Quasi-quadrics and related structures

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Abstract

In a projective space $PG(n, q)$ a *quasi-quadric* is a set of points that has the same intersection numbers with respect to hyperplanes as a non-degenerate quadric in that space. Of course, non-degenerate quadrics themselves are examples of quasi-quadrics, but many other examples exist. In the case that n is odd, quasi-quadrics have two sizes of intersections with hyperplanes and so are *two-character sets*. These sets are known to give rise to strongly regular graphs, two-weight codes, difference sets, SDP-designs, Reed-Muller codes and bent functions. When n is even, quasi-quadrics have three sizes of intersection with respect to hyperplanes. Certain of these may be used to construct antipodal distance regular covers of complete graphs. The aim of this paper is to draw together many of the known results about quasi-quadrics, as well as to provide some new geometric construction methods and theorems.

1 Introduction and preliminaries

In this section we briefly recall some definitions from graph theory, coding theory and design theory that will be used in later sections.

A *graph* is a pair $\Gamma = (V, E)$ where V is a non-empty set of *vertices* and E is a collection of 2-subsets of V, called *edges*. A *path of length i* joining two vertices $\gamma, \delta \in \Gamma$ is a sequence $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_i = \delta$ of vertices such that $\{\gamma_j, \gamma_{j+1}\}$ is an edge for $j = 0, 1, \ldots, i - 1$. We will only be concerned with *connected graphs*, that is, graphs in which each pair of vertices is joined by a path. The *distance* $d(\gamma, \delta)$ of two vertices γ, δ is the length of a shortest path joining them, and the *diameter* d of Γ is the maximal distance occurring between two vertices in Γ. For $i = 1, 2, \ldots, d$, the *distance-i graph* Γ_i is the graph with vertex set V and with edges the pairs of vertices which are at distance i in Γ.

The graph Γ is *antipodal* of diameter $d > 1$ if the distance-d graph Γ_d is a disjoint union of cliques. In this case, we define a new graph $\overline{\Gamma}$ whose vertices are the maximal cliques of Γ_d, and two vertices are adjacent if their union contains an edge of Γ. If each vertex $\gamma \in \Gamma$ has the same valency as the vertex of $\overline{\Gamma}$ which is the maximal clique containing γ, then Γ is called an *antipodal covering graph* of $\overline{\Gamma}$. If, in addition, all maximal cliques of Γ_d have the same size r then Γ is an *antipodal r-cover* of $\overline{\Gamma}$.

The graph Γ is *distance regular* if there are integers b_{i-1}, c_i (for $i = 1, 2, \ldots, d$) such that for any two vertices $\gamma, \delta \in \Gamma$ at distance $i = d(\gamma, \delta)$, there are precisely c_i neighbours of δ at distance $i - 1$ from γ, and precisely b_i neighbours of δ at distance $i + 1$ from γ. In particular Γ is regular of valency k, and its *intersection array* is the sequence $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$.

The graph Γ is *strongly regular* with parameters $(|V|, k, \lambda, \mu)$ if it is regular with valency k and if the number of vertices joined to two given adjacent vertices is λ and the number of vertices joined to two given non-adjacent vertices is μ, i.e. it is a distance regular graph with intersection array $\{k, k - 1 - \lambda; 1, \mu\}$.

For a positive integer e, a *perfect e-code* in a graph Γ is a non-empty subset $\mathcal{C}$ of the vertex set with the property that any vertex lies at distance at most e from a unique vertex in $\mathcal{C}$.

Let X be a finite set. An *association scheme with d classes* ([3, 2.1]) is a pair $(X, \mathcal{R})$ such that:
(i) $\mathcal{R} = (R_0, R_1, \ldots, R_d)$ is a partition of $X \times X$;
(ii) $R_0 = \{(x, x) : x \in X\}$;
(iii) if $(x, y) \in R_i$ then $(y, x) \in R_i$ for $i = 0, 1, \ldots, d$;
(iv) there are integers p_{ij}^k such that for any pair $(x, y) \in R_k$ the number of $z \in X$ such that $(x, z) \in R_i$ and $(y, z) \in R_j$ equals p_{ij}^k.

An *$[n, k]$-code* $\mathcal{C}$ over $\mathrm{GF}(q)$ is a k-dimensional subspace of $\mathrm{GF}(q)^n$. The *weight* $wt(x)$ of a vector $x \in \mathrm{GF}(q)^n$ is the number of its non-zero entries, and $\mathcal{C}$ is a *two-weight* code if $|\{i : i \neq 0 \text{ and there exists a vector } x \in \mathcal{C} \text{ with } wt(x) = i\}| = 2$.

Let V be an m-dimensional vector space over $\mathrm{GF}(2)$. The *First-order Reed-Muller code* $\mathcal{R}(1, m)$ is the subspace of the vector space of functions $V \to \mathrm{GF}(2)$

that consists of all polynomial functions in monomials x_i, that is,

$$\mathcal{R}(1, m) = \langle x_i \colon i = 1, \ldots, m \rangle .$$

Let G be an additively written group of order v. A k-subset D of G is a $(v, k, \lambda; n)$ *difference set of order* $n = k - \lambda$ if every non-zero element of G has exactly λ representations as a difference $d - d'$ for $d, d' \in D$. The difference set is called *elementary Abelian* if G is elementary Abelian.

Let Λ be a proper set of non-zero vectors in a vector space V over $\mathrm{GF}(q)$. Then Λ is a $\{\lambda_1, \lambda_2\}$-difference set over $\mathrm{GF}(q)$ if $\mathrm{GF}(q)^*\Lambda = \Lambda$ and if for $v \in V \setminus \{0\}$, the set $\{(x, y) : x, y \in \Lambda \text{ and } x - y = v\}$ has cardinality λ_1 if $v \in \Lambda$ and λ_2 if $v \notin \Lambda \setminus \{0\}$.

A symmetric (v, k, λ)-design satisfies the *symmetric difference property* (see [11]) if for every three blocks B, C, D then $B \triangle C \triangle D$ (where $X \triangle Y = (X \cup Y) \setminus (X \cap Y)$ is the symmetric difference of X and Y) is either a block or the complement of a block. Such a design is called an *SDP-design*.

Let V be an m-dimensional vector space over $\mathrm{GF}(2)$ and let $P(x)$ be a function from V to $\mathrm{GF}(2)$. Then $P(x)$ is called a *bent function* if the Fourier coefficients of $(-1)^{P(x)}$ are all 1 [14].

2 A construction method

In this section we begin by outlining a general construction method in projective geometries. Various instances of the method are then examined. In subsection 2.1 it is shown how the construction method gives rise to antipodal distance regular covers of complete graphs. Theorems are also proved about feasible parameters for the sets used to construct the covers, and an example is given that has "non-standard" parameters. In subsection 2.2 sets in projective spaces with two intersection numbers are examined. A very brief survey of results connecting these sets to strongly regular graphs, two-weight codes, difference sets, SDP-designs, Reed-Muller codes and bent functions is given.

