Self-converse Mendelsohn designs with odd block size

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Abstract

A Mendelsohn design $MD(v, k, \lambda)$ is a pair $(X, \mathcal{B})$, where X is a v-set together with a collection $\mathcal{B}$ of ordered k-tuples from X such that each ordered pair from X is contained in exactly λ k-tuples of $\mathcal{B}$. An $MD(v, k, \lambda)$ is said to be *self-converse*, denoted by $SCMD(v, k, \lambda) = (X, \mathcal{B}, f)$, if there is an isomorphism f from $(X, \mathcal{B})$ to $(X, \mathcal{B}^{-1})$, where $\mathcal{B}^{-1} = \{\langle x_k, x_{k-1}, ..., x_2, x_1\rangle ; \langle x_1, ..., x_k\rangle \in \mathcal{B}\}$. The existence of $SCMD(v, 3, \lambda)$, $SCMD(v, 4, 1)$, $SCMD(v, 5, 1)$ and $SCMD(v, 4t+2, 1)$ has been completely settled, where $2t + 1$ is a prime power. But up to now, there is no result about odd block size larger than five. In this paper, we give a constructive proof for the existence of k-$SCMD(v)$, where k is odd and $k > 5$, $v \equiv 1 \pmod{k}$.

1 Introduction

Let X be a v-set and $3 \le k \le v$. A *cyclic k-tuple* from X is a collection of k ordered pairs $(x_0, x_1), (x_1, x_2), \cdots, (x_{k-2}, x_{k-1})$ and (x_{k-1}, x_0), where $x_0, x_1, \cdots, x_{k-1}$ are distinct elements of X. It is denoted by $\langle x_0, x_1, \cdots, x_{k-1}\rangle$ or any cyclic shift $\langle x_i, x_{i+1}, \cdots, x_{k-1}, x_0, \cdots, x_{i-1}\rangle$. A (v, k, λ)-*Mendelsohn design* is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is a collection of cyclic $k-$tuples (called *blocks*) from X, such that each ordered pair of distinct elements of X belongs to exactly λ blocks of $\mathcal{B}$. It is denoted by $MD(v, k, \lambda)$.

Let $(X, \mathcal{B})$ be an $MD(v, k, \lambda)$. Define

$$\mathcal{B}^{-1} = \{\langle x_{k-1}, x_{k-2}, \cdots, x_1, x_0\rangle ; \langle x_0, x_1, \cdots, x_{k-1}\rangle \in \mathcal{B}\}.$$

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Obviously $(X, \mathcal{B}^{-1})$ is also an $MD(v,k,\lambda)$, which is called the *converse* of $(X, \mathcal{B})$. If there exists an isomorphism f from $(X, \mathcal{B})$ to $(X, \mathcal{B}^{-1})$, then the $MD(v,k,\lambda)$ is called *self-converse* and this is denoted by $SCMD(v,k,\lambda) = (X, \mathcal{B}, f)$. In particular, it is denoted by k-$SCMD(v)$ when $\lambda = 1$. It is easy to show that necessary conditions for the existence of both an $MD(v,k,\lambda)$ and an $SCMD(v,k,\lambda)$ are

$$\lambda v(v-1) \equiv 0 \pmod{k}.$$

C.J. Colbourn and A. Rosa [2] posed the open problem about the existence of self-converse $MD(v,\ 3,\lambda)$. Yanxun Chang, Guihua Yang and Qingde Kang [3] have solved the case for $\lambda = 1$. Jie Zhang [4] solved the case for any λ. Qingde Kang et al. [5],[6] have completely solved the cases for $k = 4, 5$ and $4t+2$, where $2t+1$ is prime power. But, up to now, there is no result about odd block size larger than five. In this paper, the following results are obtained.

Theorem 1.1 *For odd $k > 5$, there exists a k-$SCMD(v)$ if $v \equiv 0 \pmod 4$ and $v \equiv 1 \pmod k$.*

Theorem 1.2 *There exists a $(2t+1)$-$SCMD(4t+2)$ for any positive integer $t > 2$.*

Theorem 1.3 *There exists a k-$SCMD(v)$ for odd $k > 5$ and $v \equiv 1 \pmod k$, except for $(4t+1)$-$SCMD(4t+2)$ for any integer $t \geq 2$.*

2 Definition and Remarks

Let Z_v be the ring of integers modulo v and let $d_1, d_2, \cdots, d_m$ be elements of $Z_v \setminus \{0\}$. The ordered sequence $D = (d_1, d_2, \cdots, d_m)$ is called a *difference tuple* on Z_v. The corresponding number tuple $(i, i+d_1, \cdots, i+\sum_{j=1}^{m} d_j)$ is denoted by $\widetilde{D_i}$, $i \in Z_v$. For convenience, we define $head(\widetilde{D_i}) = i$ and $tail(\widetilde{D_i}) = i + \sum_{j=1}^{m} d_j$. Furthermore, define:

$$\begin{aligned}
-D &= (-d_1, -d_2, \cdots, -d_m);\\
D^{-1} &= (d_m, d_{m-1}, \cdots, d_1);\\
A(D) &= (d_1, -d_2, d_3, \cdots, (-1)^{m-1}d_m);\\
-A(D)^{-1} &= -(A(D))^{-1};\\
abs(D) &= \{|d_1|, |d_2|, \cdots, |d_m|\};\\
abs(\widetilde{D_i}) &= \{|i|, |i+d_1|, \cdots, |i+\textstyle\sum_{j=1}^{m} d_j|\};\\
S[a, a+3] &= (a+2, -(a+1), a, -(a+3));\\
S[a, a+4t-1] &= (S[a, a+3], \cdots, S[a+4t-4, a+4t-1]);\\
S[a, a+4t-1]^{-1} &= (S[a, a+4t-1])^{-1}.
\end{aligned}$$

If the points in $\widetilde{D_0}$ are distinct, then D is called a *difference path* and this is denoted by $DP(D)$. Obviously, if D is a DP then $-D$ and D^{-1} are both DPs. If the head and the tail of $\widetilde{D_0}$ both are 0, then D is called a *difference cycle* and this is denoted by $DC(D)$. It is easy to see that if M is a DC, then $-M^{-1}$ is a DC too. Let a, b, d, k be positive integers with $a < d < b$. Define:

$[a,b]_k = (a, a+k, \cdots, b)$,
$[a,b]_k \setminus \{d\} = (a, \cdots, d-k, d+k, \cdots, b)$,
where $a \equiv b \equiv d \pmod{k}$ and the subscript k may be omitted when $k = 1$.

For given point set $X = Z_v$, let $\mathcal{T}$ be the set of all cyclic k-tuples on X and let f be a bijection on X. Denote $\mathcal{P} = \{(x,y)\,;\, x,y \in X,\ x \neq y\}$. The point set X is partitioned into *point-orbits* under the action of f. Two distinct points x and y are in the same point-orbit if and only if there is a positive integer s such that $f^s(x) = y$. The point-orbit containing the point x is denoted by $O_f(x)$. The number of points in $O_f(x)$ is called the *length* of $O_f(x)$. The mappings induced by f are $f((x,y)) = (f(x), f(y))$ on $\mathcal{P}$ and $f(\langle x_0, x_1, \cdots, x_{k-1}\rangle) = \langle f(x_0), f(x_1), \cdots, f(x_{k-1})\rangle$ on $\mathcal{T}$. Denote

$$Rf(P) = (f(P))^{-1},\ \forall P \in \mathcal{P}; \quad Rf(B) = (f(B))^{-1},\ \forall B \in \mathcal{T}.$$

Then the finite permutation group on $\mathcal{P}$ (or $\mathcal{T}$) generated by Rf gives an orbit partition of $\mathcal{P}$ (or $\mathcal{T}$). Each orbit in $\mathcal{P}$ containing the pair P is called the *pair-orbit* $O(P)$. Each orbit in $\mathcal{T}$ containing the block B is called the *block-orbit* $O(B)$. The number of pairs (or blocks) in a pair-orbit (or block-orbit) is called the *length* of the orbit. Call a block B *self-converse* if $Rf(B) = B$.

Let d (or D) be the difference (or difference cycle) corresponding to pair P (or block B). Then, the difference $-d$ (or difference cycle $(-D)^{-1}$) corresponds to $Rf(P)$ (or $Rf(B)$). As well, $O(P) = O(Rf(P))$ and $O(B) = O(Rf(B))$.

Let $x, y \in X$, $x \neq y$ and $y - x = d$. Suppose the length of $O_f(x)$ is m. If x and y are in the same point-orbit, then we have

	Number of pair-orbits	length of each pair-orbit
m odd, $\forall d$	1	$2m$
m even, $d \neq \frac{m}{2}$	2	m
$m \equiv 0 \pmod 4$, $d = \frac{m}{2}$	1	m
$m \equiv 2 \pmod 4$, $d = \frac{m}{2}$	2	$\frac{m}{2}$

Otherwise, let the length of $O_f(y)$ be n. Then, we have

	Number of pair-orbits	length of each pair-orbit
m odd, n odd	1	$2[m,n]$
else	2	$[m,n]$

If there are two pair-orbits corresponding to d, we call them *complementary*. As examples, we give the following:

1. if $X = Z_4$ and $f = (0,1,2,3)$, the pair-orbit corresponding to the difference 2 under Rf is $\{(0,2), (3,1), (2,0), (1,3)\}$;
2. if $X = Z_6$ and $f = (0,1,2,3,4,5)$, the pair-orbits corresponding to the difference 3 under Rf are $\{(0,3), (4,1), (2,5)\}$ and $\{(1,4), (5,2), (3,0)\}$;
3. if $|X| = 5$ and $f = (0,1,2)(\bar{0},\bar{1})$, the pair-orbits corresponding to the mixed difference 1 under Rf (see Section 5) are $\{(0,\bar{1}), (\bar{0},1), (2,\bar{1}), (\bar{0},0), (1,\bar{1}), (\bar{0},2)\}$ and $\{(1,\bar{0}), (\bar{1},2), (0,\bar{0}), (\bar{1},1), (2,\bar{0}), (\bar{1},0)\}$.

In particular, we discuss the case $X = Z_v$ and $f = (0, 1, \cdots, v-1)$, where $v \equiv 0 \pmod 4$. Then, it is easy to see that $\{(2i, 2i+d), (2i+d+1, 2i+1)\,;\, 0 \le i \le \frac{v}{2}-1\}$ and $\{(2i+1, 2i+d+1), (2i+d, 2i)\,;\, 0 \le i \le \frac{v}{2}-1\}$, where $d \in Z_v^*$, are complementary pair-orbits. The two orbits are the same when $d = \frac{v}{2}$, i.e., $\{(i, i+\frac{v}{2})\,;\, 0 \le i \le v-1\}$.

Let D be a DC on Z_v. Define the following DCs.

1 $\underline{SDC(D)}$, where D satisfies:

(1) D contains exactly one $\frac{v}{2}$.

(2) For any two pairs in $\widetilde{D}_0$, their pair-orbits are distinct.

(3) For any pair $P = (x, x+d)$ in $\widetilde{D}_0$, $d \ne \frac{v}{2}$, there exists a pair P' in $\widetilde{D}_0$ such that $O(P)$ and $O(P')$ are complementary.

2 $\underline{CDC(D)}$, where D satisfies:

(1) $\frac{v}{2} \notin D$.

