A degree condition for the codiameter

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Abstract

In this paper a degree condition for the codiameter is presented.

1 Introduction

In [1], Hikoe Enomoto proved the following theorem.

Theorem 1. [1] *Let G be a 3-connected graph with n vertices such that $\sigma_2 \geq m$. Then $d^*(G) \geq \min\{n-1, m-2\}$.*

In [2], Nathaniel Dean obtained the following result.

Theorem 2. [2]*Let G be a 2-connected graph with vertex set $\{x_1, x_2, \cdots, x_n\}$ and edge set E. Suppose G satisfies the following property for a given positive integer m: for all positive integers j and k such that $j < k$, $x_j x_k \notin E$; $d(x_j) \leq j$ and $d(x_k) \leq k-1$, we have*

(1) $d(x_j) + d(x_k) \geq m$ whenever $j + k \geq n$,

(2) $d(x_j) + d(x_k) \geq \min\{k+1, m\}$ whenever $j + k < n$.

Then $c(G) \geq \min\{m, n\}$.

The main theorem in this paper is as follows:

Theorem 3. *Let G be a 3-connected graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ where $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$. Suppose for every pair of characteristic vertices v_a and v_b we have*

(1) $d(v_a) + d(v_b) \geq m$ whenever $a + b \geq n$,

(2) $d(v_a) + d(v_b) \geq \min\{b+3, m\}$ whenever $a + b < n$.

Then $d^*(G) \geq \min\{n-1, m-2\}$.

Clearly, Theorem 1 is an immediate corollary of Theorem 3.

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2 Notation and Preliminaries

In this paper we denote the neighbour set of the vertex x by $N(x)$, and put $\Gamma(x) = \{x\} \bigcup N(x)$. For a path $P = (u_1, u_2, \cdots, u_p)$, let $u_j^+ = u_{j+1}$, $u_j^- = u_{j-1}$ and $|P| = p$. If C is a cycle in G, then $|C|$ stands for the number of vertices contained in C.

Definition 1. Assume G is a connected graph. For every two vertices x and y in G, their codistance is defined by $d^*(x,y) = \max\{|P|-1 \mid P \text{ is a } (x,y) \text{ path}\}$ and the codiameter of G is defined by $d^*(G) = \min\{d^*(x,y) \mid \{x,y\} \subset V(G)\}$.

Lemma 1. *Assume $Q = \{x_1 \cdots x_t\}$ is a path in a 2-connected graph G and $N(x_1) \subset V(Q)$, $N(x_t) \subset V(Q)$. Then for every two vertices x_a and x_b in Q, there must exist a path R in G with x_a and x_b as its ends and $|R| \geq \min\{d(x_1)+1, d(x_t)+1\}$.*

Proof. Suppose $a < b$. Let $c = \max\{i \mid x_i \in N(x_1)\}$ and $d = \min\{i \mid x_i \in N(x_t)\}$. We consider three cases.

(I) $a < c \leq b$. Let $a' = \min\{i \mid x_i \in N(x_1),\ i > a\}$. Then $R = \left(x_a \overleftarrow{Q}\, x_1 x_{a'} \overrightarrow{Q}\, x_b\right)$ includes x_1 and all its neighbours. Therefore $|R| \geq d(x_1) + 1$.

(II) $c \leq a < b$. Since G is a 2-connected graph, there exists at least one path $P_1(x_{f_1}, x_{l_1})$ satisfying that $V(Q) \bigcap V(P_1) = \{x_{f_1}, x_{l_1}\}$ $(f_1 < c < l_1)$. Choose P_1 so as to maximize l_1. If $l_1 > a$, then stop, or else we take a similar path $P_2(x_{f_2}, x_{l_2})$ satisfying that $f_2 < l_1 < l_2$ where l_2 is maximized. If $l_2 > a$, then stop. Otherwise, repeat the above procedure until we obtain paths P_r, $r = 1, 2, \cdots, q$ such that $f_r < l_{r-1} < l_r$ and $f_1 < c \leq a < l_q$.

Let $f_1' = \min\{i \mid x_i \in N(x_1),\ i > f_1\}$. It is easy to show that, if q is an odd number, then the path $R = \left(x_a \overleftarrow{Q}\, x_{l_{q-1}} \overleftarrow{P}_{q-1}\, x_{f_{q-1}} \cdots x_{l_2} \overleftarrow{P}_2\, x_{f_2} \overleftarrow{Q}\, x_{f_1'} x_1 \overrightarrow{Q}\, x_{f_1} \overrightarrow{P}_1\, x_{l_1} \cdots x_{f_q} \overrightarrow{P}_q\, x_{l_q} Q x_b\right)$ contains x_1, x_b and all the neighbours of x_1 and hence $|R| \geq d(x_1)+2$. If q is an even number, then the path $R = \left(x_a \overleftarrow{Q}\, x_{l_{q-1}} \overleftarrow{P}_{q-1}\, x_{f_{q-1}} \cdots x_{l_1} \overleftarrow{P}_1\, x_{f_1} \overleftarrow{Q}\, x_1 x_{f_1'} \overrightarrow{Q}\, x_{f_2} \overrightarrow{P}_2\, x_{l_2} \cdots x_{f_q} \overrightarrow{P}_q\, x_{l_q} Q x_b\right)$ has length at least $d(x_1) + 2$.