Let $\mathcal{K}$ be a set of points in $\mathrm{PG}(n, q)$, and embed $\mathrm{PG}(n, q)$ as a hyperplane Σ_∞ in $\mathrm{PG}(n+1, q)$. Define a graph $\Gamma(\mathcal{K})$ as follows: the *vertices* are the points of the affine space $\mathrm{AG}(n+1, q) = \mathrm{PG}(n+1, q) \setminus \Sigma_\infty$ and two vertices are *adjacent* if the line of $\mathrm{PG}(n+1, q)$ joining them meets Σ_∞ in a point of $\mathcal{K}$. In fact it is true in general that $\Gamma(\mathcal{K})$ is the point graph of the geometry that is a *linear representation* of $\mathcal{K}$, where the geometry is constructed as follows: the points of the geometry are the points of the affine space $\mathrm{AG}(n+1, q) = \mathrm{PG}(n+1, q) \setminus \Sigma_\infty$ and the lines of the geometry are the lines of $\mathrm{PG}(n+1, q)$ not in Σ_∞ and which meet Σ_∞ in a point of $\mathcal{K}$ (see [5]).

Example 1 (Thas [16], see also [3, Section 12.5.1])
Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in $\mathrm{PG}(n, q)$ or an oval in $\mathrm{PG}(2, q)$, where n and q are both even. Then $\Gamma(\mathcal{Q})$ is an antipodal distance-regular graph with intersection array $\{q^n - 1, q^n - q^{n-1}, 1; 1, q^{n-1}, q^n - 1\}$ and diameter 3. Further, a subset C of the vertex set of $\Gamma(\mathcal{Q})$ is a perfect 1-code if and only if C plus the nucleus form a line of $\mathrm{PG}(n+1, q)$.

Example 2 (Ahrens and Szekeres [1], see also [13, Section 3.1.3])
Let $\mathcal{H}$ be a hyperoval in $\mathrm{PG}(2,q)$, where q is even. Then $\Gamma(\mathcal{H})$ (often denoted $\Gamma(T_2^*(\mathcal{H}))$) is the point graph of the *Ahrens-Szekeres* generalized quadrangle $T_2^*(\mathcal{H})$, of order $(q-1,q+1)$, the points of the generalized quadrangle being the points of $\mathrm{AG}(3,q) = \mathrm{PG}(3,q) \setminus \mathrm{PG}(2,q)$, and the lines of the generalized quadrangle being the lines of $\mathrm{PG}(3,q)$ not contained in $\mathrm{PG}(2,q)$ and meeting $\mathrm{PG}(2,q)$ in a point of $\mathcal{H}$.

2.1 Equitable partition case

In this section we introduce a general setting which includes example 1.

Let $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_d)$ be a partition of the set of points of $\mathrm{PG}(n,q)$. Then Ω induces an equivalence relation on the set of hyperplanes of $\mathrm{PG}(n,q)$, in which hyperplanes H_1, H_2 are *equivalent* if $|H_1 \cap \Omega_i| = |H_2 \cap \Omega_i|$ for $i = 1, 2, \ldots, d$. Let $\Omega^\wedge$ be the partition induced by this equivalence relation on the set of hyperplanes of $\mathrm{PG}(n,q)$. It is known that $\Omega^\wedge$ has at least as many parts as Ω (by [2]), and we say that Ω is *equitable* if $\Omega^\wedge$ has the same number of parts as Ω.

We remark that Bridges and Mena proved that equitable partitions also give rise to association schemes.

Theorem 1 ([2])
Let Σ_∞ be a hyperplane in $\mathrm{PG}(n+1,q)$ and let $\Omega = (\Omega_1, \ldots, \Omega_d)$ be a partition of Σ_∞. Then Ω is an equitable partition if and only if the graphs $\Gamma(\Omega_1), \ldots, \Gamma(\Omega_d)$ form an association scheme with d classes.

As in example 1, let $\mathcal{Q}$ be a non-singular (parabolic) quadric in $\mathrm{PG}(n,q)$ or an oval in $\mathrm{PG}(2,q)$, where n and q are both even, and let N be the nucleus of $\mathcal{Q}$. Since every hyperplane on N meets $\mathcal{Q}$ in the same number of points, and hyperplanes not on N meet $\mathcal{Q}$ in one of two possible numbers of points, it is straightforward to verify that the partition $(\mathcal{Q}, \mathrm{PG}(n,q) \setminus (\mathcal{Q} \cup \{N\}), N)$ is equitable. In fact this equitable partition has extra properties, and Godsil [8] has shown that such equitable partitions always give rise to antipodal distance-regular covers of complete graphs, as follows.

Theorem 2 ([8, Theorem 6.1])
Let Σ_∞ be a hyperplane in $\mathrm{PG}(n+1,q)$ and let $(\Omega_1, \Omega_2, \Omega_3)$ be an equitable partition of Σ_∞. Then $\Gamma(\Omega_1)$ is distance-regular with q^{n+1} vertices and is an antipodal q^{t+1}-cover of a complete graph on q^{n-t} vertices, if and only if:

(1) Ω_1 is not a subspace of Σ_∞;

(2) Ω_3 is a subspace of dimension t;

(3) each $(t+1)$-dimensional subspace of Σ_∞ containing Ω_3 meets Ω_1 in exactly one point.

In view of theorems 1 and 2, we wish to investigate equitable partitions of projective spaces which satisfy the conditions of theorem 2. Without loss of generality, we can fix in $\mathrm{PG}(n,q)$ a subspace N of dimension t and say that a disjoint set $\mathcal{K}$ of points is a *pseudo-complement* for N if:

(i) each $(t+1)$-dimensional subspace of $\mathrm{PG}(n,q)$ containing N meets $\mathcal{K}$ in exactly one point;

(ii) each hyperplane not containing N meets $\mathcal{K}$ in either a or b points, where $0 \le a < b$.

The set $\mathcal{K}$ will play the role of Ω_1 and N will play the role of Ω_3. It is straightforward to verify that if $\mathcal{K}$ is not a subspace then the partition $(\mathcal{K}, \mathrm{PG}(n,q) \setminus (\mathcal{K} \cup \{N\}), N)$ is equitable. We do not include the restriction that $\mathcal{K}$ not be a subspace, for it is more convenient in the analysis and these examples can just be discarded later.

We remark that $|\mathcal{K}| = (q^{n-t} - 1)/(q-1)$.

Example 3

Let $n = 2m$ and q be even. A *parabolic quasi-quadric* with nucleus the point N in $\mathrm{PG}(n,q)$ is a set $\mathcal{K}$ of $(q^n - 1)/(q-1)$ points such that each line on N contains a unique point of $\mathcal{K}$ and each hyperplane not on N meets $\mathcal{K}$ in either $(q^m + 1)(q^{m-1} - 1)/(q-1)$ or $(q^m - 1)(q^{m-1} + 1)/(q-1)$ points. A parabolic quasi-quadric with nucleus N is a pseudo-complement for N. An example of a parabolic quasi-quadric is a non-singular parabolic quadric $\mathcal{Q}$ with nucleus N, since each hyperplane not on N meets $\mathcal{Q}$ in a non-singular elliptic or hyperbolic quadric. Further examples will be constructed in section 3.1.