(2) All the elements in $abs(D)$ are distinct.

From the above definitions and the discussion of pair-orbits, we have the following lemma.

Lemma 2.1 *Under the action of the group generated by* Rf,

(1) *there is only one block-orbit with length* v *corresponding to* $SDC(D)$;

(2) *there are two block-orbits with length* v *corresponding to* $CDC(D)$.

Obviously, $\{\widetilde{D_{2i}}, \widetilde{D_{2i+1}}^{-1}\,;\, 0 \le i \le \frac{v}{2}-1\}$ and $\{\widetilde{D_{2i+1}}, \widetilde{D_{2i}}^{-1}\,;\, 0 \le i \le \frac{v}{2}-1\}$ are the complementary block-orbits corresponding to $CDC(D)$. But $\{\widetilde{D_{2i}}, \widetilde{D_{2i+1}}^{-1}\,;\, 0 \le i \le \frac{v}{2}-1\}$ is the only orbit corresponding to $SDC(D)$.

3 Some sub-structures

A *k-cycle decomposition* of a complete graph K_v is a collection of undirected cycles with length k, whose (undirected) edges partition all the edges of K_v. Writing each k-cycle twice, once in a certain order and the other in the reverse order, then a k-$CS(v)$ gives a k-$SCMD(v)$, where the mapping f is the identity mapping. It is known [1] that there exists a k-cycle decomposition of K_v if k is odd and $v \equiv 1, k \pmod{2k}$. Thus, in order to investigate the existence of k-$SCMD(v)$ for odd k and $v \equiv 1 \pmod k$, we only need to discuss the cases $v \equiv k+1$ and $3k+1 \pmod{4k}$. In this section, we suppose $v \equiv 0 \pmod 4$ and $v \equiv 1 \pmod k$.

First, from [5] (Lemma 1 and Corollary 2), we have Lemma 3.1 and Corollary 3.2.

Lemma 3.1 *For* $DP(D) = (x_1, x_2, \cdots, x_m)$, $0 < x_1 < \cdots < x_m \le \frac{v}{2}$, $\pm A(D)$ *are* DP*s.*

Corollary 3.2 *Let* $0 < d < m$, $a > 0$ *and* $a + km \le \frac{v}{2}$. *If* $D = [a, a+km]_k$ *or* $D = [a, a+km]_k \setminus \{a+kd\}$, *then* $\pm A(D)$ *are* DP*s.*

For convenience, we give the following table (where $D = \pm A[a, a+m]$).

Table A

m	$sgn(D)$	$tail(\widetilde{D_0})$	$\{\widetilde{D_0}\}$
even	$+$	$a+\frac{m}{2}$	$[-\frac{m}{2},0]\cup[a,a+\frac{m}{2}]$
even	$-$	$-(a+\frac{m}{2})$	$[0,\frac{m}{2}]\cup[-(a+\frac{m}{2}),-a]$
odd	$+$	$-\frac{m+1}{2}$	$[-\frac{m+1}{2},0]\cup[a,a+\frac{m-1}{2}]$
odd	$-$	$\frac{m+1}{2}$	$[0,\frac{m+1}{2}]\cup[-(a+\frac{m-1}{2},-a]$

Remark In the above table, $head(\widetilde{D_0})=0$. This table can be used for $D=\pm A[a,a+km]_k$ or $D=\pm A([a,a+km]_k\setminus\{a+kd\})$, where all numbers m,d and 1 are replaced by km, kd and k respectively, while a and 2 are kept fixed. And the intervals $[*,*]$ become $[*,*]_k$.

Lemma 3.3 *Let $D=[a,a+4t-1]$, $a>0$ and $t\geq 1$. Then $\pm S(D)$ are DPs.*

Proof. Let $N=S(D)$. From the definition of $S(D)$,

$$\widetilde{N_0}=[a+1,a+2t]\cup([-2t,1]\setminus\{-(2t-1)\}).$$

So $S(D)$ is a DP and $-S(D)$ is a DP too. □

Lemma 3.4 *Let M be a DP on Z_v and $\frac{v}{2}\notin M$. If M satisfies $(\widetilde{M_0}\setminus\{0\})\cap(\widetilde{M_0}+(-1)^{i+1}\frac{v}{4})=\emptyset$, $\frac{v}{2}\notin\widetilde{M_0}$ and $\sum_{d\in M}d\equiv(-1)^i\frac{v}{4}$ (mod v) for $i=0$ or 1, then $(M,\frac{v}{2},M)$ is a DC. Furthermore, $(M,\frac{v}{2},M)$ is an SDC when $\frac{v}{4}$ is odd.*

Proof. Let $\widetilde{M_0}=(0,x_1,\cdots,x_m)$, where $m=|M|$ and $0<|x_i|<\frac{v}{2}$. Obviously, $x_m=\sum_{d\in M}d\equiv(-1)^i\frac{v}{4}$ (mod v). If $D=(M,\frac{v}{2},M)$, then

$$\widetilde{D_0}=(0,x_1,\cdots,x_m,\tfrac{v}{2}+x_m,\tfrac{v}{2}+x_m+x_1,\cdots,\tfrac{v}{2}+2x_m).$$

$\widetilde{D_0}$ is closed since $x_m\equiv(-1)^i\frac{v}{4}$ (mod v). Furthermore, since M is a DP and $(\widetilde{M_0}\setminus\{0\})\cap(\widetilde{M_0}+(-1)^{i+1}\frac{v}{4})=\emptyset$, D is a DC. If $\frac{v}{4}$ is odd, we can show that D is an SDC by the definition of SDC. □

Corollary 3.5 *If $v\equiv 4$ (mod 8), $1\leq t<t+m<\frac{v}{4}$, $t+m<a<\frac{v}{2}$ and*

$$a\equiv\pm\tfrac{v}{4}-(-1)^m\lceil\tfrac{m}{2}\rceil-\tfrac{1+(-1)^m}{2}t\ (mod\ v),$$

then $(A[t,t+m],a,\frac{v}{2},A[t,t+m],a)$ is an SDC.

Proof. Letting $D=A[t,t+m]$, then D is a DP by Corollary 3.2 and $tail(\widetilde{D_0})=(-1)^m\lceil\frac{m}{2}\rceil+\frac{1+(-1)^m}{2}t$ from Table A. Let $M=(D,a)$. From Table A we can see that $|x|<\frac{v}{4}$ for all $x\in\widetilde{D_0}$. For $1\leq t<t+m<\frac{v}{4}$ and $t+m<a<\frac{v}{2}$, $(\widetilde{M_0}\setminus\{0\})\cap(\widetilde{M_0}+(-1)^{i+1}\frac{v}{4})$ is empty. Thus, $(M,\frac{v}{2},M)$ is an SDC by Lemma 3.4. □

Lemma 3.6 *Let M be a DP on Z_v and $\frac{v}{2}\notin M$. If $\frac{v}{2}\notin\widetilde{M_0}$, $|\sum_{d\in M}d|\equiv\frac{v}{4}$ (mod v) and all elements of $abs(\widetilde{M_0})$ are distinct, then $(M,\frac{v}{2},M^{-1})$ is a DC. Furthermore, $(M,\frac{v}{2},M^{-1})$ is an SDC when $\frac{v}{4}$ is even and all differences in M are odd.*

Proof. Let $\widetilde{M_0}=(0,x_1,\cdots,x_m)$, where $m=|M|$ and $0<|x_i|<\frac{v}{2}$. Denote $D=(M,\frac{v}{2},M^{-1})$. Since $|x_m|=|\sum_{d\in M}d|=\frac{v}{4}$, it is easy to see that

$$\widetilde{D_0} = (0, x_1, x_2, \cdots, x_m, -x_m, -x_{m-1}, \cdots, -x_1)$$

is closed. Because all the elements in $abs(\widetilde{M_0})$ are distinct, D is indeed a DC. If $\frac{v}{4}$ is even and all the differences in M are odd, D is an SDC by the definition of SDC. □

Corollary 3.7 *Let a, b, m, t be odd and satisfy the following conditions: $1 \le t < t + 2m < a < \frac{v}{4}$, $b \in [1, \frac{v}{2} - 1] \setminus [t, t + 2m]_2$ and $a \neq b$. Let $T = A[t, t + 2m]_2$ and $M = (T, a, b)$. If $v \equiv 0 \pmod 8$, $|a + b + tail(\widetilde{T_0})| \equiv \frac{v}{4} \pmod v$, then $(M, \frac{v}{2}, M^{-1})$ is an SDC.*

Proof. From Table A we see that $tail(\widetilde{T_0}) = -(m + 1)$. Since a, b, t are all odd and satisfy the conditions, the elements in $abs(\widetilde{M_0})$ are all distinct. Then $(M, \frac{v}{2}, M^{-1})$ is an SDC by Lemma 3.6. □

Lemma 3.8
(1) *Let $s, t \ge 1$ and $v = (8s + 1)(8t + 7) + 1$. If $N_1 = A[1, 4t - 1]_2$, $N_2 = (4t + 2, A([2, 4t]_2^{-1}))$ and $N_3 = (4t + 1, \frac{v}{4} - 4t - 3)$, then $(N_1, \frac{v}{2}, (N_1 \setminus \{1\})^{-1}, N_2, N_3, 1, N_2^{-1}, N_3)$ is an SDC;*
(2) *Let $s \ge 0$, $t \ge 1$ and $v = (8s + 5)(8t + 3) + 1$. If $N_1 = A[1, 4t - 1]_2$ and $N_2 = (\frac{v}{4} + 4t, A[2, 4t]_2)$, then $(N_1, \frac{v}{2}, (N_1 \setminus \{1\})^{-1}, N_2, 1, N_2^{-1})$ is an SDC.*

Proof.
(1) Let $D = (N_1, \frac{v}{2}, (N_1 \setminus \{1\})^{-1}, N_2, N_3, 1, N_2^{-1}, N_3)$. Then $\widetilde{D_0} = [-2t, 0]_2 \cup [1, 2t - 1]_2 \cup [\frac{v}{2} - 4t + 1, \frac{v}{2} - 2t] \cup [\frac{v}{2} - 6t + 1, \frac{v}{2} - 4t - 1]_2 \cup [\frac{v}{2} - 2t + 1, \frac{v}{2} + 1]_2 \cup [-\frac{v}{4} - 4t, -\frac{v}{4} + 2]_2 \cup \{\frac{v}{2} + 2t + 2, -\frac{v}{4} - 2t - 1, -\frac{v}{4} + 4t + 3\}$ by Table A. It is easy to see that $|\sum_{d \in D} d| \equiv 0 \pmod v$ and all elements in $\widetilde{D_0}$ are distinct except for the head = the tail. So, D is a DC. By the definition of SDC, D is an SDC.
(2) Let $D = (N_1, \frac{v}{2}, (N_1 \setminus \{1\})^{-1}, N_2, 1, N_2^{-1})$. Then $\widetilde{D_0} = [-2t, 0]_2 \cup [1, 2t - 1]_2 \cup [\frac{v}{2} - 4t + 2, \frac{v}{2} - 2t]_2 \cup [\frac{v}{2} - 6t + 1, \frac{v}{2} - 4t - 1]_2 \cup [-\frac{v}{4} - 2t - 1, -\frac{v}{4} + 2t - 1]_2 \cup [-\frac{v}{4} - 6t, -\frac{v}{4} - 2t]_2$ by Table A. The rest of the proof is similar to (1). □