(III) $a < b < c$. If $d \geq a$, from (I) and (II) we can see that there exists a path in G with x_a and x_b as its ends and length at least $d(x_t) + 1$. Therefore we can assume $d < a < b < c$ and $N(x_1) \bigcap \{x_{a+1}, \cdots, x_{b-1}\} \neq \emptyset$. Now $N(x_t) \bigcap \{x_{a+1}, \cdots, x_{b-1}\} \neq \emptyset$. (Otherwise, $N(x_1) \bigcap \{x_{a+1}, \cdots, x_{b-1}\} = \emptyset$. Then the path $\left(x_a \overleftarrow{Q}\, x_1 x_c \overleftarrow{Q}\, x_b\right)$ has length at least $d(x_1) + 1$). Let

$$a' = \min\{i \mid x_i \in N(x_q),\ i > a\},\ b' = \max\{i \mid x_i \in N(x_t),\ i < b\},$$

$$f = \max\{i \mid x_i \in N(x_1),\ i < b\},\ h = \min\{i \mid x_i \in N(x_t),\ i > a\}.$$

If $b' \geq f$, then the path $\left(x_a \overleftarrow{Q}\, x_1 x_{a'} \overrightarrow{Q}\, x_{b'} x_t \overleftarrow{Q}\, x_b\right)$ contains x_1 and all its neighbours. Therefore the length of the path is at least $d(x_1) + 1$.

If $b' < f$, then the path $\left(x_a \overleftarrow{Q}\, x_1 x_f \overleftarrow{Q}\, x_h x_t \overleftarrow{Q}\, x_b\right)$ has length at least $d(x_t) + 2$.

3 Proof of Theorem 1

Definition 2. Let G be a graph with n vertices $V(G) = \{v_1, v_2, \cdots, v_n\}$, where $d(v_1) \le d(v_2) \le \cdots \le d(v_n)$. We call v_a and v_b $(a < b)$ a pair of characteristic vertices in G if I) $v_a v_b \notin E$ and II) $d(v_a) \le a+1$, $d(v_b) \le b$.

Theorem 1. *Let G be a 3-connected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d(v_1) \le d(v_2) \le \cdots \le d(v_n)$. Then passing through every edge of G there exists a cycle of length at least $d(v_a)+d(v_b)-1$ for some pair v_a and v_b of characteristic vertices.*

The rest of this section is devoted to the proof of Theorem 1.

We will prove it by contraposition. Suppose the theorem is false. Throughout the proof, let e be an arbitrary edge in G and $P = (v_j \cdots v_k)$ be one of the longest paths passing through e, chosen so as to maximize $j+k$. In addition, $e = v_f v_h$. It is easily seen that $N(v_j) \bigcup N(v_k) \subseteq V(p)$ and $v_j v_k \notin E$. Suppose $j < k$ without loss of generality.

Proposition 1. *If $v_i^+ v_j \in E$ and $i \ne f$, then $i \le j$.*

Proof. If $v_i^+ v_i \in E$ and $i \ne f$, then $P' = v_i \overleftarrow{P} v_j v_i^+ \overrightarrow{P} v_k$ is one of the longest paths passing through e. So $i + k \le j + k$, i.e. $i \le j$, which completes our proof.

Similarly, we can prove that if $v_i^- v_k \in E$ and $i \ne h$, then $i \le k$.

Proposition 2. *$d(v_j) \le j+1$ and $d(v_k) \le k$.*

Proof. $d(v_j) = |N^-(v_j)| \le |\{v_i \mid v_i^+ v_j \in E,\ i \ne f\}| + 1 \le |\{v_i \mid v_i \in V(G),\ i \le j\}| + 1 = j+1$.

Similarly, $d(v_k) = |N^+(v_k)| \le |\{v_i \mid v_i^- v_k \in E,\ i \ne h\}| + 1 \le |\{v_i \mid v_i \in V(G),\ i \ne h\}| = k$. This completes the proof of Proposition 2.

From Proposition 1 and Proposition 2, we know that v_j and v_k are a pair of characteristic vertices.

Renumber the vertices of P as $P = x_1 x_2 \cdots x_t$ so that $N(x_1) \bigcup N(x_t) \subseteq P$. Put $p = \max\{i \mid x_i \in N(x_1)\}$, $q = \min\{i \mid x_i \in N(x_t)\}$ and $e = x_{s-1} x_s$. We consider two cases: 1) $p \le q$ and 2) $p > q$.

Case 1. $p \le q$.

In this case, we distinguish two subcases depending on the position of the edge e: I) $2 \le s \le p$ and II) $p+1 \le s \le q$.

Subcase 1.1. $2 \le s \le p$.

Let $i_0 = \max\{i \mid x_i \in N(x_1),\ i \le s-1\}$.

Algorithm 1.1

Step 0. Set $S = \{x_1, \cdots, x_{i_0 - 1}\} \bigcup \{x_s, \cdots, x_{p-1}\}$, $R = \{x_{p+1}, \cdots, x_t\}$ and $W = \{x_{i_0}, \cdots, x_{s-1}\}$.

$r \leftarrow 0$, $l_0 \leftarrow p$.

Step 1. Find a path from S to R, $P_r(x_{f_r}, x_{l_r})$, such that $P_r \bigcap P = \{x_{f_r}, x_{l_r}\}$ with the maximal l_r.

Step 2. i) If $l_r \le l_{r-1}$, then stop and set $l_{r-1} = c$.

ii) If $l_r > l_{r-1}$ and $l_r \le q$, then $S \leftarrow S \bigcup \{x_{l_{r-1}}, \cdots, x_{l_r - 1}\}$, $R \leftarrow \{x_{l_r+1}, \cdots, x_t\}$ and return to step 1.

iii) If $l_r > q$, go to the next step.

Step 3. Let $f_1' = \min\{i \mid x_i \in N(x_1),\ i > f_1\}$, $l_r' = \max\{i \mid x_i \in N(x_t),\ i < l_r\}$, $C_0 = x_1 \overrightarrow{P} x_{f_1'} x_1$, $C_i = P(x_{f_i}, x_{l_i}) P_i x_{f_i}$, $i = 1, 2, \cdots, r$, $C_{r+1} = P(x_{l_r'}, x_t) x_{l_r'}$ and $C = \sum_0^{r+1} C_l$, where $\sum$ stands for symmetric difference. Clearly, the cycle C passes through e and contains all the elements in $\Gamma(x_1) \bigcup \Gamma(x_t)$. Since $N(x_1) \bigcap N(x_t) \subseteq \{x_p\}$, we have $|C| \geq d(x_1) + d(x_t) + 1$, which contradicts the maximality of $j + k$. Hence, the Algorithm 1.1 will not stop at iii) of step 2.