We now collect results and further study the pseudo-complements for subspaces in $\mathrm{PG}(n,q)$.

Theorem 3 ([8, 7.1, Corollary])

If q is odd then any pseudo-complement for a t-dimensional subspace N of $\mathrm{PG}(n,q)$ is an $(n - t - 1)$-dimensional subspace not through N.

Theorem 4

Let N be an $(n-2)$-dimensional subspace of $\mathrm{PG}(n,q), n \ge 2$, and let $\mathcal{K}$ be a pseudo-complement for N. Then either

(i) $\mathcal{K}$ is a line; or

(ii) $n = 2$, q is even and $\mathcal{K}$ is an oval with nucleus N.

Proof. First, if $a > 0$ then $\mathcal{K}$ is a set of $q + 1$ points which is met by every hyperplane; so $\mathcal{K}$ is a line [9, Section 3.5]. Next, suppose $a = 0$ and let t_b denote the number of hyperplanes not on N meeting $\mathcal{K}$ in b points. It is straightforward to verify that

$$bt_b = (q+1)\left(\frac{q^n - 1}{q-1} - 1\right) \quad \text{and} \quad b(b-1)t_b = (q+1)q\left(\frac{q^{n-1} - 1}{q-1}\right);$$

so $b = 2$ and $t_a \ne 0$. If $n = 2$ then $\mathcal{K}$ is an oval with nucleus the point N and q is even. Otherwise, suppose $n \ge 3$. Let Σ be a hyperplane not on N and not meeting

$\mathcal{K}$, and let Σ_{n-2} be a hyperplane of Σ meeting N in the $(n-3)$-dimensional subspace $\Sigma \cap N$. One hyperplane on Σ_{n-2} contains N and hence meets $\mathcal{K}$ in one point, and we suppose m hyperplanes on Σ_{n-2} meet $\mathcal{K}$ in two points. Then $1 + 2m = q + 1$; so q is even. Now let Σ_{n-2} be an $(n-2)$-dimensional subspace of $\mathrm{PG}(n,q)$ meeting N in an $(n-4)$-dimensional subspace, and let $|\Sigma_{n-2} \cap \mathcal{K}| = m$. Since no hyperplane on Σ_{n-2} meets $\mathcal{K}$ in one point, if $m = 0$ then 2 divides $|\mathcal{K}| = q + 1$, a contradiction. If $m \geq 1$ then $m + (q+1)(2-m) = q+1$, so $m = (q+1)/q$; also a contradiction. □

Theorem 5
Let N be an $(n-3)$-dimensional subspace of $\mathrm{PG}(n,q), n \geq 3$, and let $\mathcal{K}$ be a pseudo-complement for N. Then either:

(a) $\mathcal{K}$ is a plane not meeting N;

(b) $q = 2$, $n = 3$ and $\mathcal{K} \cup \{N\}$ is the complement for a plane in $\mathrm{PG}(3,2)$;

(c) q is an even square, $a = q - \sqrt{q} + 1$ and $b = q + \sqrt{q} + 1$.

Proof. For ease of notation we write $\tau_d = (q^d - 1)/(q - 1)$, the number of points in a $(d-1)$-dimensional subspace of $\mathrm{PG}(n,q)$. In particular, $\tau_d = q\tau_{d-1} + 1$. Let $\mathcal{K}$ be a pseudo-complement for N, and let t_a, t_b denote the numbers of hyperplanes not on N meeting $\mathcal{K}$ in a or b points, respectively. It easily follows from the definitions that

$$t_a + t_b = \tau_{n+1} - \tau_3 \tag{1}$$

$$at_a + bt_b = \tau_3(\tau_n - \tau_2) \tag{2}$$

$$a(a-1)t_a + b(b-1)t_b = \tau_3(\tau_3 - 1)(\tau_{n-1} - 1). \tag{3}$$

Thus,

$$t_a = \frac{b(\tau_{n+1} - \tau_3) - \tau_3(\tau_n - \tau_2)}{(b-a)} \quad \text{and} \quad t_b = \frac{\tau_3(\tau_n - \tau_2) - a(\tau_{n+1} - \tau_3)}{b-a}.$$

Since $t_a \geq 0$, it follows that $b \geq q + 2$. Similarly, $t_b \geq 0$ implies that $a \leq q + 1$. Further, neither of t_a, t_b can be zero (else, for example, $t_b = 0$ implies $a = \tau_3/q$, which is impossible).

If we take $ab(1) + (1 - a - b)(2) + (3)$ we obtain:

$$0 = ab(\tau_{n+1} - \tau_3) + (1 - a - b)\tau_3(\tau_n - \tau_2) + \tau_3(\tau_3 - 1)(\tau_{n-1} - 1)$$
$$0 = abq - (a + b)(q^2 + q + 1) + (q^2 + q + 1)(q + 2). \tag{4}$$

Now either $a + b = q + 2$ or $q^2 + q + 1$ divides ab (as the greatest common divisor $(q^2 + q + 1, q) = 1$). However if $a + b = q + 2$, then $ab = 0$ and $q^2 + q + 1$ divides ab trivially. Thus we can suppose that $q^2 + q + 1$ divides ab, and we let $r \geq 0$ be the integer such that $ab = r(q^2 + q + 1)$.

On substituting $ab = r(q^2 + q + 1)$ into equation (4), we obtain $a + b = rq + q + 2$. It follows that a and b are the roots of the quadratic polynomial equation

$$x^2 - (rq + q + 2)x + r(q^2 + q + 1) = 0 \tag{5}$$

and so the discriminant of this polynomial must be a square. The discriminant is

$$(rq + q + 2)^2 - 4r(q^2 + q + 1) = q^2 (r - 1)^2 + 4 (q + 1 - r).$$

Now $b \leq q^2 + q + 1$ and $a \leq q + 1$, so $ab = r(q^2 + q + 1) \leq (q + 1)(q^2 + q + 1)$. Hence $r \leq q + 1$. If $r = q + 1$, then the discriminant is a square. Assume $r \leq q$, then $q^2 (r - 1)^2 + 4 (q + 1 - r)$ is a square and must be at least $(q(r - 1) + 1)^2$. If equality holds then it is easy to show that r is not an integer. Hence

$$q^2(r - 1)^2 + 4(q + 1 - r) \geq (q(r - 1) + 2))^2$$
$$\Rightarrow 4(q + 1 - r) \geq 4q(r - 1) + 4$$
$$\Rightarrow r \leq 2q/(q + 1)$$
$$\Rightarrow r \leq 1.$$

Hence r is either 0, 1 or $q + 1$. We examine each of these cases.