Let a, b, c, i be positive integers and b be even. Denote

$$U_i(a, b, c) = (A[a + b(i - 1), a + bi - 1], -(c + i), c - i, \tfrac{b}{2} + 2i);$$
$$V_i(a, b, c) = (A[a + b(i - 1), a + bi - 1], -(c + i), c - i + 1, \tfrac{b}{2} + 2i - 1).$$

Lemma 3.9 *Let a, b, c, s, i, v be positive integers and v, b be even. If the following conditions are satisfied: $c + \frac{b}{2} + s < \frac{v}{2}$, $c > 2s - 1$ and $0 < a < a + bs - 1 < \frac{v}{2}$, then U_i and V_i defined above are CDCs for any $1 \le i \le s$.*

Proof. Denote $T = A[a + b(i - 1), a + bi - 1]$. By Table A, $\{\widetilde{T_0}\} = [a + b(i - 1), a + bi - 1 - \frac{b}{2}] \cup [-\frac{b}{2}, 0]$ and $tail(\widetilde{T_0}) = -\frac{b}{2}$. Since $a + bs - 1 < \frac{v}{2}$, T is a DP on Z_v by Corollary 3.2. So, in the corresponding number tuple of U_i with head 0, all elements are distinct. Thereby, all the U_i (or V_i) are DCs on Z_v. By the definition of CDC, they are all CDCs. □

4 The Proof of Theorem 1.1

In this section, we will give several classes of constructions for a k-$SCMD(v) = (X, \mathcal{B}, f)$ for $v \equiv 0 \pmod 4$ and $v \equiv 1 \pmod k$, where the point set X is Z_v, the mapping f is $i \to i+1$ for $i \in Z_v$. Also, each block set $\mathcal{B}$ consists of one SDC and n CDCs, where $\frac{v(v-1)}{k} = (2n+1)v$ by Lemma 2.1. So, the number n of CDCs is $\frac{v-k-1}{2k}$. Furthermore, in order to verify the correctness of the given construction, we only need to show:

(1) Each given DC is an $SDC(D)$ or a $CDC(D)$, (using the conclusion in Section 3 or direct examination).

(2) The differences in all the DCs form a partition of $[1, \frac{v}{2}]$. (Note that, in an SDC, each difference except for $\frac{v}{2}$ appears twice and is calculated only once.)

Theorem 4.1 *There exists a $(4t+3)$-$SCMD((4s+1)(4t+3)+1)$ for positive integers s, t with the same parity.*

Construction. Let $v = (4s+1)(4t+3)+1$ and $X = Z_v$.

(I) $SDC(N,\ \frac{v}{2},\ N)$, where $N = (A[1, 2t], \frac{v}{4}+t)$.

(II) $CDC(D)$, where D is taken as follows.

(1) $U_i(2t+2s+1,\ 4t,\ \frac{v}{4}+t)$, $1 \le i \le s$;

(2) $V_i(\frac{v}{4}+3s+t+1,\ 4t,\ \frac{v}{4}+2s+t)$, $1 \le i \le s$.

Proof. Obviously, $\frac{v}{4}$ is odd for $v = (4s+1)(4t+3)+1$. By Corollary 3.5, (I) is an SDC. Moreover, the difference tuples in (II) are CDCs by Lemma 3.9. The differences in (I) and (II) form a partition of $[1, \frac{v}{2}]$. In addition, the number of blocks is $v + 2s \times v \times 2 = (4s+1)v$, as expected. □

Theorem 4.2 *There exists a $(4t+3)$-$SCMD((4s+1)(4t+3)+1)$ for odd integer $t \ge 3$ and even integer $s \ge 2$.*

Construction. Let $t = 2m+1$ and $s = 2n$. Then, the design will be $(8m+7)$-$SCMD(v)$, where $v = (8n+1)(8m+7)+1$.

(I) $SDC(N_1,\ \frac{v}{2},\ (N_1 \setminus \{1\})^{-1},\ N_2,\ N_3,\ 1,\ N_2^{-1},\ N_3)$,
where $N_1 = A[1, 4m-1]_2$, $N_2 = (4m+2, -A([2, 4m]_2^{-1}))$, $N_3 = (4m+1, \frac{v}{4} - 4m - 3)$.

(II) $CDC(D)$, where D is taken as follows.

(1) $U_i(4m+4n+3,\ 8m+4,\ \frac{v}{4} - 4m - 3)$, $1 \le i \le 2n-1$;

(2) $V_i(\frac{v}{4} + 6n + 2m + 2,\ 8m+4,\ \frac{v}{4} - 4m + 4n - 3)$, $1 \le i \le 2n$;

(3) $(A[\frac{v}{4} - 6m - 2n - 3, \frac{v}{4} - 4m - 2n - 4], A[\frac{v}{4} - 4m + 6n - 2, \frac{v}{4} + 2m + 6n + 1], -(\frac{v}{4} + 2n - 4m - 3), \frac{v}{4} - 4m - 2n - 3, 4m + 4n + 2)$.

Proof. By Lemma 3.8(1), (I) is an SDC. By Lemma 3.9 or direct examination, the difference tuples in (II) are CDCs. The differences in (I) and (II) form a partition of $[1, \frac{v}{2}]$. In addition, the number of blocks is $v + (1 + 2n - 1 + 2n) \times 2v = (8n+1)v$, as expected. For $4m - 1 \ge 1$, we need $m \ge 1$, i.e., $t \ge 3$. □

Theorem 4.3 *There exists a 7-$SCMD(56t+8)$ for any positive integer t.*

Construction.

(I) $SDC(1, -2, 28t+4, 1, 14t+3, 14t+3, -2)$.

(II) $CDC(D)$, where D is taken as follows.

(1) $U_i(4t+3,\ 4,\ 14t+3),\quad 1 \le i \le 2t$;

(2) $V_i(20t+4,\ 4,\ 18t+4),\quad 1 \le i \le 2t$.

Theorem 4.4 *There exists a $(4t+3)$-$SCMD((4s+1)(4t+3)+1)$ for odd integer $s \ge 1$ and even integer $t \ge 2$.*

Construction. Let $t = 2m$ and $s = 2n+1$. Then the design will be $(8m+3)$-$SCMD(v)$, where $v = (8n+5)(8m+3)+1$.

(I) $SDC(N_1, \frac{v}{2}, (N_1 \setminus \{1\})^{-1}, N_2, 1, N_2^{-1})$, where
$N_1 = A[1, 4m-1]_2$, $N_2 = (\frac{v}{4}+4m, A[2,4m]_2)$.

(II) $CDC(D)$, where D is taken as follows.

(1) $U_i(4m+4n+3,\ 8m,\ \frac{v}{4}+4m),\quad 1 \le i \le 2n+1$;

(2) $V_i(\frac{v}{4}+6n+4m+4, 8m, \frac{v}{4}+4m+4n+2)$, $1 \le i \le 2n$;

(3) $(A[a, a+2m-1], A[\frac{v}{2}-6m, \frac{v}{2}-1], -(b+2n+1),\ b-2n, 4m+4n+1)$,
where $a = \frac{v}{4}+2m-2n-1, b = \frac{v}{4}+4m+4n+2$.

Proof. Similar to the proof of Theorem 4.2. □

Theorem 4.5 *There exists a $(4t+3)$-$SCMD(4t+4)$ for any positive integer t.*

Construction.

(1) $t \equiv 0 \pmod 2$: $SDC(A[1, 2t+2], S[1, 2t]^{-1}, 2t+1)$.

(2) $t \equiv 3 \pmod 4$: $SDC(A[1, 2t+1]_2, 2t+2, A[3, 2t+1]_2^{-1}, M, 1, M^{-1})$,
where $M = (-2t, -(t+1), A[t+3, 2t-2]_2^{-1}, 2, A[4, t-1]_2^{-1})$.

(3) $t \equiv 1 \pmod 4$ and $t > 1$: $SDC(A[1, 2t+1]_2, 2t+2, -A[3, 2t+1]_2^{-1}, -2t, -(t+1),$
$A[4, 2t]_2^{-1}, 1, A[2, t-1]_2, -2, A[t+5, 2t-2]_2, -(t+3))$.

(4) $t = 1$: $SDC(1, -2, 4, 1, -2, 3, 3)$.

Theorem 4.6 *There exists a $(4t+1)$-$SCMD((4s+3)(4t+1)+1)$ for integers $s \ge 0$ and $t \ge 1$ with the same parity.*

Construction. Let $v = (4s+3)(4t+1)+1$.

(I) $SDC(N, \frac{v}{2}, N)$, where $N = (A[1, 2t-1], \frac{v}{4}-t)$.

(II) $CDC(D)$, where D is taken as follows.

(1) $U_i(2t+2s+1,\ 4t-2,\ \frac{v}{4}-t),\quad 1 \le i \le s$;

(2) $V_i(4st+4s+2t+4,\ 4t-2,\ 4st+3s+2t+2),\quad 1 \le i \le s+1$.

Proof. By Corollary 3.5, (I) is an SDC. By Lemma 3.9, all the difference tuples in (II) are CDCs. The differences in (I) and (II) form a partition of $[1, \frac{v}{2}]$. In addition, the number of blocks is $v + (2s+1) \times 2v = (4s+3)v$, as expected. □

Theorem 4.7 *There exists a $(4t+1)$-$SCMD((4s+3)(4t+1)+1)$ for odd integer $t \ge 3$ and even integer $s \ge 2$.*

Construction. First, we give the construction for $t = 3$ and $s = 2$, i.e., 13-$SCMD(144)$.

(1) $SDC(M, 72, M^{-1})$, where $M = (A[65, 71]_2, 35, 5)$;

(2) $CDC(A[6, 14] \setminus \{8\}, A[64, 70]_2, 8)$;

(3) $CDC(A[15, 24], 36, 3, -34)$;

(4) $CDC(A[25, 32], 38, -39, 37, 1, -33)$;

(5) $CDC(A[40,49], 62, 4, -61)$;

(6) $CDC(A[50,59], 63, 2, -60)$.

Then, let $t = 2m+1$ and $s = 2n$, where $m = 1,\ n \geq 2$ or $m > 1,\ n \geq 1$. The design will be $(8m+5)$-$SCMD(v)$, where $v = (8n+3)(8m+5)+1$.

(I) $SDC(N,\ \frac{v}{2},\ N^{-1})$.

(II) $CDC(D)$, where D is taken as follows.

(1) $(A[2, 8m]_2, a-1, -(a+1), A[c, c+4m-2]_2, 8m+3, -(8m+5), A[c+1, c+4m-1]_2, 8m+4)$;

(2) $(8m+1, -(8m+2), A([b-2m-2, b+2m+2] \setminus \{b\}), A[c+4m, c+4m+3], A[d-4m+9, d], -(c+4m+4), c-1, 8m+6)$;

(3) $(A[x_{1+(8m+2)(i-1)}, x_{(8m+2)i}], -(c+4m+4+i), c-1-i, 8m+2i+6)$, $1 \leq i \leq 2n-1$;

(4) $(A[p+(8m+2)(i-1), p-1+(8m+2)i], -(b+2m+2+i), b-2m-2-i, 8m+2i+5)$, $1 \leq i \leq 2n$,

where $a = 8m+4n+7, b = 16nm+6n+2m-3, c = 16nm+10n+4m, d = 16nm+12n+12m-5$, $p = 16nm+16n+12m+7, N = (A[1, 8m-1]_2, a, b)$, $M = [a+2, p-1] \setminus [b-2m-2n-2, d] = (x_1, x_2, \cdots, x_{|M|})$ and $x_i < x_{i+1}$ for $1 \leq i \leq |M|$.

