $d = 3$ and finally that the induced mapping $g_f : V(P(n,2)) \to W$ is bijective. This completes the proof of the theorem.

4. Open problems

In view of Theorem 1 and the result [4] that $P(n,k)$ is $((7n+4)/2, 1)$-antimagic if $n \ge 4$ is even and $k \le n/2 - 1$, we conjecture that the generalized Petersen graph $P(n,k)$ is (a,d)-antimagic for all feasible values of a and d. More specifically, we put forward the following three conjectures.

Conjecture 2. *If n is even, $n \ge 6$ and $2 \le k \le \frac{n}{2} - 1$, then the generalized Petersen graph $P(n,k)$ is $\left(\frac{3n}{2} + 3, 3\right)$-antimagic.*

Conjecture 3. *If n is odd, $n \geq 5$ and $2 \leq k \leq \frac{n-1}{2}$, then the generalized Petersen graph $P(n,k)$ is $\left(\frac{5n+5}{2}, 2\right)$-antimagic.*

Conjecture 4. *For $n \equiv 1 \pmod 2$, $n \geq 7$ and $1 \leq k \leq \frac{n-1}{2}$, the generalized Petersen graph $P(n,k)$ has a $\left(\frac{n+7}{2}, 4\right)$-antimagic labeling.*

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