Algorithm 1.2.

Step 0. Let $r \leftarrow 0$, $Q_0 = x_{c+1} \cdots x_t$ and $y_0 \leftarrow x_{c+1}$.

Step 1. If $N(y_r) \backslash \{x_c\} \subseteq Q_r$, go to Step 3. Otherwise, go to the next step.

Step 2. If $(N(y_r) \backslash \{x_c\}) \backslash Q_r \neq \emptyset$, then choose v_l in the set $(N(y_r) \backslash \{x_c\}) \backslash Q_r$, so as to maximize l. Let $v_l = y_{r+1}$, $y_{r+1} Q_r = Q_{r+1}$ and $r \leftarrow r + 1$. Then go to Step 1.

Step 3. Let $Q_r = (v_l \cdots v_k)$. If $i \leq l$ for every $v_i \in (N(v_l) \bigcap Q^r)^-$, then $r^* \leftarrow r$ and stop. Otherwise, choose $v_i \in (N(v_l) \bigcap Q_r)^-$, so as to maximize i. Set $v_i = y_{r+1}$, $Q_{r+1} = v_i \overleftarrow{Q_r} v_l v_i^+ \overleftarrow{Q_r} v_k$, $r \leftarrow r + 1$ and $Q_r \leftarrow Q_{r+1}$. Return to Step 1.

It follows from $c \leq q$ and $N(x_t) \subseteq V(Q_r) \bigcup \{x_c\}$ that $V(Q_r) \supseteq \{x_{c+1}, x_{c+2}, \cdots, x_t\}$ for $r = 0, 1, \cdots, r^*$.

In the following, we prove $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$ by recursive reasoning. When $r = 0$, $V(Q_0) = \{x_{c+1}, x_{c+2}, \cdots, x_t\}$ and the equality clearly holds. Now suppose the equality is true for some r ($r < r^*$). That is to say, $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$. We will verify the case $r+1$. For this purpose, let $Q_r = w_1 w_2 \cdots w_\alpha$, i.e. $w_1 = y_r$, $w_\alpha = x_t$, $h = \max\{i \mid w_i \in N(w_1)\}$ and $h' = \min\{i \mid w_i \in N(w_\alpha)\}$.

Since G is 3-connected and $x_c \notin V(Q_r)$, there exist two $\{x_1, x_2, \cdots, x_{c-1}\} - V(Q_r)$ chains $\mu_1(x_{i_1}, x_{j_1})$ and $\mu_2(x_{i_2}, x_{j_2})$ in $G \backslash \{x_c\}$ with empty intersection. Suppose $j_1 < j_2$. Choose μ_1 and μ_2 so as to maximize j_2. By Algorithm 1.1, $\{x_{i_1}, x_{i_2}\} \subseteq W$. Without loss of generality, suppose $i_1 < i_2$ and let $R_1 = x_{i_1} \overleftarrow{P} x_1 x_p \overleftarrow{P} x_{i_2}$. Then R_1 contains e and all the elements in $\Gamma(x_1)$. Hence $|R_1| \geq d(x_1) + 1$.

Proposition 3. $h' < j_1 < j_2$.

Proof. Suppose $h' \geq j_1$. Since $G \backslash \{x_c\}$ is 2-conncected and $N(w_\alpha) \subseteq V(Q_r) \bigcup \{x_c\}$, applying the proof of Lemma 1 (i) and (ii) to $G \backslash \{x_c\}$ and $\overleftarrow{Q_r}$, we know from Algorithm 1.1 and choice of μ_1 and μ_2 that there exists a (w_{j_1}, w_{j_2}) path R_2 such that

i) $V(R_2) \supseteq \Gamma(w_\alpha) \backslash \{x_c\}$;

ii) $R_2 \bigcap (R_1 \bigcup \mu_1 \bigcup \mu_2) \subseteq \{w_{j_1}, w_{j_2}\}$.

Let $C = R_1 \bigcup R_2 \bigcup \mu_1 \bigcup \mu_2$, then C is a cycle passing through e with length $d(x_1) + d(x_t) + 1$. This contradicts the maximality of $j + k$.

Proposition 4. $h > j_1$.

Proof. Suppose $h \leq j_1$. Similar to the proof of Proposition 3 with condition $N(w_1) \subseteq V(Q_r) \bigcup \{x_c\}$, we can find a (w_{j_1}, w_{j_2}) path such that

i) $V(R_2) \supseteq \Gamma(w_\alpha) \setminus \{x_c\}$;

ii) $R_2 \bigcap (R_1 \bigcup \mu_1 \bigcup \mu_2) \subseteq \{w_{j_1}, w_{j_2}\}$.

Let $C = R_1 \bigcup R_2 \bigcup \mu_1 \bigcup \mu_2$. Then C is a cycle passing through e with length $d(x_1) + d(x_t) + 1$. This contradicts the maximality of $j + k$.

Proposition 5. $N(y_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$.