Proof. Here, we only give the proof for $(t,s) \neq (3,2)$. Let $v = (8n+3)(8m+5)+1$. By Corollary 3.7, (I) is an SDC. By Lemma 3.9 or direct examination, the difference tuples in (II) are CDCs. The differences in (I) and (II) form a partition of $[1, \frac{v}{2}]$. In addition, the number of blocks is $v+(1+1+2n-u-1+1+u-1+2n)\times 2v = (8n+3)v$, as expected. It is easy to see that $|M| = (8m+2)(2n-1)$. Then, from the definition of M, we need $b-2m-2n-2 \geq a+2$, i.e., $16mn+4n-5 \geq 8m+4n+9$, which implies $n \geq \frac{4m+7}{8m}$. This inequality holds when $m = 1,\ n \geq 2$ or $m > 1,\ n \geq 1$, i.e., $t \geq 3,\ s \geq 2$ and $(s,t) \neq (2,3)$. But the construction for $(s,t) = (2,3)$ has been given above. □

Theorem 4.8 *There exists a $(8t+5)$-$SCMD(24t+16)$ for any positive integer t.*

Construction.

(I) $SDC(M, 12t+8, M^{-1})$, where
$M = (A[1, 4t-1]_2, A[8t+9, 12t+7]_2, -(8t+5), 6t+1)$;

(II) $CDC(A[2,4t]_2, A[8t+8, 12t+6]_2, -A([4t+1, 6t]^{-1}), -A([6t+3, 8t+7] \setminus \{8t+5\})^{-1}, 6t+2)$.

Theorem 4.9 *There exists a $(4t+1)$-$SCMD((4s+3)(4t+1)+1)$ for even integer $t \geq 4$ and odd integer $s \geq 5$.*

Construction. First, we give the construction for $t = 4$ and $s = 5$, i.e., 17-$SCMD(392)$.

(1) $SDC(M, 196, M^{-1})$, where $M = (1, -3, A[189, 195]_2, -31, -61)$;

(2) $CDC(2, -4, A[188, 194]_2, -A[5, 10]^{-1}, -A[42, 45]^{-1}, 11)$;

(3) $CDC(A[27, 30], A[51, 60], -46, 41, 12)$;

(4) $CDC(A[62+14(i-1), 61+14i], -(46+i), 41-i, 12+2i)$, $1 \leq i \leq 4$;

(5) $CDC(A[118+14(i-1), 117+14i], -(31+i), 27-i, 11+2i)$, $1 \leq i \leq 5$.

Then, let $t = 2m$ and $s = 2n+1$, where $m = 2,\ n \geq 3$ or $m > 2,\ n \geq 2$. The design will be $(8m+1)$-$SCMD(v)$, where $v = (8n+7)(8m+1)+1$.

(I) $SDC(N, \frac{v}{2}, N^{-1})$.

(II) $CDC(D)$, where D is taken as follows.

(1) $(A[2, 4m-4]_2, A[\frac{v}{2}-4m, \frac{v}{2}-2]_2, -A[4m-3, 6m-2]^{-1},$
$-A([6m, 8m-1]^{-1}), 6m-1)$;

(2) $(A([a-2m, a+2m] \setminus \{a\}), A[b, b+4m-3], -(d-1), 12m+8n+7, 8m+4n)$;

(3) $(A[x_{1+(8m-2)(i-1)}, x_{(8m-2)i}], -(c+i), c-4m-1-i, 8m+2i),\ 1 \leq i \leq 2n-1$;

(4) $(8m+4n+2, -(8m+4n+3), b-2, -(b-1), A[e, e+8m-7], -c, c-4m-1, 8m)$;

(5) $(A[p+(8m-2)(i-1), p-1+(8m-2)i], -(a+2m+i), a-2m-i, 8m+2i-1)$,
$1 \leq i \leq 2n+1$,

where $a = 10m+6n+5$, $b = 12m+10n+10$, $c = 16m+10n+8$, $d = 16m+12n+9$, $e = 16nm+8n+8m+12$, $p = 16nm+16m+8n+6$, $N = (A[1, 4m-5]_2, A[\frac{v}{2}-4m+1, \frac{v}{2}-1]_2, -a, -(16nm-4n-1))$, $M = [d, d+(8m-2)(2n-1)] \setminus \{16mn-4n-1\} = (x_1, x_2, \cdots, x_{|M|})$ and $x_i < x_{i+1}$ for $1 \leq i \leq |M|$.

Proof. Here, we only give the proof for $(t,s) \neq (4,5)$. By Lemma 3.6, (I) is an SDC. By Lemma 3.9 or direct examination, the difference tuples in (II) are CDCs. The differences in (I) and (II) form a partition of $[1, \frac{v}{2}]$. In addition, the number of blocks is $v + (1+1+2n-1+1+2n+1) \times 2v = (8n+7)v$, as expected. It is easy to see that $|M| = (8n-2)(2n-1)$. Then from the definition of M, we need $16mn-4n-1 \geq d$, i.e., $16mn-4n-1 \geq 16mn+12n+9$, which implies $n \geq 1+\frac{13}{8m-8}$. This inequality holds when $m = 2,\ n \geq 3$ or $m > 2,\ n \geq 2$, i.e., $t \geq 4,\ s \geq 5$ and $(s,t) \neq (5,4)$. But the construction for $(s,t) = (5,4)$ has been given above. □

Theorem 4.10 *There exists a $(8t+1)$-$SCMD(56t+8)$ for any positive integer t.*

Construction.

(1) $SDC(M, 28t+4, M^{-1})$, where $M = (A[6t+1, 14t-5]_2, 18t+1, 28t+3)$;

(2) $CDC(A[6t, 14t-6]_2, A[22t+1, 26t+2], 6t-1)$;

(3) $CDC(A[3, 6t-2], 14t-4, -(14t-3), A[26t+3, 28t+2], -18t, 22t, -1)$;

(4) $CDC(A[14t-1, 18t-2], A[18t+2, 22t-1], -(14t-2), 18t-1, -2)$.

Theorem 4.11 *There exists a $(8t+1)$-$SCMD(120t+16)$ for any integer $t \geq 2$.*

Construction.

(1) $SDC(M, 60t+8, M^{-1})$, where $M = (A[42t+7, 50t+1]_2, 34t+3, 60t+7)$;

(2) $CDC(A[38t+5, 42t+6], A[42t+8, 50t+2]_2, 6t-1)$;

(3) $CDC(A[6t+(8t-2)(i-1), 6t-1+(8t-2)i], -(34t+3-i), 38t+1+i,$
$-(2i-1)), 1 \leq i \leq 3$;

(4) $CDC(A[7, 6t-2], A[50t+3, 52t+8], -(30t-6), 34t-1, -6)$;

(5) $CDC(A[30t-3, 34t-4], A[34t+4, 38t+1], -(30t-5), 34t-2, -4)$;

(6) $CDC(A[52t+9, 60t+6], -(30t-4), 34t-3, -2)$.

Proof. In this construction, we need $6 < 6t-1$, i.e., $t \geq 2$. □

Theorem 4.12 *There exists a 9-$SCMD(72s+64)$ for any nonnegative integer s.*

Construction.

(I) $SDC(M, 36s+32, M^{-1})$, where $M = (1, -3, 18s+19, 36s+31)$.

(II) $CDC(D)$, where D is taken as follows

(1) $(2, -4, 6, -7, A([18s+17, 18s+21] \setminus \{18s+19\}), 5)$;

(2) $(A[4s+6i+4, 4s+6i+9], -(18s+21+i), 18s+17-i, 2i+7), 1 \le i \le 2s+1$;

(3) $(A[24s+6i+23, 24s+6i+28], -(22s+27+i), 22s+24-i, 2i+6), 1 \le i \le 2s$;

(4) $(A[22s+24, 22s+27], 36s+29, -(36s+30), -(24s+28), 20s+23, 4s+8)$.

The proof of Theorem 1.1:

According to the range of k and v, there are the following cases :

(1) $k = 4t+3, \quad v = (4s+1)(4t+3)+1, \ t > 0$.

If $s = 0$, see Theorem 4.5. If $s > 0$ and s, t have the same parity, see Theorem 4.1; otherwise see Theorem 4.2 ($t > 1$ odd), Theorem 4.3 ($t = 1$) and Theorem 4.4 (t even).

(2) $k = 4t+1, \quad v = (4s+3)(4t+1)+1, \ t > 1$.

If s, t have the same parity, see Theorem 4.6; if s is even and t is odd, see Theorem 4.7–4.8; if s is odd and t is even, see Theorem 4.11–4.12. □

5 The proof of Theorem 1.2

In this section, we will give several classes of constructions for a k-$SCMD(2k) = (X, \mathcal{B}, f)$, where the point set X is $\{\infty_1, \infty_2\} \cup (Z_{k-1} \times Z_2)$ and the mapping f is

$$(\infty_1)(\infty_2) \prod_{i \in Z_{k-1}} (i, \bar{i}).$$

For brevity, the points in $Z_{k-1} \times Z_2$ are denoted by $x = (x, 0)$ or $\bar{x} = (x, 1)$. The difference between points of $Z_{k-1} \times \{0\}$ (or of $Z_{k-1} \times \{1\}$) is said to be *pure*, and is denoted by d (or $\bar{d}$). The difference between x and $\bar{x}$ is said to be *mixed*, and is denoted by d_{01} (for the ordered pairs $(x, \overline{x+d})$) or d_{10} (for the ordered pairs $(\bar{x}, x+d)$). Define:

$\overline{A}[a, a+d] = (\bar{a}, -(\overline{a+1}), \cdots, (-1)^d(\overline{a+d}))$;

$MA[a, a+d] = (a_{01}, -(a+1)_{10}, \cdots, (-1)^d(a+d)_{ij})$, where $(i, j) = (1, 0)$ for odd d, or $(0, 1)$ for even d;

$\overline{M}A[a, a+d] = (a_{10}, -(a+1)_{01}, \cdots, (-1)^d(a+d)_{ij})$, where $(i, j) = (0, 1)$ for odd d, or $(1, 0)$ for even d.

And, define the following block families:

$SDC_\infty(M) = (M, 0_{ij}, -\overline{M}^{-1})$=$D$, where $i = 0$ or 1 follows $tail(\widetilde{M_0}) \in Z_{k-1} \times \{0\}$ or $Z_{k-1} \times \{1\}$. The corresponding number tuple family is $\{(\infty, \widetilde{D_a})$; $a \in Z_{k-1} \times \{t\}\}$, where t is determined by the first difference in M;

$CDC_\infty(M)$, where M is a DP, the head and the tail of $\widetilde{M_a}$ belong to different point-orbits, and $\sum_{d \in M} d$ is odd. The corresponding number tuple family is $\{(\infty, \widetilde{M_a})$; $a \in Z_{k-1} \times \{t\}\}$, where t is determined by the first difference in M.