(a) $r = q + 1$. The solutions to (5), assuming $a < b$, are then $a = q + 1$ and $b = q^2 + q + 1$. Every hyperplane of $\mathrm{PG}(n, q)$ (including those containing N) then meets $\mathcal{K}$ in $q + 1$ or $q^2 + q + 1$ points. If follows (see for example [9, Section 3.3.7]) that $\mathcal{K}$ is the set of points of a plane.

(b) $r = 0$. Then $a = 0$ and $b = q + 2$. Let Σ be a hyperplane not containing N and not meeting $\mathcal{K}$. Then Σ meets N in a subspace Σ_{n-4} of dimension $n - 4$ (for $n = 3$ this corresponds to Σ being disjoint from N). Choose a subspace Σ_{n-2} of Σ containing Σ_{n-4}. We count the points of $\mathcal{K}$ on hyperplanes containing Σ_{n-2}. The hyperplane given by the span $\langle N, \Sigma_{n-2} \rangle$ meets $\mathcal{K}$ in $q + 1$ points. There are q other hyperplanes on Σ_{n-2} each meeting $\mathcal{K}$ in 0 or $q + 2$ points. Suppose m of them meet it in $q + 2$ points, then $(q + 1) + m(q + 2) = q^2 + q + 1$, giving $m = q^2/(q + 2)$. Hence $q = 2$ and $m = 1$.

When $n = 3$, we are in $\mathrm{PG}(3, 2)$ and Σ_{n-2} is a line ℓ disjoint from the point N. There are exactly three planes on ℓ. One is given by the span of N and ℓ. It meets $\mathcal{K}$ in three points not on ℓ, that is, exactly the three points of the plane not equal to N and not on ℓ. Another plane meets $\mathcal{K}$ in the four points not on ℓ. The last plane is disjoint from $\mathcal{K}$. It follows that $\mathcal{K}$ is the complement for a plane not on N.

For $n > 3$, consider a subspace Σ_{n-2} meeting N in a subspace of dimension $n - 5$ (for $n = 4$ it is disjoint). Note that the span of Σ_{n-2} and N is the whole space, and so no hyperplane of $\mathrm{PG}(n, 2)$ on Σ_{n-2} contains N. Hence each of the three hyperplanes on Σ_{n-2} meets $\mathcal{K}$ in either 0 or 4 points. Let m be the size of the intersection of Σ_{n-2} with $\mathcal{K}$. If $m = 0$ we obtain a contradiction since 4 does not divide $|\mathcal{K}|$. If $m \neq 0$ each of the three hyperplanes meets $\mathcal{K}$ in $4 - m$ points, and we obtain $m + 3(4 - m) = 7$, i.e. $2m = 5$. Hence such sets do not exist for $n > 3$.

(c) $r = 1$. The solutions to (5) are $a = q - \sqrt{q} + 1$ and $b = q + \sqrt{q} + 1$. By theorem 3, q must be even. □

The next example shows that the possibility in (c) can occur.

Example 4

Consider the action of a cyclic subgroup G of order 7 of $\mathrm{PGL}(4, 4)$ (such a group is unique up to conjugacy). Let N be the unique point fixed by G and let Σ be the

unique plane fixed by G. The point orbits of G in Σ are three Fano planes which we denote π_1, π_2, π_3. For $i = 1, 2$ and 3, G also fixes the cone with vertex N and base π_i; the point orbits of G on the cone are N, π_i and three 7-caps $C_{i,j}$ for $j = 1, 2, 3$. Without loss of generality, let $\mathcal{K} = \pi_1 \cup C_{2,j} \cup C_{3,k}$ for some $j, k \in \{1, 2, 3\}$. Then $\mathcal{K}$ is a pseudo-complement for N with $a = 3$ and $b = 7$; in fact all the pseudo-complements $\mathcal{K}$ for N constructed in this way are equivalent under the action of $\mathrm{PGL}(4, 4)$.

Theorem 6
Let $\mathcal{K}$ be a pseudo-complement for a point N in $\mathrm{PG}(n, q)$, with hyperplane intersection sizes equal to a or b. Then $ab = r\tau_n$ where either $r = \tau_{n-1}$ or $r \leq 2(\tau_{n-1} - 1)/(q + 1)$, with $\tau_d = (q^d - 1)/(q - 1)$.

Proof. The arguments are analogous to those used in the proof of theorem 5. With the same notation, the basic equations are:

$$t_a + t_b = q^n \tag{6}$$
$$at_a + bt_b = q^{n-1}\tau_n \tag{7}$$
$$a(a-1)t_a + b(b-1)t_b = q^{n-2}\tau_n(\tau_n - 1). \tag{8}$$

Later in the argument we find that $q^2 \left(r - \tau_{n-2}\right)^2 + 4\left(\tau_{n-1} - r\right)$ must be a square. We know that $a \leq \tau_{n-1}$, and that $b \leq \tau_n$, hence $ab \leq \tau_{n-1}\tau_n$ so that $r \leq \tau_{n-1}$. Suppose that $\tau_{n-1} - r \geq 1$. Then $q^2(r - \tau_{n-2})^2 + 4(\tau_{n-1} - r)$ is a square, so must be at least $(q(r - \tau_{n-2}) + 2)^2$. Hence

$$q^2(r - \tau_{n-2})^2 + 4(\tau_{n-1} - r) \geq (q(r - \tau_{n-2}) + 2)^2$$
$$\Rightarrow \tau_{n-1} - r \geq 1 - q\tau_{n-2} + rq$$
$$\Rightarrow \tau_{n-1} - 1 + q\tau_{n-2} \geq r + rq$$
$$\Rightarrow r \leq \frac{2(\tau_{n-1} - 1)}{(q + 1)}.$$

It follows that if $r \neq \tau_{n-1}$ then $r \leq 2(\tau_{n-1} - 1)/(q + 1)$, as required. □

2.2 Two-character set case

A set $\mathcal{K}$ of points in $\mathrm{PG}(n, q)$ is a *two-character set*, with *characters* h_1, h_2, if every hyperplane meets $\mathcal{K}$ in either h_1 or h_2 points.

Theorem 7 (Delsarte, see [4])
Let $\mathcal{K} = \{P_i : i = 1, 2, \ldots, |\mathcal{K}|\}$, where each P_i is an element of $\mathrm{GF}(q)^{n+1}$, be a two-character set in $\mathrm{PG}(n, q)$, with characters h_1, h_2. Then

1. the graph $\Gamma(\mathcal{K})$ is a strongly regular graph;

2. the code $\{(x \cdot P_1, x \cdot P_2, \ldots, x \cdot P_{|\mathcal{K}|}) : x \in \mathrm{GF}(q)^{n+1}\}$ is a linear two-weight $[|\mathcal{K}|, n + 1]$-code with weights $|\mathcal{K}| - h_1, |\mathcal{K}| - h_2$;

3. the set $\mathcal{D} = \{v \in \mathrm{GF}(q)^{n+1} : \langle v \rangle \in \mathcal{K}\}$ is a $\{\lambda_1, \lambda_2\}$-difference set over $\mathrm{GF}(q)$, for some $\{\lambda_1, \lambda_2\}$.