Proof. $V(Q_r)\bigcap\{x_1,x_2,\cdots,x_{c-1}\}=\emptyset$ by assumption. From the connectedness and $V(Q_r)\supseteq\{x_{c+1},x_{c+2},\cdots,x_t\}$, it follows that $N(y_r)\bigcap\{x_1,x_2,\cdots,x_{c-1}\}\subseteq W$ by Algorithm 1.1. Next we prove $N(y_r)\bigcap W=\emptyset$. Suppose $N(y_r)\bigcap W\neq\emptyset$. Let $x_f\in N(y_r)\bigcap W$. Since at least one of x_{i_1} and x_{i_2} is not x_f, we assume $x_{i_1}\neq x_f$ with $f>i_1$ without loss of generality. By Proposition 3, there exists $j_1'=\min\{i\mid i<j_1,\ w_i\in N(w_\alpha)\}$ such that $C=x_{i_1}\overleftarrow{P}x_1x_p\overleftarrow{P}x_fw_1\overrightarrow{Q_r}w_{j_1'}w_\alpha\overleftarrow{Q_r}w_{j_1}\overleftarrow{\mu_1}x_{i_1}$ passes through e and $|C|\geq d(x_1)+d(x_2)+1$. This contradicts the maximality of $j+k$.

Similarly, we have the following result.

Proposition 6. $N(y_r)\bigcap(\mu_1\bigcup\mu_2)=\emptyset$.

By Proposition 5 and Algorithm 1.2, $V(Q_{r+1})\bigcap\{x_1,x_2,\cdots,x_{c-1}\}=\emptyset$. Hence, $V(Q_r)\bigcap\{x_1,x_2,\cdots,x_{c-1}\}=\emptyset$ for $r=0,1,2,\cdots,r^*$ and Propositions 3–6 hold for every $r\in\{0,1,2,\cdots,r^*\}$.

Since G is a finite graph, there must exist a path $r^*=(v_l,\cdots,v_k)$ with $v_k=x_t$ by the use of Algorithm 1.2. Assume $j<l$ (in the case $j>l$, the proof is similar). Then, we have $d(v_l)\leq|N(v_l)\setminus\{x_c\}|+1=|N^-(v_l)\bigcap Q_r|+1=|\{v_i\mid v_i^+v_l\in E\}|+1\leq|\{v_i\mid v_i\in V(G),\ i\leq l\}|=l$. In addition, $d(v_j)\leq j+1$. So v_j and v_l are a pair of characteristic vertices. Let $r^*=a_1a_2\cdots a_\alpha=v_l\cdots v_k$, $a_1=v_l$, $a_\alpha=v_k=x_t$ and $G'=G\setminus\{x_c\}$. Then there exist two $\{x_1,x_2,\cdots,x_{c-1}\}-V(r^*)$ chains $\mu_1(x_{i_1},a_{j_1})$ and $\mu_2(x_{i_2},a_{j_2})$ in G' with empty intersection. By Lemma 1 and the choice of μ_1 and μ_2, there exists a (a_{j_1},a_{j_2}) path R_2 such that

i) $R_2\bigcap R_1=\emptyset$ and $R_2\bigcap(\mu_1\bigcup\mu_2)\subseteq\{a_{j_1},a_{j_2}\}$;

ii) $V(R_2)\supseteq\Gamma(v_l)\setminus\{x_l\}$ or $V(R_2)\supseteq\Gamma(v_k)\setminus\{x_c\}$.

Let $C=R_1\bigcup R_2\bigcup\mu_1\bigcup\mu_2$. Then C is a cycle passing through e with length at least $d(v_j)+d(v_k)+1$ or $d(v_j)+d(v_l)+1$. This contradicts the maximality of $j+k$ or $j+l$.

Subcase 1.2. $p+1\leq s\leq q$.

Let $G'=G\setminus\{x_c\}$. Then G' is a 2-connected graph. There exists a $\{x_1,x_2,\cdots,x_{p-1}\}-\{x_{p+1},\cdots,x_t\}$ path $P_1(x_{f_1},x_{l_1})$ such that l_1 is maximized. If $l_1>s$, then stop. Otherwise, find a similar path $P_2(x_{f_2},x_{l_2})$ in G' with $f_2<l_1<l_2$. If $l_2>s$, then stop. Repeat this procedure. Finally, we get a path $P_r(x_{f_r},x_{l_r})$ $(r=1,2,\cdots,d)$ with $f_r<l_{r-1}<l_r$ and $f_1<p<s<l_d$. Clearly $l_d\leq q$. (Otherwise, let $C_0=x_1\overrightarrow{P}x_{f_1'}x_1$ with $f_1'=\min\{i\mid i>f_1,\ x_i\in N(x_1)\}$, $C_i=x_{f_i}\overrightarrow{P}x_{l_i}\overleftarrow{P_i}x_{f_i}$ $(i=1,2,\cdots,d)$ and $C_{d+1}=x_{l_d'}Px_tx_{l_d'}$ where $l_d'=\max\{i\mid x_i\in N(x_t),\ i<l_d\}$. Then the symmetric difference $C=\sum\limits_{i=0}^{d+1}C_i$ is a cycle passing through e with length at least $d(x_1)+d(x_t)+2$. This contradicts the maximality of $j+k$.

Algorithm 1.3

Step 0. Set $S=\{x_s,\cdots,x_{l_d-1}\}$, $R=\{x_{l_d+1},\cdots,x_t\}$, $W=\{x_{l_d-1},\cdots,x_{s-1}\}$ and $r\leftarrow d$.

Step 1. Find a path $P_r(x_{f_r},x_{l_r})$ from S to R so as to maximize l_r.

Step 2. i) If $l_r\leq l_{r-1}$, then stop and set $l_{r-1}=c$.

ii) If $l_r>q$, then stop.

iii) If $l_r>l_{r-1}$ and $l_r\leq q$, then $S\leftarrow S\bigcup\{x_{l_{r-1}},\cdots,x_{l_r-1}\}$, $R\leftarrow\{x_{l_r+1},\cdots,x_t\}$ and return to Step 1.

This algorithm will not stop at ii) of step 2. Otherwise it is easily proven that $C = \sum_{i=0}^{r+1} C_i$ is a cycle passing through the edge e with length at least $d(x_l) + d(x_t) + 2$, where C_i is identical to C_i in Algorithm 1.1. This contradicts the maximality of $j + k$.