For convenience, the subscript a in the above number tuple families is called the *starter*. Note that the terminology SDC_∞ (and CDC_∞) excludes DC, since $\widetilde{M_a}$ cannot be closed. Under the mapping f in this section, there are $k-1$ block-orbits corresponding to each SDC_∞ and $\frac{k-1}{2}$ block-orbits corresponding to each CDC_∞. There is only one self-converse block in each block-orbit of $SDC_\infty(M)$. But there are two blocks in each block-orbit of $CDC_\infty(M)$ of which one is the f-converse of the other. Briefly, we gather all the blocks in these orbits and called them the *block family* corresponding to $SDC_\infty(M)$ or $CDC_\infty(M)$. The following Lemmas 5.1–5.2 are obvious.

Lemma 5.1 *Let t be a positive integer. The point set is $X = \{\infty_1, \infty_2\} \cup (Z_{4t+2} \times Z_2)$ and the mapping is $f = (\infty_1)(\infty_2)\prod_{i=0}^{4t+1}(i, \bar{i})$. Then the following DCs are both SDC_∞s and the blocks in the block family cover each pair with pure difference in $[1, 2t]$ and in $[\bar{1}, \overline{2t}]$ exactly once.*

(1) *$SDC_{\infty_1}(A[1, 2t])$ with starters in $[0, 4t+1]$;*

(2) *$SDC_{\infty_2}(\overline{A}[1, 2t])$ with starters in $[\bar{0}, \overline{4t+1}]$.*

Lemma 5.2 *Let t be a positive integer. The point set is $X = \{\infty_1, \infty_2\} \cup (Z_{4t} \times Z_2)$ and the mapping is $f = (\infty_1)(\infty_2)\prod_{i=0}^{4t-1}(i, \bar{i})$. Then, the following DCs in (1)–(4) are all SDC_∞s and the blocks in the block family cover each pair with pure difference in $[1, 2t] \setminus \{2t-1\}$ and $[\bar{1}, \overline{2t}] \setminus \{\overline{2t-1}\}$ exactly once.*

(1) *$SDC_{\infty_1}((2t)_{01}, \overline{A}[1, 2t-2])$ with starters in $[0, 2t-1]$;*

(2) *$SDC_{\infty_2}(\overline{2t}, \overline{A}[1, 2t-2])$ with starters in $[\overline{2t}, \overline{4t-1}]$;*

(3) *$SDC_{\infty_1}(2t, A[1, 2t-2])$ with starters in $[2t, 4t-1]$;*

(4) *$SDC_{\infty_2}((2t)_{10}, A[1, 2t-2])$ with starters in $[\bar{0}, \overline{2t-1}]$.*

Note In the following theorems, the mentioned replacement for a block should be done for its f-converse as well.

Theorem 5.3 *There exists an $(8t+3)$-$SCMD(16t+6)$ for any positive integer t.*

Construction. $X = \{\infty_1, \infty_2\} \cup (Z_{8t+2} \times Z_2)$, $f = (\infty_1)(\infty_2)\prod_{i=0}^{8t+1}(i, \bar{i})$.

(1) $SDC_{\infty_2}(\overline{A}[1, 4t])$ with starters in $[\bar{0}, \overline{8t+1}]$, where the block with starter $\overline{4t}$, i.e., $\langle \infty_2, \overline{4t}, \cdots \rangle$, is replaced by $\langle \infty_1, \overline{4t}, \cdots \rangle$;

(2) $SDC_{\infty_1}(A[1, 4t])$ with starters in $[0, 8t+1]$;

(3) $CDC_{\infty_1}(\overline{M}A[1, 4t], \overline{4t+1}, -\overline{M}A[2, 4t]^{-1}, 4t+1)$ with starters in $[\bar{1}, \overline{8t+1}]_2$, where the block with starter $\overline{8t+1}$, i.e., $\langle \infty_1, \overline{8t+1}, \cdots, \bar{1}, 8t+1, 4t \rangle$, is replaced by $B = \langle \infty_1, \overline{8t+1}, \cdots, \bar{1}, 4t, \infty_2 \rangle$;

(4) $CDC_{\infty_2}((4t+1)_{01}, \overline{M}A[1, 4t-2], (4t+1)_{10}, (-1)_{01}, -\overline{M}A[2, 4t]^{-1}, -(4t)_{01})$ with starters in $[1, 8t+1]_2$;

(5) $C = \langle 8t+1, a_0, b_0, a_1, b_1, \cdots, a_{4t}, b_{4t} \rangle$, where $a_i = 4(i+1)t$, $b_i = \overline{4(i+1)t+1}$, $0 \le i \le 4t$ and all a_i, b_i are in Z_{8t+2}.

Proof. The number of the blocks is $(8t+2) + (8t+2) + (4t + 4t + 1 + 1) \times 2 = \frac{(16t+6)(16t+5)}{8t+3}$, as expected.

By Lemma 5.1, (1) and (2) are both SDC_∞s. By direct checking, (3) and (4) are both CDC_∞s. Obviously, if $d \ne 1_{01}$ and $(4t-1)_{10}$, each pair with difference $(\pm d)_{01}, (\pm d)_{10}, \pm d, \pm\bar{d}$ appears exactly once in (1)–(6) except for the pairs

$(\bar{1}, 8t+1)$,$(8t+1, 4t)$,$(\overline{4t}, \overline{8t+1})$ and $(\overline{8t+1}, 1)$. If $d = 1_{01}$ or $(4t-1)_{10}$, each pair with difference d appears exactly once in (1)–(6) except for the pairs in the set $S = \{(i, \overline{i+1}), (i+1, \bar{i});\ i \in [0, 8t]_2\} \cup \{(\bar{i}, i+4t-1), (\overline{i+4t-1}, i);\ i \in [3, 8t+1]_2\}$. Since $\gcd(1+4t-1, 8t+2) = 2$, all a_i, b_i in the construction can form directed cycle $D = \langle a_0, b_0, \cdots, a_{4t}, b_{4t}\rangle$ and its f-converse. Let $C = \langle a_0, b_0, \cdots, a_{4t}, b_{4t}, 8t+1\rangle$. Note that $(b_{4t}, 8t+1, a_0) = (\bar{1}, 8t+1, 4t)$. Then, obviously, C and its f-converse cover all pairs in S and the pairs $(\bar{1}, 8t+1)$,$(8t+1, 4t)$,$(\overline{4t}, \overline{8t+1})$ and $(\overline{8t+1}, 1)$.

From the definition of SDC_∞ and CDC_∞, we can see that the construction is an $(8t+3)$-$SCMD(16t+6)$. □

Theorem 5.4 *There exists an* $(8t+7)$-$SCMD(16t+14)$ *for any positive integer* t.

Construction. $X = \{\infty_1, \infty_2\} \cup (Z_{8t+6} \times Z_2)$, $f = (\infty_1)(\infty_2)\prod_{i=0}^{8t+5}(i, \bar{i})$.

(1) $SDC_{\infty_1}(A[1, 4t+2])$ with starters in $[0, 8t+5]$;

(2) $SDC_{\infty_2}(\overline{A}[1, 4t+2])$ with starters in $[\bar{0}, \overline{8t+5}]$, where the block with starter $\overline{4t+2}$, i.e., $\langle\infty_2, \overline{4t+2}, \cdots\rangle$, is replaced by $\langle\infty_1, \overline{4t+2}, \cdots\rangle$;

(3) $CDC_{\infty_1}(\overline{4t+3}, \overline{M}A[1, 4t+2], -\overline{M}A[2, 4t+2]^{-1}, 4t+3)$ with starters in $[\bar{1}, \overline{8t+5}]_2$, where the block with starter $\overline{8t+5}$, i.e., $\langle\infty_1, \overline{8t+5}, \cdots, \bar{1}, 8t+5, 4t+2\rangle$, is replaced by $B = \langle\infty_1, \overline{8t+5}, \cdots, \bar{1}, 4t+2, \infty_2\rangle$;

(4) $CDC_{\infty_2}(1_{01}, -\overline{M}A[2, 4t+2]^{-1}, -(4t+2)_{01}, (4t+3)_{10}, MA[3, 4t]^{-1}, (4t+3)_{01}, (-1)_{10}, 2_{01})$ with starters in $[1, 8t+5]_2$;

(5) $C = \langle 8t+5, a_0, b_0, a_1, b_1, \cdots, a_{4t+2}, b_{4t+2}\rangle$, where a_i, $b_i \in Z_{8t+6}$, $a_i = (i+1)(4t+2)$, $b_i = \overline{(i+1)(4t+2)+1}$, $0 \le i \le 4t+2$.

Proof. Similar to the proof of Theorem 5.3. □

Theorem 5.5 *There exists a* 7-$SCMD(14)$.

Construction. $X = \{\infty_1, \infty_2\} \cup (Z_6 \times Z_2)$, $f = (\infty_1)(\infty_2)\prod_{i=0}^{5}(i, \bar{i})$.

(1) $SDC_{\infty_1}(A[1, 2])$ with starters in $[0, 5]$;

(2) $SDC_{\infty_2}(\overline{A}[1, 2])$ with starters in $[\bar{0}, \bar{5}]$, where the block with starter $\bar{2}$, i.e., $\langle\infty_2, \bar{2}, \bar{3}, \bar{1}, 1, 3, 2\rangle$, is replaced by $\langle\infty_1, \bar{2}, \bar{3}, \bar{1}, 1, 3, 2\rangle$;

(3) $CDC_{\infty_1}(\bar{3}, 1_{10}, (-2)_{01}, (-2)_{10}, 3)$ with starters in $[\bar{1}, \bar{5}]_2$, where the block with starter $\bar{5}$, i.e., $\langle\infty_1, \bar{5}, \bar{2}, 3, \bar{1}, 5, 2\rangle$, is replaced by $B = \langle\infty_1, \bar{5}, \bar{2}, 3, \bar{1}, 2, \infty_2\rangle$;

(4) $CDC_{\infty_2}(3_{01}, 2_{10}, 2_{01}, 3_{10}, 1_{01})$ with starters in $[1, 5]_2$;

(5) $C = \langle 5, 2, \bar{3}, 4, \bar{5}, 0, \bar{1}\rangle$.

Theorem 5.6 *There exists an* $(8t+5)$-$SCMD(16t+10)$ *for any positive integer* $t \ge 3$.

Construction. $X = \{\infty_1, \infty_2\} \cup (Z_{8t+4} \times Z_2)$, $f = (\infty_1)(\infty_2)\prod_{i=0}^{8t+3}(i, \bar{i})$.