In view of this theorem we are interested in investigating two-character sets in $\mathrm{PG}(n, q)$. See [4] for a comprehensive survey of two-character sets, two-weight codes and $\{\lambda_1, \lambda_2\}$-difference sets.

Example 5

Let $n = 2m + 1$ be odd. An *elliptic quasi-quadric* in $\mathrm{PG}(n, q)$ is a set $\mathcal{K}$ of $\frac{(q^m-1)(q^{m+1}+1)}{(q-1)}$ points such that each hyperplane meets it in either $\frac{q^{2m}-1}{q-1}$ or $\frac{q(q^m+1)(q^{m-1}-1)}{q-1} + 1$ points. A *hyperbolic quasi-quadric* in $\mathrm{PG}(n, q)$ is a set $\mathcal{K}$ of $\frac{(q^m+1)(q^{m+1}-1)}{(q-1)}$ points such that each hyperplane meets it in either $\frac{q^{2m}-1}{q-1}$ or $\frac{q(q^{m-1}+1)(q^m-1)}{q-1} + 1$ points. We denote an elliptic quasi-quadric by $\mathcal{K}^-$ and a hyperbolic quasi-quadric by $\mathcal{K}^+$. A non-degenerate elliptic quadric is an elliptic quasi-quadric and a non-degenerate hyperbolic quadric is a hyperbolic quasi-quadric. Further examples will be constructed in section 3.2 below.

In addition to the structures mentioned above, in the case of $q = 2$ two-character sets also give rise to certain symmetric SDP-designs, Reed-Muller codes and bent functions, as follows.

Theorem 8 ([12])

Let $\mathcal{K}^\epsilon$ be an elliptic or hyperbolic quasi-quadric in $\mathrm{PG}(n, 2)$, where $n = 2m + 1$ and $\epsilon \in \{-, +\}$. Embed $\mathrm{PG}(n, 2)$ as a hyperplane Σ_∞ in $\mathrm{PG}(n + 1, 2)$. The symmetric differences of the hyperplanes of Σ_∞ with $\mathcal{K}^\epsilon$ are then of two types, those of size $|\mathcal{K}^\epsilon|$, and those of size $2^{n+1} - |\mathcal{K}^\epsilon|$. Define a design with points the points of the affine space $\mathrm{AG}(n + 1, 2) = \mathrm{PG}(n + 1, 2) \setminus \Sigma_\infty$ and blocks the affine cones projecting the symmetric differences of the first type. This gives a symmetric SDP-design with parameters

$$(v, k, \lambda) = (2^{n+1}, 2^n + \epsilon 2^{(n-1)/2}, 2^{n-1} + \epsilon 2^n).$$

Note that the SDP-designs constructed from elliptic and hyperbolic quasi-quadrics are complementary to one another and that the number of SDP-designs grows exponentially with n [12].

Theorem 9 ([7], see [18, Section V.1.90])

The rows of the incidence matrix of any symmetric SDP-design are the minimum weight codewords in a binary linear code spanned by the first order Reed-Muller code $\mathrm{RM}(1, 2m)$ and the characteristic function of an elementary Abelian difference set in $\mathrm{AG}(2m, 2)$ (or, equivalently, the vector of values of a bent function on $2m$ variables).

3   Constructing quasi-quadrics

In this section we construct new quasi-quadrics from quadrics in projective spaces. In subsection 3.1, a geometric method of "pivotting" parabolic quadrics to construct new parabolic quasi-quadrics is provided, hence by the results of the previous section giving antipodal distance regular covers of complete graphs. In subsection 3.2,

several geometric constructions of elliptic and hyperbolic quasi-quadrics are given, so yielding strongly regular graphs, SDP-designs and so forth.

For a good introduction to quadrics and their properties, see [10].

3.1 Constructing parabolic quasi-quadrics

Let $Q(2m, q)$ be a non-degenerate parabolic quadric in $\mathrm{PG}(2m, q)$, q even, $m > 1$. Let Σ_k be a subspace of dimension k contained in $Q(2m, q)$, $k < m - 1$. The polar space $\Sigma_k^\perp$ of Σ_k with respect to the quadratic form for $Q(2m, q)$ is then of dimension $2m - k - 1$. The radical of the quadratic form of $Q(2m, q)$ restricted to $\Sigma_k^\perp$ is then just Σ_k. Further, the factor space $\Sigma_k^\perp / \Sigma_k$ has (even) dimension $2m - 2k - 2$. It follows that $\Sigma_k^\perp \cap Q(2m, q)$ is the cone $\Sigma_k Q(2m - 2k - 2, q)$ with vertex Σ_k and base a non-degenerate parabolic quadric $Q(2m - 2k - 2, q)$ in some subspace $\Sigma_{2m-2k-2}$ of dimension $2m - 2k - 2$ disjoint from Σ_k.

Suppose that $Q(2m - 2k - 2, q)$ has nucleus N'. Let Q' be a parabolic quasi-quadric in $\Sigma_{2m-2k-2}$ with the same parameters as $Q(2m - 2k - 2, q)$ and with the same nucleus N'. We then call the set $Q(2m, q) - \Sigma_k Q(2m - 2k - 2, q) \cup \Sigma_k Q'$ a *pivotted set* of $Q(2m, q)$ with respect to Σ_k. Note that the size of a pivotted set is the same as the size of $Q(2m, q)$.

Theorem 10
Every pivotted set of $Q(2m, q)$, q even, is a parabolic quasi-quadric with the same intersection numbers as those of $Q(2m, q)$.

Proof. Using the notation of the previous two paragraphs, we show that every hyperplane of $\mathrm{PG}(2m, q)$ not on the nucleus N of $Q(2m, q)$ meets the pivotted set in either $|Q^-(2m - 1, q)|$ or $|Q^+(2m - 1, q)|$ points.