Algorithm 1.4

Step 0. Set $x_{c+1} = y_0$, $Q_0 \leftarrow P(x_{c+1}, x_t)$ and $i \leftarrow 0$.

Step 1. If $N(y_r) \setminus \{x_c\} \subseteq V(Q_r)$, go to Step 3. Otherwise, go to Step 2.

Step 2. If $N(y_r) \setminus (V(Q_r) \bigcup \{x_c\}) \neq \emptyset$, choose v_l so as to maximize the index l. Set $v_l = y_{r+1}$, $y_{r+1}Q_r = Q_{r+1}$ and $r \leftarrow r+1$. Then return to Step 1.

Step 3. Let $Q_r = (v_l \cdots v_k)$. If $i < l$ for every $v_i \in [N(y_r) \bigcap Q_r]^-$, then stop. Otherwise, choose v_i in $[N(y_r) \bigcap Q_r]^-$ so as to maximize the index i. Let $Q_{r+1} = \left(v_i \overleftarrow{Q_r}\, v_l v_i^+ \overrightarrow{Q_r}\, v_k\right)$ and $r \leftarrow r+1$. Return to Step 1.

Now it is easily seen that $V(Q_r) \supseteq \{x_{c+1}, x_{c+2}, \cdots, x_t\}$ for $r = 0, 1, 2, \cdots, r^*$.

Since $c \leq q$, we have $N(x_t) \subseteq V(Q_r) \bigcup \{x_c\}$. In the following, we prove $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$ in a recursive way.

$V(Q_0) = \{x_{c+1}, x_{c+2}, \cdots, x_t\}$. So the equality holds when $r = 0$.

Assume the equality is true for some r ($r < r^*$). We consider the case $r+1$.

Let $Q_r = w_1 w_2 \cdots w_\beta$. Then $w_1 = y_r$, $w_\beta = x_t$, $h = \max\{i \mid w_i \in N(w_1)\}$ and $h' = \min\{i \mid w_i \in N(w_\beta)\}$. By assumption, $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$. Since G is 3-connected and $x_c \notin V(Q_r)$, there exist two $\{x_1, x_2, \cdots, x_{c-1}\} - V(Q_r)$ chains $\mu_1(x_{i_1}, w_{j_1})$ and $\mu_2(x_{i_2}, w_{j_2})$ in $G \setminus \{x_c\}$ without intersection. We assume $j_1 < j_2$. Choose μ_1 and μ_2 so as to maximize j_2. Then $\{x_{i_1}, x_{i_2}\} \subseteq W$. Now suppose $i_1 < i_2$ without loss of generality. Let

$$R_1 = \begin{cases} x_{i_1} \overleftarrow{P} x_{l_{d-1}} \overleftarrow{P}_{d-1} x_{f_{d-1}} \cdots x_{l_2} \overleftarrow{P}_2 x_{f_2} \overrightarrow{P}\, x_{f_2}\, \overrightarrow{P}\, x_{f_1'} x_1 P x_{f_1}\, \overrightarrow{P}_1\, x_{l_1} \cdots x_{f_d}\, \overleftarrow{P}_d\, x_{l_d} P x_{i_2} \\ \qquad \text{if } d \text{ is odd} \\ x_{i_1}\, \overleftarrow{P}\, x_{l_{d-1}}\, \overleftarrow{P}_{d-1}\, x_{f_{d-1}} \cdots x_{l_1}\, \overleftarrow{P}_1\, x_{f_1}\, \overleftarrow{P}\, x_1 x_{f_1'}\, \overrightarrow{P}\, x_{f_2}\, \overrightarrow{P}_2\, x_{l_2} \cdots x_{f_d}\, \overrightarrow{P}_d\, x_{l_d} P x_{i_2} \\ \qquad \text{if } d \text{ is even,} \end{cases}$$

and $f_1' = \min\{i \mid i > f_1,\ x_i \in N(x_1)\}$. Clearly, R_1 contains the edge e and all the elements in $\Gamma(x_1)$. Thus $|R_1| \geq d(x_1) + 1$.

Proposition 7. $h' < j_1 < j_2$.

Proof. Suppose $h' \geq j_1$. Since $G \setminus \{x_c\}$ is 2-connected and $N(w_\beta) \subseteq V(Q_r) \bigcup \{x_c\}$, applying the proof of Lemma (i) and (ii) to $G \setminus \{x_c\}$ and $\overrightarrow{Q_r}$, we know from Algorithm 1.4 and the choice of μ_1 and μ_2 that there exists a path R_2 such that

(i) $V(R_2) \supseteq \Gamma(w_\beta) \setminus \{x_c\}$

(ii) $R_2 \bigcap (R_1 \bigcup \mu_1 \cup \mu_2) \subseteq \{w_{j_1}, w_{j_2}\}$.

Let $C = R_1 \bigcup R_2 \bigcup \mu_1 \bigcup \mu_2$. Then C is a cycle passing through e with length at least $d(x_1) + d(x_t) + 1$. This contradicts the maximality of $j + k$.

Proposition 8. $h > j_1$.

Proposition 9. $N(y_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$.