(1) $SDC_{\infty_1}((4t+2)_{01}, \overline{A}[1, 4t])$ with starters in $[0, 4t+1]$, where the block with starter 0, i.e., $\langle\infty_1, 0, \cdots\rangle$, is replaced by $\langle\infty_2, 0, \cdots\rangle$;

(2) $SDC_{\infty_1}(4t+2, A[1, 4t])$ with starters in $[4t+2, 8t+3]$;

(3) $SDC_{\infty_2}((4t+2)_{10}, A[1, 4t])$ with starters in $[\bar{0}, \overline{4t+1}]$;

(4) $SDC_{\infty_2}(\overline{4t+2}, \overline{A}[1, 4t])$ with starters in $[\overline{4t+2}, \overline{8t+3}]$;

(5) $CDC_{\infty_2}((4t-2)_{01}, \overline{-(4t+1)}, -\overline{M}A^{-1}[1, 4t+1], -MA[1, 4t-2], -(4t+1), \overline{-(4t-1)_{01}})$ with starters in $[1, 8t+3]_2$, where the block with starter 1, i.e., $\langle\infty_2, 1, \overline{4t-1}, \overline{8t+2}, \cdots\rangle$, is replaced by $B = \langle\infty_1, \infty_2, 1, \overline{8t+2}, \cdots\rangle$;

(6) $CDC_{\infty_1}(\overline{M}A^{-1}[2,4t-3], -\overline{M}A^{-1}[4t-1,4t+1], -(4t+1), MA^{-1}[4t-1, 4t+1], \overline{M}A[4,4t-2], 1_{01}, -\overline{4t+1}, 2_{10}, (4t+1)_{01}, -(4t)_{10})$ with starters in $[\bar{1}, \overline{8t+3}]_2$;

(7) $C = \langle \overline{4t-1}, a_0, b_0, a_1, b_1, \cdots, a_{4t+1}, b_{4t+1}\rangle$, where $a_i = \overline{-2(i+1)}$, $b_i = -2(i+1)+1$, $0 \le i \le 4t+1$ and a_i, b_i are in Z_{8t+4}.

Proof. Similar to the proof of Theorem 5.3. □

Theorem 5.7 *There exists a* $13\text{-}SCMD(26)$.

Construction. $X = \{\infty_1, \infty_2\} \cup (Z_{12} \times Z_2)$, $f = (\infty_1)(\infty_2)\prod_{i=0}^{11}(i, \bar{i})$.

(1) $SDC_{\infty_1}(6_{01}, \overline{A}[1,4])$ with starters in $[0,5]$, where the block with starter 0, i.e., $\langle \infty_1, 0, \cdots\rangle$, is replaced by $\langle \infty_2, 0, \cdots\rangle$;

(2) $SDC_{\infty_1}(6, A[1,4])$ with starters in $[6,11]$;

(3) $SDC_{\infty_2}(6_{10}, A[1,4])$ with starters in $[\bar{0}, \bar{5}]$;

(4) $SDC_{\infty_2}(\bar{6}, \overline{A}[1,4])$ with starters in $[\bar{6}, \overline{11}]$;

(5) $CDC_{\infty_2}(2_{01}, \overline{-5}, -\overline{M}A^{-1}[1,5], -5, (-3)_{01}, 4_{10}, (-1)_{01})$ with starters in $[1,11]_2$, where the block with starter 11, i.e., $\langle \infty_2, 11, \bar{1}, \bar{8}, \cdots\rangle$, is replaced by $B = \langle \infty_1, \infty_2, 11, \bar{8}, \cdots\rangle$;

(6) $CDC_{\infty_1}(2_{10}, (-5)_{01}, 1_{10}, 5_{01}, (-4)_{10}, (-4)_{01}, \overline{-5}, 3_{10}, -5, 1_{01}, 2_{10})$ with starters in $[\bar{1}, \overline{11}]_2$;

(7) $C = \langle \bar{1}, \bar{8}, 1, \overline{10}, 3, \bar{0}, 5, \bar{2}, 7, \bar{4}, 9, \bar{6}, 11\rangle$.

Theorem 5.8 *There exists a* $21\text{-}SCMD(42)$.

Construction. $X = \{\infty_1, \infty_2\} \cup (Z_{20} \times Z_2)$, $f = (\infty_1)(\infty_2)\prod_{i=0}^{19}(i, \bar{i})$.

(1) $SDC_{\infty_1}(10_{01}, \overline{A}[1,8])$ with starters in $[0,9]$, where the block with starter 0, i.e., $\langle \infty_1, 0, \cdots\rangle$, is replaced by $\langle \infty_2, 0, \cdots\rangle$;

(2) $SDC_{\infty_1}(10, A[1,8])$ with starters in $[10,19]$;

(3) $SDC_{\infty_2}(10_{10}, A[1,8])$ with starters in $[\bar{0}, \bar{9}]$;

(4) $SDC_{\infty_2}(\overline{10}, \overline{A}[1,8])$ with starters in $[\overline{10}, \overline{19}]$;

(5) $CDC_{\infty_2}(6_{01}, \overline{-9}, -\overline{M}A^{-1}[1,9], -MA[1,6], -9, (-7)_{01})$ with starters in $[1,19]_2$, where the block with starter 1, i.e., $\langle \infty_2, 1, \bar{7}, \overline{18}, \cdots\rangle$, is replaced by $B = \langle \infty_1, \infty_2, 1, \overline{18}, \cdots\rangle$;

(6) $CDC_{\infty_1}(\overline{M}A^{-1}[2,5], -\overline{M}A^{-1}[7,9], -9, -\overline{M}A[7,9], \overline{-9}, 2_{10}, 1_{01}, \overline{M}A[4,6], 9_{01}, (-8)_{10})$ with starters in $[\bar{1}, \overline{19}]_2$;

(7) $C = \langle \bar{7}, a_0, b_0, a_1, b_1, \cdots, a_4, b_4\rangle$, where $a_i = \overline{-2(i+1)}$, $b_i = -2(i+1)+1$, $0 \le i \le 4$ and all a_i, b_i are in Z_{20}.

Theorem 5.9 *There exists an* $(8t+1)\text{-}SCMD(16t+2)$ *for any integer* $t \ge 2$.

Construction. $X = \{\infty_1, \infty_2\} \cup (Z_{8t} \times Z_2)$, $f = (\infty_1)(\infty_2)\prod_{i=0}^{8t-1}(i, \bar{i})$.

(1) $SDC_{\infty_1}((4t)_{01}, \overline{A}[1,4t-2])$ with starters in $[0, 4t-1]$;

(2) $SDC_{\infty_1}(4t, A[1,4t-2])$ with starters in $[4t, 8t-1]$, where the block with starter $4t$, i.e., $\langle \infty_1, 4t, \cdots\rangle$, is replaced by $\langle \infty_2, 4t, \cdots\rangle$;

(3) $SDC_{\infty_2}((4t)_{10}, A[1,4t-2])$ with starters in $[\bar{0}, \overline{4t-1}]$;

(4) $SDC_{\infty_2}(\overline{4t}, \overline{A}[1,4t-2])$ with starters in $[\overline{4t}, \overline{8t-1}]$;

(5) $CDC_{\infty_2}((4t-4)_{01}, \overline{-(4t-1)}, \overline{M}A^{-1}[1,4t-1], 1_{01}, \overline{4t-1}, \overline{M}A[2,4t-3])$ with starters in $[1, 8t-1]_2$, where the block with starter 1, i.e., $\langle \infty_2, 1, \overline{4t-3}, \overline{8t-2}, \cdots\rangle$, is replaced by $B = \langle \infty_1, \infty_2, 1, \overline{8t-2}, \cdots\rangle$;

(6) $CDC_{\infty_1}(-\overline{M}A^{-1}[2,4t-5], -\overline{M}A[4t-2,4t-1], -\overline{M}A[4t-2,4t-1], -(\overline{4t-1}), (-2)_{10}, 1_{01}, \overline{M}A[4,4t-3], (4t-1)_{10}, (4t-2)_{01}, \overline{4t-1}, -(4t-3)_{10})$ with starters in $[\bar{1}, \overline{8t-1}]_2$;

(7) $C = \langle \overline{4t-3}, a_0, b_0, a_1, b_1, \cdots, a_{4t-1}, b_{4t-1}\rangle$, where $a_i = \overline{-2(i+1)}$, $b_i = -2(i+1)+1$, $0 \le i \le 4t-1$ and all a_i, b_i are in Z_{8t}.

Proof. In the construction, we need $4t-4>0$, i.e., $t \ge 2$. The rest of the proof is similar to Theorem 5.3. □

Theorem 5.10 There exists a 9-$SCMD(18)$.

Construction. $X = \{\infty_1, \infty_2\} \cup (Z_8 \times Z_2)$, $f = (\infty_1)(\infty_2)\prod_{i=0}^{7}(i, \bar{i})$.

(1) $SDC_{\infty_1}(4_{01}, \overline{A}[1,2])$ with starters in $[0,3]$, where the block with starter 2, i.e., $\langle \infty_1, 2, \cdots\rangle$, is replaced by $\langle \infty_2, 2, \cdots\rangle$;

(2) $SDC_{\infty_1}(4, A[1,2])$ with starters in $[4,7]$;

(3) $SDC_{\infty_2}(4_{10}, A[1,2])$ with starters in $[\bar{0}, \bar{3}]$;

(4) $SDC_{\infty_2}(\bar{4}, \overline{A}[1,2])$ with starters in $[\bar{4}, \bar{7}]$;

(5) $CDC_{\infty_2}(2_{01}, \overline{-3}, 3_{10}, 1_{01}, (-2)_{10}, (-3)_{01}, \bar{3})$ with starters in $[1,7]_2$, where the block with starter 1, i.e., $\langle \infty_2, 1, \bar{3}, \bar{0}, 3, \bar{4}, 2, \bar{7}, \bar{2}\rangle$, is replaced by $B = \langle \infty_1, \infty_2, 1, \bar{0}, 3, \bar{4}, 2, \bar{7}, \bar{2}\rangle$;

(6) $CDC_{\infty_1}((-3)_{10}, 2_{01}, \bar{3}, (-1)_{10}, 3_{01}, \overline{-3}, (-2)_{10})$ with starters in $[\bar{1}, \bar{7}]_2$;

(7) $C = \langle 1, \bar{3}, \bar{0}, 7, \bar{6}, 5, \bar{4}, 3, \bar{2}\rangle$.

The proof of Theorem 1.2:

Let $k = 2t+1$ $(t > 2)$. According to the value of t modulo 4 we have following cases:

If $t \equiv 1 \pmod 4$, see Theorem 5.3;
If $t \equiv 3 \pmod 4$, see Theorem 5.4 and Theorem 5.5;
If $t \equiv 2 \pmod 4$, see Theorem 5.6–5.8;
If $t \equiv 0 \pmod 4$, see Theorem 5.9 and Theorem 5.10. □

6 The proof of Theorem 1.3

Let $D = (d_1, d_2, \cdots, d_k)$ be a CDC. If there are $d_i, d_{i+1}, d_{i+2}, d_{i+3} \in D$ such that $d_i \equiv d_{i+2}$, $d_{i+1} \equiv d_{i+3}$ and $d_i \not\equiv d_{i+1} \pmod 2$, then the CDC is said to be of *ALT-type* and $(d_i, d_{i+1}, d_{i+2}, d_{i+3})$ is called the *ALT-piece.*

Lemma 6.1 *Among the CDCs of each construction given in Section 4, there is at least one ALT-type CDC.*

Proof. In section 4, we need to investigate all the constructions except for Theorem 4.5 in which there is no CDC. In fact we can point out the following ALT-type CDCs.