Let Σ_{2m-1} be a hyperplane of $\mathrm{PG}(2m, q)$ that does not contain N. Then Σ_{2m-1} meets $\Sigma_k^\perp$ in a hyperplane Σ_{2m-k-2} of $\Sigma_k^\perp$. There are two cases to consider.
(i) Assume $\Sigma_k \not< \Sigma_{2m-k-2}$. Then $\Sigma_k \cap \Sigma_{2m-k-2}$ is a hyperplane Σ_{k-1} of Σ_k. Dimensional arguments then show that $\Sigma_{2m-k-2} \cap \Sigma_k Q(2m - 2k - 2, q) = \Sigma_{k-1} Q(2m - 2k - 2, q)$. Similarly, it follows that $\Sigma_{2m-k-2} \cap \Sigma_k Q' = \Sigma_{k-1} Q'$. Since Q' and $Q(2m - 2k - 2, q)$ have the same size it follows that Σ_{2m-1} meets $Q(2m, q)$ and the pivotted set in the same number of points.
(ii) Assume $\Sigma_k < \Sigma_{2m-k-2}$. Note that the nucleus N of $Q(2m, q)$ is contained in the subspace $\langle N', \Sigma_k \rangle$. Hence Σ_{2m-k-2} meets $\Sigma_{2m-2k-2}$ in a hyperplane of $\Sigma_{2m-2k-2}$ not on N'. It follows that $\Sigma_{2m-k-2} \cap Q(2m - 2k - 2, q)$ is either a $Q^-(2m - 2k - 3, q)$ or a $Q^+(2m - 2k - 3, q)$. Hence we have two cases to consider.
(a) Assume $\Sigma_{2m-k-2} \cap \Sigma_k^\perp = \Sigma_k Q^-(2m - 2k - 3, q)$. Now Σ_{2m-1} meets $Q(2m, q)$ in either a $Q^-(2m - 1, q)$ or a $Q^+(2m - 1, q)$. But $Q^+(2m - 1, q)$ does not contain a surface isomorphic to $\Sigma_k Q^-(2m - 2k - 3)$ [10, Corollary 2 to Theorem 22.8.3]. So in this case it must be that $\Sigma_{2m-1} \cap Q(2m, q)$ is isomorphic to a $Q^-(2m - 1, q)$.

Since the intersection sizes with respect to hyperplanes not on the nucleus are the same for $Q(2m - 2k - 2, q)$ and Q' it follows that $|\Sigma_{2m-1} \cap Q'|$ is equal to either

$$|Q^-(2m - 1, q)| - |\Sigma_k Q^-(2m - 2k - 3, q)| + |\Sigma_k Q^-(2m - 2k - 3, q)|$$

or

$$|Q^-(2m - 1, q)| - |\Sigma_k Q^-(2m - 2k - 3, q)| + |\Sigma_k Q^+(2m - 2k - 3, q)|$$

which, with some algebra, is easily shown to be either $|Q^-(2m - 1, q)|$ or $|Q^+(2m - 1, q)|$.

(b) Assume $\Sigma_{2m-k-2} \cap \Sigma_k^\perp = \Sigma_k Q^+(2m - 2k - 3, q)$. As before Σ_{2m-1} meets $Q(2m, q)$ in either a $Q^-(2m - 1, q)$ or a $Q^+(2m - 1, q)$. But $Q^-(2m - 1, q)$ does not contain a surface isomorphic to $\Sigma_k Q^+(2m - 2k - 3)$ [10, Corollary 2 to Theorem 22.8.3]. So in this case it must be that $\Sigma_{2m-1} \cap Q(2m, q)$ is isomorphic to a $Q^+(2m - 1, q)$.

Similarly to case (a), $|\Sigma_{2m-1} \cap Q'|$ is given by

$$|Q^+(2m - 1, q)| - |\Sigma_k Q^+(2m - 2k - 3, q)| + |\Sigma_k Q^-(2m - 2k - 3, q)|$$

or

$$|Q^+(2m - 1, q)| - |\Sigma_k Q^+(2m - 2k - 3, q)| + |\Sigma_k Q^+(2m - 2k - 3, q)|$$

which can be shown to be either $|Q^-(2m - 1, q)|$ or $|Q^+(2m - 1, q)|$. □

3.2 Constructing elliptic and hyperbolic quasi-quadrics

We begin this section by defining pivotting for elliptic and hyperbolic quadrics in a similar way to that of parabolic quadrics. For a related method that constructs quasi-quadrics from quadrics see [6]. In that paper, De Clerck and Delanote show that their method, in case $q = 2$, corresponds to Seidel switching in the associated strongly regular graphs.

Let $Q^-(2m + 1, q)$ be a non-degenerate elliptic quadric in $\mathrm{PG}(2m + 1, q)$, $m > 1$. Let X be a point of $Q^-(2m+1, q)$. The tangent (polar) space $X^\perp$ of X with respect to the quadratic form for $Q^-(2m+1, q)$ is then of dimension $2m$, and $X^\perp \cap Q^-(2m+1, q)$ is the cone $XQ^-(2m-1, q)$ with vertex X and base a non-degenerate elliptic quadric $Q^-(2m - 1, q)$ in some subspace Σ_{2m-1} of $X^\perp$ of dimension $2m - 1$ disjoint from X.

Let Q' be an elliptic quasi-quadric in Σ_{2m-1} with the same parameters as $Q^-(2m - 1, q)$. We then call the set $Q^-(2m + 1, q) - XQ^-(2m - 1, q) \cup XQ'$ a *pivotted set* of $Q^-(2m + 1, q)$ with respect to X. Note that the size of a pivotted set is the same as the size of $Q^-(2m + 1, q)$.

Theorem 11
Every pivotted set with respect to a point of $Q^-(2m+1, q)$ is an elliptic quasi-quadric with the same intersection numbers as those arising from $Q^-(2m + 1, q)$.

Proof. Let $Q'' = Q^-(2m + 1, q) - XQ^-(2m - 1, q) \cup XQ'$ be a pivotted set with respect to X as above, with $Q^-(2m - 1, q)$ and Q' contained in a hyperplane Σ_{2m-1} (not on X) of $X^\perp$. Then clearly any hyperplane $H \neq X^\perp$ not on X meets Q'' in a set of size $|Q(2m, q)|$ or $|YQ^-(2m - 1, q)|$.

Suppose H contains X, then $H \cap \Sigma_{2m-1}$ is a hyperplane of Σ_{2m-1}, and so $H \cap Q^-(2m-1, q)$ is either of type $Q(2m-2, q)$ or is a cone $ZQ^-(2m-3, q)$. We consider these two cases.

(i) Assume H meets $Q^-(2m-1, q)$ in a cone $ZQ^-(2m-3, q)$. Now a parabolic quadric $Q(2m, q)$ does not contain a surface isomorphic to $XZQ^-(2m - 3, q)$ [10, Lemma

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22.8.3], and so H must meet $Q^-(2m+1,q)$ in a cone with some point as vertex and a $Q^-(2m-1,q)$ as base. It follows that H meets Q'' in either $|XQ^-(2m-1,q)|$ or

$$|XQ^-(2m-1,q)| - |XZQ^-(2m-3,q)| + |XQ(2m-2,q)| = |Q(2m,q)|$$

points.