Proof. $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$ by assumption. From the connectedness and $V(Q_r) \supseteq \{x_{c+1}, x_{c+2}, \cdots, x_t\}$, it follows that $N(y_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} \subseteq W$

by Algorithm 1.3. Next we prove $N(y_r) \bigcap W = \emptyset$. Suppose $N(y_r) \bigcap W \neq \emptyset$. Let $x_f \in N(y_r) \bigcap W$. Since at least one of x_{i_1} and x_{i_2} is not x_f, we assume $x_{i_1} \neq x_f$ with $f > i_1$. By Proposition 7, there exist $j_1' = \max\{i \mid i < j_1,\ w_i \in N(w_\beta)\}$ and $f_1' = \min\{i \mid i > f_1,\ x_i \in N(x_1)\}$ such that

$$C = \begin{cases} x_{i_1} \overleftarrow{P} x_{l_{d-1}} \overleftarrow{P}_{d-1} x_{f_{d-1}} \cdots x_{l_2} \overleftarrow{P}_2 x_{f_2} \overleftarrow{P} x_{f_1'} x_1 \overrightarrow{P} x_{f_1} \overrightarrow{P}_1 x_{l_1} \cdots x_{f_d} \overrightarrow{P}_d x_{l_d} \overleftarrow{P} x_f \\ y_r \overrightarrow{Q}_r w_{j_1'} w_\beta \overrightarrow{Q}_r w_{j_1} \overleftarrow{\mu}_1 x_{i_1} & \text{if } d \text{ is odd} \\ x_{i_1} \overleftarrow{P} x_{l_{d-1}} \overleftarrow{P}_{d-1} x_{f_{d-1}} \cdots x_{l_1} \overleftarrow{P}_1 x_{f_1} \overleftarrow{P} x_1 x_{f_1'} \overrightarrow{P} x_{f_2} \overrightarrow{P}_2 x_{l_2} \cdots x_{f_d} \overrightarrow{P}_d x_{l_d} \overleftarrow{P} x_f \\ y_r \overrightarrow{Q}_r w_{j_1'} w_\beta \overrightarrow{Q} w_{j_1} \overleftarrow{\mu}_1 x_{i_1} & \text{if } d \text{ is even} \end{cases}$$

is a cycle passing through e with length at least $d(x_1) + d(x_t) + 1$. This contradicts the maximality of $j + k$.

Similarly, we have

Proposition 10. $N(y_r) \bigcap (\mu_1 \bigcup \mu_2) = \emptyset$.

From Proposition 9 and Algorithm 1.4, we know that

$$V(Q_{r+1}) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset.$$

Therefore $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$ for $r = 0, 1, 2, \cdots, r^*$ and hence Propositions 7–10 hold for every r.

Let $r^* = b_1 b_2 \cdots b_\beta = v_l \cdots v_k$ and $j < l$. Then $d(v_j) \leq j + 1$ and $d(v_l) \leq l$, i.e. v_j and v_l are a pair of characteristic points. Let $G' = G \setminus \{x_c\}$. Then there exist two $\{x_1, x_2, \cdots, x_{c-1}\} - V(r^*)$ chains $\mu_1(x_{i_1}, b_{j_1})$ and $\mu_2(x_{i_2}, b_{j_2})$ with empty intersection.

By Lemma 1 and the choice of μ_1 and μ_2, there exists a (b_{j_1}, b_{j_2}) path R_2 such that

i) $R_2 \bigcap R_1 = \emptyset$ and $R_2 \bigcap (\mu_1 \bigcup \mu_2) \subseteq \{b_{j_1}, b_{j_2}\}$;

ii) $V(R_2) \supseteq \Gamma(v_l) \setminus \{x_c\}$ or $V(R_2) \supseteq \Gamma(v_k) \setminus \{x_c\}$.

Let $C = R_1 \bigcup R_2 \bigcup \mu_1 \bigcup \mu_2$. Then C is a cycle passing through e with length at least $d(v_j) + d(v_k) + 1$ or $d(v_j) + d(v_l) + 1$. This contradicts the maximality of $j + k$ or $j + l$.

Case 2. $p > q$.

In this case, we may suppose that there exists a pair of positive integers p' and q' such that $x_{p'} \in N(x_1)$, $x_{q'} \in N(x_t)$, $p' > q'$ and $p' - q'$ is minimized.

Lemma 2. $q' + 1 \leq s \leq p'$.

Proof. Firstly, $N(x_1) \bigcap N^+(x_t) \subseteq \{x_s\}$ (If there exist some $i \neq s$ such that $x_i \in N(x_1) \bigcap N^+(x_t)$, then the cycle $\left(x_1 \overrightarrow{P} x_{i-1} x_t \overleftarrow{P} x_i x_1\right)$ is a hamiltonian cycle passing through e, which leads to a contradiction). If $s \leq p'$ or $s \geq q' + 1$, the cycle $C = \left(x_1 \overrightarrow{P} x_{q'} x_t \overleftarrow{P} x_{p'} x_1\right)$ contains e and all the elements in $\{x_1\} \bigcup N(x_1) \bigcup (N^+(x_t) \setminus \{x_{q'+1}\})$. Therefore $|C| \geq d(x_l) + d(x_t) - 1$. This contradicts the maximality of $j + k$. Now it is easy to see that $N(x_1) \bigcap N(x_t) \subseteq \{x_p, x_q\}$. When $q \neq q'$, $N(x_1) \bigcap \{x_{q+1}, \cdots, x_{q'}\} = \emptyset$. When $p \neq p'$, $N(x_t) \bigcap \{x_{p'}, \cdots, x_{p-1}\} = \emptyset$.

Algorithm 2.1

Let $i_0 = \max\{i \mid x_i \in N(x_1),\ i \leq s - 1\}$ and $j_0 = \min\{i \mid x_i \in N(x_t),\ i > p\}$.

Step 0. Set $S = \{x_1, \cdots, x_{i_0-1}\} \bigcup \{x_s, \cdots, x_{p-1}\}$, $R = \{x_{p+1}, \cdots, x_t\}$, $r \leftarrow 1$ and $W = \{x_{i_0}, \cdots, x_{s-1}\}$.

Step 1. Find a path $P_r(x_{f_r}, x_{l_r})$ from S to R such that the intersection of $P_r(x_{f_r}, x_{l_r})$ and P is $\{x_{f_r}, x_{l_r}\}$ and l_r is maximized.