Theorem 4.1—$U_1(2t+2s+1, 4t, \frac{v}{4}+t)$ (for $t \ge 1$);
Theorem 4.2—$U_1(4m+4n+3, 8m+4, \frac{v}{4}-4m-3)$ (for $m \ge 1$);
Theorem 4.3—$U_1(4t+3, 4, 14t+3)$ (for $t \ge 1$);
Theorem 4.4—$U_1(4m+4n+3, 8m, \frac{v}{4}+4m)$ (for $m \ge 1$);

Theorem 4.6—$U_1(2t+2s+1, 4t-2, \frac{v}{4}-t)$ (for $t \geq 1$);

Theorem 4.7—(3) in the first construction,

the difference cycle with $i=1$ in part(II) (4) (for $m \geq 1$) in the second construction;

Theorem 4.8—the only CDC (for $t \geq 1$, so there is an ALT-piece in $-A([6t+3, 8t+7] \setminus \{8t+5\})^{-1}$);

Theorem 4.9—(3) in the first construction,

the difference cycle with $i=1$ in part(II) (5) (for $m \geq 2$) in the second construction;

Theorem 4.10—(2) (for $t \geq 1$);

Theorem 4.11—(1) (for $t \geq 2$);

Theorem 4.12—part(II) (2). □

Theorem 6.2 *For odd $k > 5$, let $v \equiv 1 \pmod{k}$, $v \neq k+1$ and $v \equiv 0 \pmod{4}$. If there exist a k-$SCMD(v)$ with ALT-type CDCs and a k-$SCMD(2k)$, then there exists a k-$SCMD(v+2k)$.*

Proof. Let $k = 2t+1$ and $t > 2$. Let $(X, \mathcal{B}, g)$ be a $(2t+1)$-$SCMD(v)$ with an ALT-type CDC and $(Y, \mathcal{C}, h)$ be a $(2t+1)$-$SCMD(4t+2)$, where $X = Z_v$, $g = (0, 1, \cdots, v-1)$, $Y = \{a_i, \overline{a_i};\ 1 \leq i \leq 2t\} \cup \{\infty_1, \infty_2\}$, $h = (\infty_1)(\infty_2)\prod_{i=1}^{2t}(a_i, \overline{a_i})$.

Let $D = (d_1, d_2, d_3, d_4, \cdots, d_{2t+1})$, where $(d_1,\ d_2,\ d_3\ , d_4)$ is an ALT-piece. And, $B = \langle x_1, x_2, x_3, x_4, x_5, \cdots, x_{2t+1}\rangle$ is a block in $\mathcal{B}$, where $x_i + d_i \equiv x_{i+1} \pmod{v}$, $i \in Z_{2t+1}$. Let $O(B)$ be the block-orbit containing B. For expressing the parity of all these x_i (suppose x_1 is even), we give the following table.

Table B

	case I	case II
(d_1, d_2, d_3, d_4)	even, odd, even, odd	odd, even, odd, even
$(x_1, x_2, x_3, x_4, x_5)$	even, even, odd, odd, even	even, odd, odd, even, even

Define the following five basic blocks:

$B_1 = \langle x_1, x_2, a_1, y_1, a_2, \cdots, y_{t-2}, a_{t-1}, y_{t-1}, a_t\rangle$;

$B_2 = \langle x_3, x_4, a_1, y_1+1, a_2, \cdots, y_{t-2}+1, a_{t-1}, y_{t-1}+1, a_t\rangle$;

$B_3 = \langle x_2, x_3, a_{t+1}, y_1, a_{t+2}, \cdots, y_{t-2}, a_{2t-1}, y_{t-1}, a_{2t}\rangle$;

$B_4 = \langle x_4, x_5, a_{t+1}, y_1+1, a_{t+2}, \cdots, y_{t-2}+1, a_{2t-1}, y_{t-1}+1, a_{2t}\rangle$;

$B_5 = \langle \infty_1, z, \infty_2, x_5, x_6, \cdots, x_{2t+1}, x_1\rangle$,

where $z = x_2$ (case I) or x_4 (case II) and these y_j, $1 \leq j \leq t-1$, are distinct elements from the set

$$\{2i;\ 0 \leq i \leq \tfrac{v-2}{2}\} \setminus \{x_1, x_2, x_3, x_3-1, x_4-1, x_5-1\}.$$

Since $\frac{v}{2} - 6 > t-1$, the required y_j can indeed be chosen. Obviously, the points in each B_i are mutually distinct. By Table B, in both cases, we have $x_1 \not\equiv x_3$, $x_2 \not\equiv x_4$, $x_3 \not\equiv x_5 \pmod{2}$ and $x_1 \equiv x_5 \equiv z \pmod{2}$. Therefore, each a_i appears in two basic blocks above, e.g. $\langle \cdots u, a_i, v, \cdots\rangle$ and $\langle \cdots u', a_i, v', \cdots\rangle$, such that $u \not\equiv u'$ and $v \not\equiv v' \pmod{2}$.

Define the mapping f on $X \cup Y$ as follows:

$$f(x) = \begin{cases} g(x); & x \in X \\ h(x); & x \in Y \end{cases}.$$

Let G be the finite permutation group generated by Rf and let $O(B_i)$ $(1 \leq i \leq 5)$ be the block-orbit containing each B_i under the action of G. Let

$$\mathcal{J} = (\mathcal{B} \backslash O(B)) \cup \mathcal{C} \cup (\bigcup_{i=1}^{5} O(B_i)).$$

Obviously, $(X \cup Y, \mathcal{J}, f)$ is a k-$SCMD(v+2k)$. □

Theorem 6.3 *There exists a* $(4t+3)$-$SCMD(12t+10)$ *for any positive integer* t.

Proof. Let $(X, \mathcal{B}, g)$ be a $(4t+3)$-$SCMD(4t+4)$ as given in Theorem 4.5, where B contains only an SDC, and let $(Y, \mathcal{C}, h)$ be a $(4t+3)$-$SCMD(8t+6)$ as given in Theorem 1.2. To avoid confusion, denote $Y = \{\infty_1, \infty_2\} \cup \{a_i, b_i;\ i \in Z_{4t+2}\}$ and $h = (\infty_1)(\infty_2) \prod_{i=0}^{4t+1} (a_i, b_i)$. Now, we construct a $(4t+3)$-$SCMD(12t+10)$ on $X \cup Y$. Define the mapping f by

$$f(x) = \begin{cases} g(x) & x \in X \\ h(x) & x \in Y \end{cases}.$$

We define five basic blocks $B_1, B_2, \cdots, B_5$ in three cases:

(1) t is even.

Let $D = SDC(A[1, 2t+2], S[1, 2t]^{-1}, 2t+1)$. There is an ALT-piece $(d_1,\ d_2,\ d_3,\ d_4) = (1, -2, 3, -4)$ in $A[1, 2t+2]$. Let $B = \langle 0, 1, 4t+3, 2, 4t+2,\ x_1, \cdots, x_{4t-2} \rangle$. Define the following five basic blocks:

$B_1 = \langle 0, 1, a_0, y_0, a_1, y_1, \cdots, a_{2t-1}, y_{2t-1}, a_{2t} \rangle$,
$B_2 = \langle 4t+3, 2, a_0, y_0+1, a_1, y_1+1, \cdots, a_{2t-1}, y_{2t-1}+1, a_{2t} \rangle$,
$B_3 = \langle 1, 4t+3, a_{2t+1}, y_0, a_{2t+2}, y_1, \cdots, a_{4t}, y_{2t-1}, a_{4t+1} \rangle$,
$B_4 = \langle 2, 4t+2, a_{2t+1}, y_0+1, a_{2t+2}, y_1+1 \cdots, a_{4t}, y_{2t-1}+1, a_{4t+1} \rangle$,
$B_5 = \langle \infty_1, 2, \infty_2, 4t+2, x_1, x_2, \cdots, x_{4t-2}, 0 \rangle$,

where $(y_0,\ y_1, \cdots, y_{2t-1}) = (2, 3, \cdots, 2t+1)$.

(2) t is odd and $t > 1$.

Now, the two SDCs both contain the interval $A[1, 2t+1]_2$. For $t \geq 3$, there is an ALT-piece $(d_1,\ d_2,\ d_3,\ d_4) = (1, -3, 5, -7)$ in $A[1, 2t+1]_2$. Let $B = \langle 0, 1, 4t+2, 3, 4t, x_1, \cdots, x_{4t-2} \rangle$. Define the following five basic blocks:

$B_1 = \langle 0, 1, a_0, y_0, a_1, y_1, \cdots, a_{2t-1}, y_{2t-1}, a_{2t} \rangle$,
$B_2 = \langle 3, 4t, a_0, y_0+1, a_1, y_1+1, \cdots, a_{2t-1}, y_{2t-1}+1, a_{2t} \rangle$,
$B_3 = \langle 1, 4t+2, a_{2t+1}, y_0, a_{2t+2}, y_1, \cdots, a_{4t}, y_{2t-1}, a_{4t+1} \rangle$,
$B_4 = \langle 4t+2, 3, a_{2t+1}, y_0+1, a_{2t+2}, y_1+1 \cdots, a_{4t}, y_{2t-1}+1, a_{4t+1} \rangle$,
$B_5 = \langle \infty_1, 4t+2, \infty_2, 4t, x_1, x_2, \cdots, x_{4t-2}, 0 \rangle$,

where $(y_0,\ y_2, \cdots, y_{2t-1}) = (4, 5, \cdots, 2t+3)$.

(3) $t = 1$.

Let $D = SDC(1, -2, 4, 1, -2, 3, 3)$ and $B = \langle 0, 1, 7, 3, 4, 2, 5 \rangle$. Define the following five basic blocks:

$B_1 = \langle 7, 3, a_0, 2, a_1, 6, a_2 \rangle$;
$B_2 = \langle 4, 2, a_0, 3, a_1, 7, a_2 \rangle$;
$B_3 = \langle 3, 4, a_3, 2, a_4, 6, a_5 \rangle$;
$B_4 = \langle 2, 5, a_3, 3, a_4, 7, a_5 \rangle$;
$B_5 = \langle \infty_1, 3, \infty_2, 5, 0, 1, 7 \rangle$.

Obviously, each a_i appears in two basic blocks above: $\langle\cdots u, a_i, v, \cdots\rangle$ and $\langle\cdots u', a_i, v', \cdots\rangle$, such that $u \not\equiv u'$ and $v \not\equiv v' \pmod 2$. The blocks $B_5 = \langle\infty_1, u, \infty_2, v, \cdots, w\rangle$ in three cases satisfy $u \equiv v \equiv w \pmod 2$. Then, let G be the finite permutation group generated by Rf. Let $O(B_i)$ be the block-orbit containing B_i under the action of G. It is easy to see that each ordered pair (x, y), which belongs to $X \times X$ or $X \times Y$, appears in exactly one block of $\bigcup_{i=1}^{5} O(B_i))$. Let

$$\mathcal{A} = \mathcal{C} \cup (\bigcup_{i=1}^{5} O(B_i)).$$

Obviously, $(X \cup Y, \mathcal{A}, f)$ is a $(4t+3)$-$SCMD(12t+10)$. $\square$

Lemma 6.4 [5] *Let $(X, \mathcal{B}, f)$ be a k-$SCMD(v)$.*

(1) *The self-converse block A in $\mathcal{B}$, i.e., $f(A)^{-1} = A$, must possess one of the following structures:*

Type I. $A = \langle a_1, a_2, \cdots, a_t, b_t, \cdots, b_2, b_1\rangle, \quad t = \frac{k}{2}$;

Type II. $A = \langle\infty, a_1, \cdots, a_t, \infty', b_t, \cdots, b_1\rangle, \quad t = \frac{k}{2} - 1$;

Type III. $A = \langle\infty, a_1, \cdots, a_t, b_t, \cdots, b_1\rangle, \quad t = \frac{k-1}{2}$,

where $f(a_i) = b_i$, $f(b_i) = a_i$ for $1 \le i \le t$ and $f(\infty) = \infty$, $f(\infty') = \infty'$.