(ii) Assume H meets $Q^-(2m-1,q)$ in a parabolic quadric $Q(2m-2,q)$. Then H meets $X^\perp \cap Q^-(2m+1,q)$ in the cone $XQ(2m-2,q)$. There are $q+1$ hyperplanes of $PG(2m+1,q)$ on the subspace containing $XQ(2m-2,q)$ each of which either meets $Q^-(2m+1,q)$ in a parabolic quadric or a cone with base an elliptic quadric in some $(2m-1)$-dimensional subspace. The union of the points of $Q^-(2m+1,q)$ in these hyperplanes is all of the points of $Q^-(2m+1,q)$. Counting the points of this in each of the $q+1$ hyperplanes on $XQ(2m-2,q)$ shows that exactly one of the hyperplanes contains a cone on an elliptic quadric in some $(2m-1)$-dimensional subspace, and the other q all contain non-degenerate parabolic quadrics in $2m$-dimensional subspaces. Hence, apart from $X^\perp$, every other hyperplane on $XQ(2m-2,q)$ meets $Q^-(2m+1,q)$ in a non-degenerate parabolic quadric. It follows that H meets Q'' in either $|Q(2m,q)|$ or

$$|Q(2m,q)| + |XZQ^-(2m-3,q)| - |XQ(2m-2,q)| = |XQ^-(2m-1,q)|$$

points. $\square$

We now obtain a similar result for hyperbolic quadrics. Let $Q^+(2m+1,q)$ be a non-degenerate hyperbolic quadric in $PG(2m+1,q)$, $m > 1$. Let X be a point of $Q^+(2m+1,q)$. Then $X^\perp \cap Q^+(2m+1,q)$ is the cone $XQ^+(2m-1,q)$ with vertex X and base a non-degenerate hyperbolic quadric $Q^+(2m-1,q)$ in some subspace Σ_{2m-1} of dimension $2m-1$ disjoint from X.

Let Q' be a hyperbolic quasi-quadric in Σ_{2m-1} with the same parameters as $Q^+(2m-1,q)$. We then call the set $Q^+(2m+1,q) - XQ^+(2m-1,q) \cup XQ'$ a *pivotted set* of $Q^+(2m+1,q)$ with respect to X.

Theorem 12
Every pivotted set with respect to a point of $Q^+(2m+1,q)$ is a hyperbolic quasi-quadric with the same intersection numbers as those arising from $Q^+(2m+1,q)$.

The proof is essentially the same as for the previous theorem.

There is more we can say in the case $q = 2$.

Theorem 13
Let $Q^-(2m+1,2)$ be a non-singular elliptic quadric in $PG(2m+1,2)$, and let Σ_{2m} be a hyperplane of $PG(2m+1,2)$ meeting $Q^-(2m+1,2)$ in a non-singular parabolic quadric Q with nucleus N. Let Q' be any parabolic quasi-quadric in Σ_{2m} with nucleus N. Then the set of points of $Q'' = (Q^-(2m+1,2)-Q)\cup Q'$ is an elliptic quasi-quadric in $PG(2m+1,2)$.

Proof. Let Σ'_{2m} be any hyperplane of $PG(2m+1,2)$. Note that Σ'_{2m} meets $Q^-(2m+1,2)$ in either a non-singular parabolic quadric $Q(2m,2)$ or in a cone

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$XQ^-(2m-1,2)$ with vertex some point X and base a non-singular elliptic quadric. We show that Σ'_{2m} meets Q'' in either $|Q(2m,2)| = 2^{2m} - 1$ or $|XQ^-(2m-1,2)| = 2^{2m} - 2^m - 1$ points.

If $\Sigma'_{2m} = \Sigma_{2m}$, then clearly $|\Sigma'_{2m} \cap Q''| = |Q(2m,2)|$.

Assume $\Sigma'_{2m} \neq \Sigma_{2m}$, then $\Sigma'_{2m} \cap \Sigma_{2m}$ is hyperplane Σ_{2m-1} of Σ_{2m}. If $N \in \Sigma_{2m-1}$ then, since Q and Q' are parabolic quasi-quadrics, Σ_{2m-1} meets both Q and Q' in $2^{2m-1} - 1$ points. So $|\Sigma'_{2m} \cap Q''| = |\Sigma'_{2m} \cap Q^-(2m+1,2)|$ and the intersection size of Σ'_{2m} with Q'' is one of the required sizes.

Assume $N \notin \Sigma_{2m-1}$. Then $\Sigma_{2m-1} \cap Q$ is either a non-degenerate hyperbolic quadric Q^+_{2m-1} or a non-degenerate elliptic quadric Q^-_{2m-1}. We consider these two cases.

(i) Assume $\Sigma_{2m-1} \cap Q = Q^+_{2m-1}$. Since a cone $XQ^-(2m-1,2)$ contains no $Q^+(2m-1,2)$ it must be that $\Sigma'_{2m} \cap Q^-(2m+1,2)$ is a parabolic quadric $Q(2m,2)$. Now Q' is a parabolic quasi-quadric and so $\Sigma_{2m-1} \cap Q'$ is either of size $|Q^+(2m-1,2)|$ or $|Q^-(2m-1,2)|$. Hence

$$|\Sigma'_{2m} \cap Q''| = |Q(2m,2)| - |Q^+_{2m-1}| + |Q^+(2m-1,2)|$$

or

$$|\Sigma'_{2m} \cap Q''| = |Q(2m,2)| - |Q^+_{2m-1}| + |Q^-(2m-1,2)|$$

which are easily shown to be of the required sizes.

(ii) Assume $\Sigma_{2m-1} \cap Q = Q^-_{2m-1}$. Now every hyperplane of $\mathrm{PG}(2m+1,2)$ on Σ_{2m-1} is either of type $Q(2m,2)$ or $XQ^-(2m-1,2)$. A simple counting of points shows that there is a unique $Q(2m,2)$ contained in $Q^-(2m+1,2)$ and containing Q^-_{2m-1}, i.e. Q. Hence $|\Sigma'_{2m} \cap Q^-(2m+1,2)|$ must be of type $XQ^-(2m-1,2)$. Thus

$$|\Sigma'_{2m} \cap Q''| = |XQ^-(2m-1,2)| - |Q^-_{2m-1}| + |Q^-(2m-1,2)|$$

or

$$|\Sigma'_{2m} \cap Q''| = |XQ^-(2m-1,2)| - |Q^-_{2m-1}| + |Q^+(2m-1,2)|$$

which are easily shown to be of the required sizes. □

Theorem 14

Let $Q^+(2m+1,2)$ be a non-singular hyperbolic quadric in $\mathrm{PG}(2m+1,2)$, and Σ_{2m} be a hyperplane of $\mathrm{PG}(2m+1,2)$ meeting $Q^+(2m+1,2)$ in a non-singular parabolic quadric Q with nucleus N. Let Q' be any parabolic quasi-quadric in Σ_{2m} with nucleus N. Then the set of points of $Q'' = (Q^+(2m+1,2) - Q) \cup Q'$ is a hyperbolic quasi-quadric in $\mathrm{PG}(2m+1,2)$.

The proof is essentially the same as in the previous theorem.

Theorem 15

Let $Q^+(2m+1,2)$ be a non-singular hyperbolic quadric in $\mathrm{PG}(2m+1,2)$, and let Σ_{2m} be a hyperplane of $\mathrm{PG}(2m+1,2)$ meeting $Q^+(2m+1,2)$ in a non-singular parabolic quadric Q with nucleus N. Let Σ_{m-1} be a generator of Q. Then the set of points of $Q'' = (Q^+(2m+1,2) - Q) \cup (\Sigma_{2m} - (Q \cup \langle N, \Sigma_{m-1} \rangle)) \cup \Sigma_{m-1}$ is a hyperbolic quasi-quadric in $\mathrm{PG}(2m+1,2)$.