Step 2. i) If $l_r \leq l_{r-1}$, then stop and set $c = l_{r-1}$.

ii) If $l_r > j_0$, then stop.

iii) If $l_r \leq j_0$, let $S \leftarrow S \bigcup \{x_{r-1}, \cdots, x_{l_r-1}\}$, $R \leftarrow \{x_{l_r+1}, \cdots, x_t\}$ and return to Step 1.

Similar to Algorithm 1.1, the above algorithm will not stop at ii) of Step 2.

Algorithm 2.2

Step 0. Set $r \leftarrow 0$, $Q_0 = x_{c+1} \overrightarrow{P} x_t$, $y_0 \leftarrow x_{c+1}$.

Step 1. If $[N(y_r) \setminus \{x_c\}] \subseteq V(Q_r)$, go to Step 3. Otherwise, go to Step 2.

Step 2. Choose v_l in $N(y_r) \setminus [V(Q_r) \bigcup \{x_c\}]$ such that the index l is maximized. Let $y_{r+1} = v_l$, $y_{r+1}Q_r = Q_{r+1}$ and $r \leftarrow r+1$. Return to Step 1.

Step 3. Let $Q_r = (v_l \cdots v_k)$. If $i \leq l$ is true for every $v_i \in [N(v_l) \bigcap Q_r]^-$, then stop. Otherwise, choose v_i in $[N(v_l) \bigcap Q_r]^-$ such that i is maximized. Let $Q_{r+1} = v_i \overleftarrow{Q}_r v_l v_i^+ \overrightarrow{Q}_r v_k$ and $r \leftarrow r+1$. Return to Step 1.

For $r = 0, 1, 2, \cdots, r^*$, $V(Q_r) = w_1 \cdots w_r$, $w_1 = y_r$ and $w_r = x_t$. It is easily seen that $V(Q_r) \supseteq \{x_{c+1}, x_{c+2}, \cdots, x_t\}$. In addition, $N(w_1) \subseteq V(Q_r) \bigcup \{x_c\}$ due to $c \leq j_0$.

We will prove $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$ recursively. When $r = 0$, $V(Q_0) = \{x_{c+1}, x_{c+2}, \cdots, x_t\}$. The equality holds trivially. Suppose the equality is true for some $r < r^*$. We consider the case $r+1$. Let $h = \max\{i \mid w_i \in N(w_1)\}$. By assumption, $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$. Since G is 3-connected and $x_c \notin V(Q_r)$, there exist two $\{x_1, x_2, \cdots, x_{c-1}\} - V(Q_r)$ chains $\mu_1(x_{i_1}, w_{j_1})$ and $\mu_2(x_{i_2}, w_{j_2})$ in $G - \{x_c\}$. Assume $i_1 < i_2$ and choose μ_1 and μ_2 to maximize j_2. We have $w_r = x_t$. By Algorithm 2.2, $\{x_{i_1}, x_{i_2}\} \subseteq W$. Assume $i_1 < i_2$ without loss of generality. Let $R_1 = x_{i_1} \overleftarrow{P} x_1 x_p \overleftarrow{P} x_{i_2}$. Then R_1 contains e and all the elements in $\Gamma(x_1)$. Thus $|R_1| \geq d(x_1) + 1$.

Proposition 11. $h > j_1$.

Proof. Suppose $h \leq j_1$. Note that $N(w_1) \subseteq V(Q_r) \bigcup \{x_c\}$ and $G - \{x_c\}$ is 2-connected. Applying the proof of i) and ii) of Lemma 1 to $G - \{x_c\}$ and $\overleftarrow{Q}_r$, we know from Algorithm 2.1 and the choice of μ_1 and μ_2 that there exists a (w_{j_1}, w_{j_2}) path R_2 such that

i) $V(R_2) \supseteq \Gamma(w_1) \setminus \{x_c\}$;

ii) $R_2 \bigcap (R_1 \bigcup \mu_1 \bigcup \mu_2) \subseteq \{w_{j_1}, w_{j_2}\}$.

Let $C = R_1 \bigcup R_2 \bigcup \mu_1 \bigcup \mu_2$. Then C is a cycle passing through e with length at least $d(x_1) + d(w_1) + 1$. This contradicts the maximality of $j + l$.

Proposition 12. $N(y_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$.

Proof. $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$ by assumption. Since Q_r is connected and $V(Q_r) \supseteq \{x_{c+1}, x_{c+2}, \cdots, x_t\}$, we have $N(y_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} \subseteq W$ by applying Algorithm 2.2. We will prove $N(y_r) \bigcap W = \emptyset$. Suppose $N(y_r) \bigcap W \neq \emptyset$. Then we take $x_f \in N(r_r) \bigcap W$.

i) $q \neq q'$. In this case, we assume $f \leq q$ without loss of generality. Let $f' =$

$\min\{i \mid x_i \in N(w_\beta\}$. Then $C = x_1 \overrightarrow{P} \cdots f_f w_1 \overrightarrow{Q_r} \cdots w_\beta x_{f'} \overrightarrow{P} x_p x_1$ contains e and all the elements in $\Gamma(x_1) \bigcup \Gamma(w_1) - \{x_c\}$. Hence $|C| \geq d(x_1) + d(w_1) - 1$.

ii) $q = q'$. If $f = q$, the proof is similar to i). If $f \neq q$,then $G'' = G - \{x_c, x_q\}$ is a connected graph. Hence there exists a $\{x_1, \cdots, x_{q-1}, x_{q+1}, \cdots, x_{c-1}\} - V(Q_r)$ chain $\mu_3(x_{i_3}, x_{j_3})$. Therefore, the cycle

$$C = x_1 \overrightarrow{P} x_{i_3} \mu_3 w_{j_3} \overleftarrow{Q_r} w_1 w_{j_3'} \overrightarrow{Q_r} \cdots w_\beta x_q \overrightarrow{P} x_p x_1$$

contains e and has length $d(x_1) + d(w_1) - 1$. This contradicts the maximality of $j + l$.