(2) *If f contains a transposition (a, b), then the block covering the ordered pair (a, b) must be self-converse of Type I (if k even) or Type III (if k odd).*

(3) *If $\mathcal{B}$ contains a self-converse block, then*

(for k even): f contains at least $\frac{k}{2}$ transpositions and $\mathcal{B}$ contains at least $\frac{k}{2}$ self-converse blocks of Type I;

(for k odd): f contains at least $\frac{k-1}{2}$ transpositions and $\mathcal{B}$ contains at least $k-1$ self-converse blocks of Type III.

Theorem 6.5 *There exists no $(4t+1)$-$SCMD(4t+2)$ for any positive integer t.*

Proof. Suppose there is a $(4t+1)$-$SCMD(4t+2) = (X, \mathcal{B}, f)$, where $|\mathcal{B}| = 4t+2 = |X|$. Obviously, the elements in any block of $\mathcal{B}$ are $X\backslash\{a\}$ and the missing element x is distinct for different block. Let B_a be the block without point a. Then $\mathcal{B} = \{B_a;\ a \in X\}$. It is easy to see that, if $f(a) = b$ then $f^{-1}(B_a) = B_b$. So we have

(1) f contains no 1-cycle.

In fact, suppose $f = (\infty)\cdots$, then B_∞ is self-converse. By Lemma 6.4 (1), f contains two fixed points. By Lemma 6.4 (3), there are at least $4t$ self-converse blocks of Type III in $\mathcal{B}$. But, this is impossible since $t \ge 2$.

(2) f contains no 2-cycle (i.e., transposition).

Suppose $f = (a, b)\cdots$, then, by Lemma 6.4 (2), there is a self-converse block containing the pair (a, b) in $\mathcal{B}$. Furthermore, by Lemma 6.4 (3), $\mathcal{B}$ contains at least $4t$ self-converse blocks of Type III, which is impossible by (1).

(3) f contains no 3-cycle.

Suppose $f = (a, b, c)\cdots$, then under the action of the derived mapping $B \to f(B)^{-1}$, $B_a \to B_b \to B_c \to B_a$. So, we have $f^3(B_a)^{-1} = B_a$. Let $B_a = \langle b, x_1, \cdots, x_m, c, y_n, \cdots, y_1\rangle$, where $x_i, y_j \notin \{a, b, c\}$, $m+n = 4t-1$ and $\{x_1, \cdots, x_m\}$ or $\{y_1, \cdots, y_n\}$ may be empty. The expression of the relation $f^3(B_a)^{-1} = \langle b, f^3(y_1), \cdots, f^3(y_n), c, f^3(x_m), \cdots, f^3(x_1)\rangle = B_a$ shows that $m = n$, which is impossible since $m+n = 4t-1$.

(4) f contains no $(2s+1)$-cycle.

When $s = 0$ or 1, the conclusion is correct by (1) or (3). Now, let $s \geq 2$ and $f = (a_0, a_1, \cdots, a_{2s}) \cdots$. Obviously, if $B_{a_0} = \langle a_1, \cdots, a_2, \cdots, a_{2s}, \cdots \rangle$, then $f^{2s+1}(B_{a_0})^{-1} = \langle a_{2s}, \cdots, a_2, \cdots, a_1, \cdots \rangle = B_{a_0}$. But, this is impossible since $2s \geq 4$.

(5) f contains no $(4s+2)$-cycle.

Suppose $f = (a_0, a_1, \cdots, a_{4s+1}) \cdots$. Let $B_x = \langle a_0, a_{2s+1}, \cdots \rangle$ be the block containing the ordered pair (a_0, a_{2s+1}). Obviously, the block $f^{2s+1}(B_x)^{-1}$ contains the ordered pair (a_0, a_{2s+1}) too. So $B_x = f^{2s+1}(B_x)^{-1}$, i.e., x must belong to an odd cycle of f, which is impossible by (4).

So f can only contain $4s$-cycles. But $4t + 2 \equiv 2 \pmod 4$, which is impossible. Thus, there exists no $(4t+1)$-$SCMD(4t+2)$. □

The proof of Theorem 1.3:

All possibilities are shown in the following table:

$k \equiv \pmod 4$	$v = (4s+1)k + 1 \equiv \pmod 4$	$v = (4s+3)k + 1 \equiv \pmod 4$
1	2	0
3	0	2

In this table, two parts of $v \equiv 0 \pmod 4$ have been solved in Theorem 1.1. By Lemma 6.1 and Theorem 6.2, the following recursive relations hold when there is at least one CDC in the original constructions:

($k \equiv 1 \pmod 4$) k-$SCMD((4s+3)k+1) \to k$-$SCMD((4s+5)k+1)$;

($k \equiv 3 \pmod 4$) k-$SCMD((4s+1)k+1) \to k$-$SCMD((4s+3)k+1)$.

While the two exceptions have been solved:

$k > 5$ and there is no CDC in the original construction, see Theorem 6.3;

there exists no $(4t+1)$-$SCMD(4t+2)$, see Theorem 6.5. □

References

[1] C.J. Colbourn and J.H. Dinitz, *The CRC Handbook of Combinatorial Designs*, *CRC* Press, Inc., 1996.

[2] C.J. Colbourn and A. Rosa, *Directed and Mendelsohn triple systems*, Chapter 4 of Contemporary Design Theory, Wiley Interscience Publication (1992), 97–136.

[3] Yanxun Chang, Guihua Yang and Qingde Kang, *The spectrum of self-converse MTS*, Ars Combinatoria, 44 (1996), 273–281.

[4] Jie Zhang, *The spectrum of simple self-converse* $MTS(v, \lambda)$, Acta Mathematic Applicate Sinica, Vol. 20, No. 4 (1997), 487–497.

[5] Qingde Kang, *Self-converse Mendelsohn designs with block size* $4t+2$, Journal of Combinatorial Designs 7 (1999), 283–310.

[6] Qingde Kang, Xiuling Shan and Qiujie Sun, *The spectrum of self-converse Mendelsohn designs with block size* 4 *and* 5, Journal of Combinatorial Designs 8 (2000), 411–418.

Appendix

1. $\underline{7\text{-}SCMD(36)}$ (Theorem 4.1, let $t = 1, s = 1$).
$X = Z_{36}$ and $f = (0, 1, \cdots, 35)$.
(I) $SDC(1, -2, 10, 18, 1, -2, 10)$.
(II) $CDC(D)$, where D is taken as follows.
(1) $(A[5, 8], -11, 9, 4)$;
(2) $(A[14, 17], -13, 12, 3)$.

2. $\underline{15\text{-}SCMD(136)}$ (Theorem 4.2, let $t = 3, s = 2$).
$X = Z_{136}$ and $f = (0, 1, \cdots, 135)$.
(I) $SDC(1, -3, 68, -3, 6, -4, 2, 5, 27, 1, 2, -4, 6, 5, 27)$.
(II) $CDC(D)$, where D is taken as follows.
(1) $(A[11, 22], -28, 26, 8)$;
(2) $(23, -24, A[34, 43], -29, 25, 10)$;
(3) $(A[44, 55], -32, 31, 7)$;
(4) $(A[56, 67], -33, 30, 9)$.

3. $\underline{7\text{-}SCMD(64)}$ (Theorem 4.3, let $t = 1$).
$X = Z_{64}$ and $f = (0, 1, \cdots, 63)$.
(I) $SDC(1, -2, 17, 32, 1, -2, 17)$.
(II) $CDC(D)$, where D is taken as follows.
(1) $(A[7, 10], -18, 16, 4)$;
(2) $(A[11, 14], -19, 15, 6)$;
(3) $(A[24, 27], -22, 21, 3)$;
(4) $(A[28, 31], -23, 20, 5)$.

4. $\underline{11\text{-}SCMD(56)}$ (Theorem 4.4, let $t = 2, s = 1$).
$X = Z_{56}$ and $f = (0, 1, \cdots, 55)$.
(1) $SDC(1, -3, 28, -3, 18, 2, -4, 1, -4, 2, 18)$;
(2) $CDC(15, -16, A[22, 27], -21, 20, 5)$;
(3) $CDC(A[7, 14], -19, 17, 6)$.

5. $\underline{5\text{-}SCMD(36)}$ (Theorem 4.6, let $t = 1, s = 1$).
$X = Z_{36}$ and $f = (0, 1, \cdots, 35)$.
(1) $SDC(1, 8, 18, 1, 8)$;
(2) $CDC(5, -6, 7, -9, 3)$;
(3) $CDC(14, -15, 11, -12, 2)$;
(4) $CDC(16, -17, 10, -13, 4)$.

6. $\underline{13\text{-}SCMD(248)}$ (Theorem4.7, let $t = 3, s = 4$).
$X = Z_{248}$ and $f = (0, 1, \cdots, 247)$.
(I) $SDC(M, 124, M^{-1})$, where $M = (A[1, 7]_2, 23, 43)$.
(II) $CDC(D)$, where D is taken as follows.
(1) $(A[2, 8]_2, 11, -13, 22, -24, 56, -58, 57, -59, 12)$;
(2) $(9, -10, A[39, 47] \setminus \{43\}, A[60, 63], -64, 55, 14)$;
(3) $(A[25, 34], -65, 54, 16)$;
(4) $(A[64, 73], -66, 60, 18)$;

(5) $(A[74,83], -67, 59, 20)$;

(6) $(A[84+10(i-1), 83+10i], -(47+i), 39-i, 13+2i)$.

7. $\underline{5\text{-}SCMD(20t+16)}$.

$X = Z_{20t+16}$ and $f = (0, 1, \cdots, 20t+15)$.

(I) $\begin{cases} SDC(1, 5t+3, 10t+8, 1, 5t+3) & t \text{ is odd} \\ SDC(1, 5t+3, 10t+8, 5t+3, 1) & t \text{ is even} \end{cases}$.

(II) $CDC(D)$, where D is taken as follows.

(1) $(2t+2i+1, -(2t+2i+2), 5t-i+3, -(5t+i+3), 2i+1)$, $1 \le i \le t$;

(2) $(8t+2i+4, -(8t+2i+5), 7t-i+5, -(7t+i+4), 2i)$, $1 \le i \le t+1$.

8. $\underline{17\text{-}SCMD(528)}$ (Theorem 4.10, let $t = 4, s = 7$).

$X = Z_{248}$ and $f = (0, 1, \cdots, 527)$.

(I) $SDC(M, 264, M^{-1})$, where $M = (1, -3, A[257, 263]_2, -43, -83)$.

(II) $CDC(D)$, where D is taken as follows.

(1) $(-2, 4, A[256, 262]_2, -A[5, 10]^{-1}, -A[12, 15]^{-1}, 11)$;

(2) $(A[39, 47] \setminus \{43\}, A[62, 67], -70, 61, 16)$;

(3) $(30, -31, 68, -69, A[77, 87] \setminus \{83\}, -71, 60, 18)$;

(4) $(A[88+14(i-1), 87+14i], -(71+i), 60-i, 18+2i)$, $1 \le i \le 5$;

(5) $(A[158+14(i-1), 157+14i], -(47+i), 39-i, 15+2i)$, $1 \le i \le 7$.

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