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Proof. First note that every hyperplane of $\mathrm{PG}(2m+1,2)$ meets $Q^+(2m+1,2)$ in either a quadric of type $Q(2m,2)$ or $XQ^+(2m-1,2)$, and these have sizes $2^{2m}-1$ and $2^{2m}-1+2^m$, respectively. One easily checks that $|Q''| = 2^{2m+1}+2^m-1$ and that that $|\Sigma_{2m} \cap Q''| = 2^{2m}-1$, which is one of the required intersection sizes.

Let H be any hyperplane different from Σ_{2m} and denote $H \cap \Sigma_{2m}$ by Σ_{2m-1}.

Suppose $N \in \Sigma_{2m-1}$. Then Σ_{2m-1} meets Q in a cone $XQ(2m-2,2)$ with vertex some point X and base $Q(2m-2,2)$. Such a cone contains $2^{2m-1}-1$ points. We consider two cases.

(i) Assume $\Sigma_{m-1} < \Sigma_{2m-1}$. Then $\langle N, \Sigma_{m-1}\rangle$ is a subspace of Σ_{2m-1} and contains $2^{m+1}-1$ points. Also $XQ(2m-2,2) - \Sigma_{m-1}$ contains $2^{2m-1}-2^m$ points. Hence

$$\begin{aligned}
|\Sigma_{2m-1} \cap Q''| &= 2^{2m}-1-(2^{m+1}-1)-(2^{2m-1}-2^m)+(2^m-1) \\
&= 2^{2m-1}-1 \\
&= |XQ(2m-2,2)|.
\end{aligned}$$

Since Σ_{2m-1} meets $Q^+(2m+1,2)$ in $XQ(2m-2,2)$ it follows that $|H \cap Q''| = |H \cap Q^+(2m+1,2)|$ and the intersection is one of the required sizes.

(ii) Assume $\Sigma_{m-1} \not< \Sigma_{2m-1}$. Then $\langle N, \Sigma_{m-1}\rangle \cap \Sigma_{2m-1}$ is a subspace of dimension $m-1$ of Σ_{2m-1} and contains 2^m-1 points. Also, $\Sigma_{m-1} \cap \Sigma_{2m-1}$ is a subspace of dimension $m-2$ and contains $2^{m-1}-1$ points. Thus $XQ(2m-2,2)-\Sigma_{m-1}$ contains $2^{2m-1}-2^{m-1}$ points. Hence

$$\begin{aligned}
|\Sigma_{2m-1} \cap Q''| &= 2^{2m}-1-(2^m-1)-(2^{2m-1}-2^{m-1})+(2^{m-1}-1) \\
&= 2^{2m-1}-1 \\
&= |XQ(2m-2,2)|.
\end{aligned}$$

Again it follows that $|H \cap Q''| = |H \cap Q^+(2m+1,2)|$ and the intersection has one of the required sizes.

For the remainder of the proof suppose $N \notin \Sigma_{2m-1}$. There are again two cases to consider.

(i) Assume $\Sigma_{2m-1} \cap Q$ is a non-degenerate hyperbolic quadric $Q^+(2m-1,2)$ which has $2^{2m-1}-1+2^{m-1}$ points.

Suppose that $\Sigma_{m-1} < \Sigma_{2m-1}$ then a counting as in the previous paragraphs shows that $|\Sigma_{2m} \cap Q''| = |\Sigma_{2m} \cap Q^+(2m+1,2)|$.

So suppose $\Sigma_{m-1} \not< \Sigma_{2m-1}$. Then, as before, $\langle N, \Sigma_{m-1}\rangle \cap \Sigma_{2m-1}$ is a subspace of dimension $m-1$ of Σ_{2m-1} and contains 2^m-1 points. As $|\Sigma_{m-1} \cap \Sigma_{2m-1}| = 2^{m-1}-1$, it follows that $(\Sigma_{2m-1} \cap Q) \setminus \Sigma_{m-1}$ contains 2^{2m-1} points. Hence

$$\begin{aligned}
|\Sigma_{2m-1} \cap Q''| &= 2^{2m}-1-(2^m-1)-(2^{2m-1})+(2^{m-1}-1) \\
&= 2^{2m-1}-1-2^{m-1}.
\end{aligned}$$

Thus in Σ_{2m-1} we have replaced a set of points of size $2^{2m-1}-1+2^{m-1}$ by a set of size $2^{2m-1}-1-2^{m-1}$.

As in the proof of theorem 14, since $H \neq \Sigma_{2m}$, the intersection $H \cap Q^+(2m+1,2)$ is a cone $XQ^+(2m-1,2)$. Hence

$$|H \cap Q''| = 2^{2m}-1+2^m-(2^{2m-1}-1+2^{m-1})+(2^{2m-1}-1-2^{m-1})$$

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$$= 2^{2m} - 1$$
$$= |Q(2m, 2)|.$$

(ii) Assume $\Sigma_{2m-1} \cap Q$ is a non-degenerate elliptic quadric $Q^-(2m-1, 2)$ which has $2^{2m-1} - 1 - 2^{m-1}$ points.

The maximal subspaces of $Q^-(2m-1, 2)$ have dimension $m - 2$ hence $\Sigma_{m-1} \not\subset H$. Also note that a hyperplane section of $Q^+(2m + 1, 2)$ of the form $XQ^+(2m - 1, 2)$ contains no $Q^-(2m-1, 2)$. It follows that H meets $Q^+(2m+1, 2)$ in a non-degenerate parabolic quadric $Q(2m, 2)$. Using this information and counting as in the previous paragraphs gives

$$|H \cap Q''| = 2^{2m} - 1 + 2^m = |XQ^+(2m - 1, 2)|.$$

$\square$

Theorem 16
Let $Q^-(2m + 1, 2)$ be a non-singular elliptic quadric in $\mathrm{PG}(2m + 1, 2)$, and let Σ_{2m} be a hyperplane of $\mathrm{PG}(2m + 1, 2)$ meeting $Q^-(2m + 1, 2)$ in a non-singular parabolic quadric Q with nucleus N. Let Σ_{m-1} be a generator of Q. Then the set of points of $Q'' = (Q^-(2m+1, 2) - Q) \cup (\Sigma_{2m} - (Q \cup \langle N, \Sigma_{m-1} \rangle)) \cup \Sigma_{m-1}$ is an elliptic quasi-quadric in $\mathrm{PG}(2m + 1, 2)$.

The proof is essentially the same as that of the previous theorem.

Remark

A quasi-quadric in $\mathrm{PG}(3, 2)$ is a quadric. In $\mathrm{PG}(5, 2)$ there are five projectively inequivalent quasi-quadrics of elliptic type and seven of hyperbolic type, see [17] for more details and [15] for a geometric treatise on some of these quasi-quadrics.

Acknowledgments

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