Similarly, the following proposition can be proven.

Proposition 13. $N(y_r) \bigcap (\mu_1 \bigcup \mu_2) = \emptyset$.

We can infer that $V(Q_{r+1}) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$ from Proposition 11 and Algorithm 2.2. Hence $V(Q_r) \bigcap \{x_1, x_2, \cdots, x_{c-1}\} = \emptyset$ for $r = 0, 1, 2, \cdots, r^*$. Meanwhile Propositions 11–13 are true for all r.

According to Algorithm 2.2, $r^* = c_1 c_2 \cdots c_r = v_l \cdots v_r$ since G is a finite graph. Assume $j < l$. Then $d(v_j) \leq j + 1$, $d(v_l) \leq l$, i.e. v_j and v_l are a pair of characteristic points. Let $G' = G \setminus \{x_c\}$. Then there exist two $\{x_1, x_2, \cdots, x_{c-1}\} - V(r^*)$ chains $\mu_1(x_{i_1}, c_{j_1})$ and $\mu_2(x_{i_2}, c_{j_2})$ with empty intersection, and we have $c_{j_2} = x_t$. By Lemma 1 and the choice of μ_1 and μ_2, there exists a (c_{j_1}, c_{j_2}) path R_2 such that

i) $R_1 \bigcap R_2 = \emptyset$, $R_2 \bigcap (\mu_1 \bigcup \mu_2) \subseteq \{c_{j_1}, c_{j_2}\}$,

ii) $V(R_2) \supseteq \Gamma(v_l) \setminus \{x_c\}$.

Let $C = R_1 \bigcup R_2 \bigcup \mu_1 \bigcup \mu_2$. Then C is a cycle passing through e with length at least $d(v_j) + d(v_l) - 1$. This contradicts the maximality of $j + l$.

4 Proof of Theorem 2 and Theorem 3

Theorem 2 *Let G be a 3-connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$. If the following hold for every pair of characteristic vertices v_a and v_b $(a < b)$:*

i) $d(v_a) + d(v_b) \geq m$ for $a + b \geq n$;

ii) $d(v_a) + d(v_b) \geq \min\{b + 3, m\}$ for $a + b < n$,

then, for every $e \in G$, there exists a cycle passing through e with length at least $m-1$.

Proof. Based on the proof of Theorem 1, we can obtain, by applying Algorithm 1.2 and 1.4 or 2.2, two pairs of characteristic points v_j, v_k and v_j, v_l either for $p > q$ or $p \leq q$ and $2 \leq s \leq q$.

For such v_j and v_l $(j < l)$, we have that

$$\begin{aligned} d(v_j) + d(v_l) &\leq |\{v_i \mid v_j v_i^+ \in E,\ i \neq f\}| + 1 + |[(N(v_l) - \{x_c\}) \textstyle\bigcap Q]^{-1}| + 1 \\ &\leq |\{v_i \mid v_i \in V(G),\ i \leq l\}| + 2 = l + 2. \end{aligned}$$

We consider two possible cases. If $j + l \geq n$, then $d(v_j) + d(v_l) \geq m$. If $j + l < n$, then $l + 2 \geq d(v_j) + d(v_l) \geq \min\{l + 3, m\}$. Hence $d(v_j) + d(v_l) \geq m$ for both cases.

If $p \leq q$ and $2 \leq s \leq p$, it is clear that $N^-(v_j) \bigcap N^+(v_k) = \emptyset$. In this case,

$$\begin{aligned} d(v_j) + d(v_k) &= |N^-(v_j)| + |N^+(v_k)| \\ &\leq |\{v_i \mid v_j v_i^+ \in E,\ i \neq f\}| + 1 + |\{v_i \mid v_k v_i^- \in E\}| \\ &\leq |\{v_i \mid v_i \in V(G),\ i \leq k\}| + 1 = k + 1. \end{aligned}$$

If $p > q$, then $|N^-(v_j) \bigcap N^+(v_k)| \leq 1$ by Lemma 2.

When $|N^-(v_j) \bigcap N^+(v_k)| = 1$, let $v_i \in N^-(v_j) \bigcap N^+(v_k)$. Then $v_i^- = x_{q'}$, $v_i^+ = x_{p'}$ and $e = v_i^- v_i$ or $v_i v_i^+$. Hence it is impossible that $v_f \in N^-(v_j) \bigcap N^+(v_k)$.

$$\begin{aligned} d(v_j) + d(v_k) &= |N^-(v_j)| + |N^+(v_k)| \\ &= |N^-(v_j) \bigcup N^+(v_k)| + |N^-(v_j) \bigcap N^+(v_k)| \\ &\leq |\{v_i \mid v_i \in V(G),\ i \leq k\}| + 1 + 1 = k + 2. \end{aligned}$$

When $N^-(v_j) \bigcap N^+(v_k) = \emptyset$,

$$\begin{aligned} d(v_j) + d(v_k) &= |N^-(v_j)| + |N^+(v_k)| \\ &\leq |\{v_i \mid v_j v_i^+ \in E,\ i \neq f\}| + 1 + |\{v_i \mid v_k v_i^- \in E,\ i \neq h\}| + 1 \\ &\leq |\{v_i \mid v_i \in V(G),\ i \leq k\}| + 2 = k + 2. \end{aligned}$$

Thus $d(v_j) + d(v_k) \geq m$ due to the condition in this theorem. Theorem 1 shows that there is a cycle of length at least $m - 1$ passing through any arbitrary edge of G, if the related condition is satisfied.

Theorem 3. *$d^*(G) \geq m - 1$ under the condition of Theorem 2.*

Proof. Given two vertices x and y, let $G' = G + xy$. Then G' satisfies the requirements in Thereom 2. Therefore, the edge xy is contained in a cycle of length at least $m - 1$. This means that x and y are connected by a path in G with length at least $m - 2$.

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